

Astrophysics

Compulsory Home Exercises. Problem Set 2.

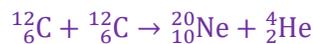
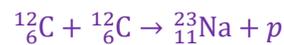
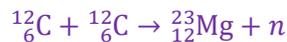
Solutions

Problem 2.1

What is the energy released by the nuclear reactions of carbon burning (fusion of 2 carbon nuclei)? Give the answer in **MeV** and **ergs per gram**.

Solution:

Carbon burning - fusion of two carbon nuclei - requires temperatures above 5×10^8 K and density $\rho > 3 \times 10^6$ g cm⁻³. Several options for carbon burning exist, with different, temperature-dependent probabilities, e.g.



But let's consider the simplest reaction



$$m({}^{12}_6\text{C}) = 12.000 \text{ amu} \quad m({}^{24}_{12}\text{Mg}) = 23.985 \text{ amu}$$

$$1 \text{ amu} = 1/12 m({}^{12}_6\text{C}) = 931.49 \text{ MeV}/c^2 = 1.660 \times 10^{-24} \text{ g} \quad (\text{Slide 220})$$

$$1 \text{ eV} = 1.602 \times 10^{-12} \text{ erg}$$

$$[2m({}^{12}_6\text{C}) - m({}^{24}_{12}\text{Mg})]c^2 = 0.015 \times 931.49 \text{ MeV} = 13.97 \text{ MeV}$$

$$= 13.97 \times 10^6 \times 1.602 \times 10^{-12} \text{ erg} = 2.24 \times 10^{-5} \text{ erg per 2 carbon nuclei (per 24 amu)}$$

Then

$$= 2.24 \times 10^{-5} / 24 / 1.660 \times 10^{-24} \text{ g} = 5.62 \times 10^{17} \text{ erg / g}$$

Problem 2.2

Calculate the mean molecular weight μ for

- 1) the completely ionized stellar interior, where we have 45% hydrogen, 52% helium, and 3% heavy elements by mass,
- 2) completely ionized hydrogen,
- 3) completely ionized helium,
- 4) **neutral** gas at the solar interior abundance, 73% hydrogen, 25% helium, and 2% heavy elements by mass.

Solution:

From Lecture 7, slide 200:

$$\mu^{-1} = 2X + \frac{3}{4}Y + \frac{1}{2}Z$$

- 1) $X=0.45, Y=0.52, Z=0.03 \rightarrow \mu=0.766$
- 2) $X=1.0 \rightarrow \mu=0.5$
- 3) $Y=1.0 \rightarrow \mu=1.33$
- 4) $X=0.73, Y=0.25, Z=0.02$

But now it is neutral gas: $\mu^{-1} = X + \frac{1}{4}Y + \frac{1}{A}Z$

A is the atomic mass of heavy elements. Most abundant heavy elements in the Milky Way are Oxygen (1.04%) and Carbon (0.46%). Then $A=(16*1.04+12*0.46)/(1.04+0.46)=14.8$

Thus, $\mu=1.26$ (and you get the same result if you assume $Z=0$).

Problem 2.3

Prove that for the case when Z is negligible, the mean molecular weight per **electron**, $\mu_e = \frac{\rho}{n_e m_H}$, can be approximately expressed as $\mu_e \approx \frac{2}{1+X}$

Solution:

$$Z=0, \quad Y=1-X$$

From Lecture 7, slide 200:

	H	He
Number density of nuclei	$\frac{X\rho}{m_H}$	$\frac{Y\rho}{4m_H}$
Number density of electrons	$\frac{X\rho}{m_H}$	$\frac{2Y\rho}{4m_H}$

$$\mu_e^{-1} = X + \frac{2}{4}Y = X + \frac{1}{2} - \frac{X}{2} = \frac{X+1}{2}$$

Problem 2.4

Does a lower Gamow energy E_G increase or decrease the probability of penetration?

Solution:

A lower E_G increases the probability of penetration: $P_{pen} \approx e^{-\sqrt{E_G/E}}$

Problem 2.5

- Calculate the Gamow energy E_G (in electronvolts) for the collision of two α -particles (helium-4 nuclei, ${}^4_2\text{He}$) and find the penetration probability P_{pen} for the typical kinetic energy of particles in the Sun's core, $E \sim 1$ keV. Compare the results with the case of two protons. Explain the result.
- What temperature is required to have the probability of penetration of two α -particles similar to that of two protons in the Sun's core?

Solution:

From Lecture 8, slide 227:

$$\begin{aligned} \text{a) } P_{pen} &\approx e^{-\sqrt{\frac{E_G}{E}}}; & E_G &= 2m_r c^2 (\pi\alpha Z_1 Z_2)^2 \\ m_r &= m_1 m_2 / (m_1 + m_2) = 16/8 \times m_p = 2m_p \\ E_G &= 2m_r c^2 (\pi\alpha Z_1 Z_2)^2 = 2 \times 2 \times 939 \text{ MeV} \times \left(\frac{3.14}{137} * 4\right)^2 = 31.6 \text{ MeV} \\ P_{pen} &\approx 6.3 \times 10^{-78} \end{aligned}$$

$$\text{b) For 2 protons (see lecture 8, slide 228): } E_G = 0.49 \text{ MeV}; P_{pen} \approx 2.4 \times 10^{-10} = P_{pp}$$

$$e^{-\sqrt{E_G(2He)/E}} = P_{pp}$$

$$E = E_G(2He) / [\ln(P_{pp})]^2 = 31.6 \text{ MeV} / 490.6 = 64400 \text{ eV} \approx 750 \times 10^6 \text{ K}$$

$$1 \text{ eV} = 11650 \text{ K (Slide 220)}$$

Problem 2.6

Prove that according to the virial theorem, the mean temperature of a star can be expressed as

$$\bar{T} \propto M^{2/3} \rho^{1/3}$$

Solution:

From Lecture 3, slide 105:

$$\bar{T} = \frac{e_G}{3} \frac{\mu m_p}{k} \frac{GM}{R}$$

$$\bar{\rho} = \frac{M}{4/3\pi R^3} \quad \rightarrow \quad R = \left(\frac{M}{4/3\pi \bar{\rho}}\right)^{1/3} \quad \rightarrow \quad \bar{T} \propto M^{2/3} \rho^{1/3}$$