

Astrophysics

Compulsory Home Exercises. Solutions of Problem Set 4.

1. Let's approximate the directional (μ) dependence of the specific intensity using the first-order Taylor expansion,

$$I(\tau, \mu) = I_0(\tau) + I_1(\tau) \mu$$

where I_0 and I_1 depend on the vertical optical depth τ but not on μ . Assuming a grey atmosphere,

- a. Derive expressions for the flux F , mean intensity J , and radiation pressure P_{rad} , in terms of I_0 and I_1 .

Solution:

The mean intensity:

$$J_\lambda = \frac{1}{4\pi} \oint I_\lambda d\omega = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_0^\pi I_\lambda \sin\theta d\theta = \frac{1}{4\pi} \int_0^{2\pi} d\varphi \int_{-1}^1 (I_0 + I_1 \mu) d\mu = I_0$$

The flux:

$$F_\lambda = \oint I_\lambda \cos\theta d\omega = \int_0^{2\pi} d\varphi \int_{-1}^1 (I_0 + I_1 \mu) \mu d\mu = \frac{4\pi}{3} I_1$$

The radiation pressure:

$$P_{rad} = \frac{1}{c} \int_0^{2\pi} d\varphi \int_{-1}^1 (I_0 + I_1 \mu) \mu^2 d\mu = \frac{4\pi}{3c} I_0$$

- b. Show that the radiation field obeys the Eddington approximation and

$$P_{rad} = \frac{4\pi}{3c} J$$

Solution:

$$\text{From the above, } I_0 = J = \frac{3cP_{rad}}{4\pi} \rightarrow P_{rad} = \frac{4\pi}{3c} J$$

- c. Within the Eddington approximation, the solution of the radiative transfer equation is

$$S = \frac{3}{4\pi} \left(\tau + \frac{2}{3} \right) F$$

Use this solution to find an expression for I_0 as a function of I_1 and τ .

Solution:

In the grey case, $S = J$.

Then, from the above expressions for J and F :

$$S = J = I_0 = \frac{3}{4\pi} \left(\tau + \frac{2}{3} \right) \frac{4\pi}{3} I_1 = \left(\tau + \frac{2}{3} \right) I_1$$

$$I_0 = \left(\tau + \frac{2}{3} \right) I_1$$

- d. Show that if $\tau \gg 1$ then $I_0 \gg I_1$ (this result justifies the use of the first-order Taylor expansion at large optical depths, showing that the radiation field becomes isotropic).

Solution:

if $\tau \gg 1$ then from $I_0 = \left(\tau + \frac{2}{3} \right) I_1 \rightarrow I_0 \approx \tau I_1 \rightarrow I_0 \gg I_1$

2. Assuming a grey atmosphere and using the Eddington approximation and an expression for the source function S as a function of vertical optical depth τ and flux F (Lecture 18),

- a. Calculate the upward specific intensity $I(\tau, \mu)$ as a function of F , τ , and direction $\mu > 0$, using the formal solution to RTE (Lecture 17)

$$I_\lambda(\tau_{\lambda,v}, \mu > 0) = \int_{\tau_{\lambda,v}}^{\infty} S_\lambda e^{(\tau_{\lambda,v}-t)/\mu} \frac{dt}{\mu},$$

Solution:

The source function S as a function of τ and F (from Lecture 18):

$$S(\tau) = \frac{3}{4\pi} \left(\tau + \frac{2}{3} \right) F$$

Substituting in the above expression (pay attention that t is a dummy integration variable) for S , we have:

$$I(\tau, \mu > 0) = \int_{\tau}^{\infty} \frac{3}{4\pi} \left(t + \frac{2}{3} \right) F e^{(\tau-t)/\mu} \frac{dt}{\mu} = \frac{3}{4\pi} F \int_{\tau}^{\infty} \left(t + \frac{2}{3} \right) e^{\frac{\tau-t}{\mu}} \frac{dt}{\mu} \Rightarrow$$

Substitute: $u = \frac{\tau-t}{\mu}$, $t = \tau - \mu u$:

$$\Rightarrow -\frac{3}{4\pi} F \int_{\tau}^{\infty} \left(\tau - \mu u + \frac{2}{3} \right) e^u du = \frac{3}{4\pi} F \int_{\tau}^{\infty} \left(\mu u e^u du - \left(\tau + \frac{2}{3} \right) e^u du \right) \Rightarrow$$

$$\Rightarrow \frac{3}{4\pi} F \left[-\left(\tau + \frac{2}{3}\right) e^{\frac{\tau-t}{\mu}} \right]_{\tau}^{\infty} + \frac{3}{4\pi} F \mu \int_{\tau}^{\infty} u e^u du \Rightarrow$$

A standard integral $\int x e^x dx = e^x(x-1)$:

$$\Rightarrow \frac{3}{4\pi} F \left(\tau + \frac{2}{3}\right) + \frac{3}{4\pi} F \left[\left(\frac{\tau-t}{\mu} - 1\right) e^{\frac{\tau-t}{\mu}} \right]_{\tau}^{\infty}$$

$$I(\tau, \mu > 0) = \frac{3}{4\pi} F \left(\tau + \frac{2}{3} + \mu\right)$$

- b. Using your expression for the upward specific intensity $I(\tau, \mu)$, evaluate the upward component of the radiative flux,

$$F^+ = 2\pi \int_0^1 I(\mu) \mu d\mu$$

Solution:

$$F^+ = 2\pi \frac{3}{4\pi} F \int_0^1 \left(\tau + \frac{2}{3} + \mu\right) \mu d\mu = \frac{3}{2} F \left[\frac{\tau\mu^2}{2} + \frac{2}{3} \frac{\mu^2}{2} + \frac{\mu^3}{3} \right]_0^1$$

$$F^+ = F \left(1 + \frac{3}{4}\tau\right)$$

- c. Find the corresponding downward component of the flux, F^- .

Solution:

$$F = F^+ - F^-$$

$$F^- = F^+ - F = F \left(1 + \frac{3}{4}\tau\right) - F = \frac{3}{4} F \tau$$