Astrophysics

Compulsory Home Exercises. Problem Set 2.

Solutions

Problem 2.1

What is the energy released by the nuclear reactions of carbon burning (fusion of 2 carbon nuclei)? Give the answer in **MeV** and **ergs per gram**.

Solution:

Carbon burning - fusion of two carbon nuclei - requires temperatures above 5×10^8 K and density $\rho > 3 \times 10^6$ g cm^{-3}. Several options for carbon burning exist, with different, temperature-dependent probabilities, e.g.

$$^{12}_{6}\mathrm{C} + ^{12}_{6}\mathrm{C} \rightarrow ^{23}_{12}\mathrm{Mg} + n$$

$$^{12}_{6}\mathrm{C} + ^{12}_{6}\mathrm{C} \rightarrow ^{23}_{11}\mathrm{Na} + p$$

$$^{12}_{6}\mathrm{C} + ^{12}_{6}\mathrm{C} \rightarrow ^{20}_{10}\mathrm{Ne} + ^{4}_{2}\mathrm{He}$$

But let's consider the simplest reaction

$$^{12}_{6}\mathrm{C} + ^{12}_{6}\mathrm{C} \rightarrow ^{24}_{12}\mathrm{Mg} + \gamma$$

$m(^{12}_{6}\mathrm{C}) = 12.000$ amu $\qquad m(^{24}_{12}\mathrm{Mg}) = 23.985$ amu

1amu =1/12 $m(^{12}\mathrm{C})$ = 931.49 MeV/c^2 =1.660 $\times 10^{-24}$ g (Slide 220)
1 eV =1.602$\times 10^{-12}$ erg

$[2m(^{12}_{6}\mathrm{C}) - m(^{24}_{12}\mathrm{Mg})]c^2 = 0.015 \times 931.49\ \mathrm{MeV} = 13.97\ \mathrm{MeV}$

$= 13.97 \times 10^6 \times 1.602 \times 10^{-12}\ \mathrm{erg} \quad = 2.24 \times 10^{-5}\ \mathrm{erg}$ per 2 carbon nuclei (per 24 amu)

Then

$= 2.24 \times 10^{-5}$ / 24 / 1.660 $\times 10^{-24}$ g = 5.62×10^{17} erg / g

Problem 2.2

Calculate the mean molecular weight μ for

1) the completely ionized stellar interior, where we have 45% hydrogen, 52% helium, and 3% heavy elements by mass,
2) completely ionized hydrogen,
3) completely ionized helium,
4) neutral gas at the solar interior abundance, 73% hydrogen, 25% helium, and 2% heavy elements by mass.

Solution:

From Lecture 7, slide 200:

$$\mu^{-1} = 2X + \frac{3}{4}Y + \frac{1}{2}Z$$

1) X=0.45, Y=0.52, Z=0.03 → μ=0.766
2) X=1.0 → μ=0.5
3) Y=1.0 → μ=1.33
4) X=0.73, Y=0.25, Z=0.02
   But now it is neutral gas: $\mu^{-1} = X + \frac{1}{4}Y + \frac{1}{A}Z$
   A is the atomic mass of heavy elements. Most abundant heavy elements in the Milky Way are Oxygen (1.04%) and Carbon (0.46%). Then A=(16*1.04+12*0.46)/(1.04+0.46)=14.8
   Thus, μ=1.26 (and you get the same result if you assume Z=0).

Problem 2.3

Prove that for the case when Z is negligible, the mean molecular weight per electron, $\mu_e = \frac{\rho}{n_e m_H}$, can be approximately expressed as $\mu_e \approx \frac{2}{1+X}$

	H	He
Number density of nuclei	$\frac{X\rho}{m_H}$	$\frac{Y\rho}{4m_H}$
Number density of electrons	$\frac{X\rho}{m_H}$	$\frac{2Y\rho}{4\,m_H}$

Solution:

Z=0, *Y*=1-*X*

From Lecture 7, slide 200:

$$\mu_e^{-1} = X + \frac{2}{4}Y = X + \frac{1}{2} - \frac{X}{2} = \frac{X+1}{2}$$

Problem 2.4

Does a lower Gamow energy E_G increase or decrease the probability of penetration?

Solution:

A lower E_G increases the probability of penetration: $P_{pen} \approx e^{-\sqrt{E_G/E}}$

Problem 2.5

a) Calculate the Gamow energy E_G (in electronvolts) for the collision of two α-particles (helium-4 nuclei, ${}^4_2\text{He}$) and find the penetration probability P_{pen} for the typical kinetic energy of particles in the Sun's core, E ~ 1 keV. Compare the results with the case of two protons. Explain the result.

b) What temperature is required to have the probability of penetration of two α-particles similar to that of two protons in the Sun's core?

Solution:

From Lecture 8, slide 227:

a) $P_{pen} \approx e^{-\sqrt{\frac{E_G}{E}}}$; $\quad E_G = 2m_r c^2 (\pi\alpha Z_1 Z_2)^2$

$m_r = m_1 m_2/(m_1 + m_2) = 16/8 \times m_p = 2m_p$

$E_G = 2m_r c^2 (\pi\alpha Z_1 Z_2)^2 = 2 \times 2 \times 939 \text{ MeV} \times \left(\frac{3.14}{137} * 4\right)^2 = 31.6 \text{ MeV}$

$P_{pen} \approx 6.3 \times 10^{-78}$

b) For 2 protons (see lecture 8, slide 228): E_G=0.49 MeV; $P_{pen} \approx 2.4 \times 10^{-10} = P_{pp}$

$e^{-\sqrt{E_G(2He)/E}} = P_{pp}$

$E = E_G(2He)/\left[\ln\left(P_{pp}\right)\right]^2 = 31.6 \text{ MeV}/490.6 = 64400 \text{ eV} \approx 750 \times 10^6 \text{ K}$

1 eV =11650 K (Slide 220)

Problem 2.6

Prove that according to the virial theorem, the mean temperature of a star can be expressed as

$$\bar{T} \propto M^{2/3} \rho^{1/3}$$

Solution:

From Lecture 3, slide 105:

$$\bar{T} = \frac{e_G}{3} \frac{\mu m_p}{k} \frac{GM}{R}$$

$\bar{\rho} = \frac{M}{4/3\pi R^3}$ → $R = \left(\frac{M}{4/3\pi\bar{\rho}}\right)^{1/3}$ → $\bar{T} \propto M^{2/3} \rho^{1/3}$