Astrophysics

Compulsory Home Exercises. Problem Set 1.

Solutions

Problem 1.1

In a star of mass M, the density decreases from the centre to the surface as a function of radial distance r, according to

$$\rho = \rho_0 \left[1 - \left(\frac{r}{R}\right)^2\right]$$

where ρ_0 is a given constant and R is the star's radius,

(a) Find $m(r)$.
(b) Derive the relation between M and R.
(c) Show that the average density of the star (total mass divided by total volume) is $0.4\rho_0$.

Solution:

From Lecture 2, slide 74:

a) $m(r) = \int_0^r 4\pi r^2 \rho(r)\, dr = 4\pi\rho_0 \int_0^r r^2 \left[1 - \left(\frac{r}{R}\right)^2\right] dr = 4\pi\rho_0 \left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right]$

b) $M = m(R) = 4\pi\rho_0 \left[\frac{R^3}{3} - \frac{R^3}{5}\right] = 8\pi\rho_0 R^3/15$

c) $\bar{\rho} = \frac{M}{V} = \frac{8\pi\rho_0 R^3}{15*4/3\pi R^3} = 0.4\rho_0$

Problem 1.2

For a star of mass M and radius R, find the central pressure and check the validity of inequality $P_c \geq \frac{3}{8\pi}\frac{GM^2}{R^4}$ for the following cases:

(a) a uniform density
(b) a density profile as in Exercise 1.1.

Solution:

From Lecture 2, slide 80:
The equation of hydrostatic equilibrium: $\frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2}\rho(r)$
We need to integrate it from the centre where P=P_c to the surface where P_S=0:

$$P = P_C - P_S = \int_0^R \frac{Gm(r)}{r^2}\rho(r)dr$$

a)

$$\rho = \bar{\rho} = \frac{M}{V} = \frac{3M}{4\pi R^3}$$

$$m(r) = \frac{4\pi r^3}{3}\bar{\rho} = \frac{Mr^3}{R^3}$$

$$P = G\int_0^R \frac{Mr^3}{R^3 r^2}\frac{3M}{4\pi R^3}dr = G\frac{3M^2}{4\pi R^6}\int_0^R r dr = \frac{3GM^2}{8\pi R^4}$$

b) From problem 1.1:

$$m(r) = 4\pi\rho_0\left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right]$$

$$M = \frac{8\pi\rho_0 R^3}{15} \quad \rightarrow \quad \rho_0 = \frac{15M}{8\pi R^3}$$

$$P = 4\pi\rho_0^2 G\int_0^R \left[\frac{r}{3} - \frac{r^3}{5R^2}\right]\left[1 - \frac{r^2}{R^2}\right]dr = 4\pi\rho_0^2 G\int_0^R \left[\frac{r}{3} - \frac{r^3}{3R^2} - \frac{r^3}{5R^2} + \frac{r^5}{5R^4}\right]dr =$$

$$= 4\pi\rho_0^2 G\int_0^R \left[\frac{r}{3} - \frac{8r^3}{15R^2} + \frac{r^5}{5R^4}\right]dr = 4\pi\rho_0^2 G\left[\frac{r^2}{6} - \frac{8r^4}{60R^2} + \frac{r^6}{30R^4}\right]_0^R$$

$$= \frac{4\pi G 15^2 M^2}{8^2\pi^2 R^6}\frac{R^2}{15} = \frac{15GM^2}{16\pi R^4} > \frac{3}{8\pi}\frac{GM^2}{R^4}$$

Problem 1.3

In Lecture 3 we obtained the mean temperature of the star to be

$$\bar{T} = \frac{e_G}{3}\frac{\mu m_p}{k}\frac{GM}{R}$$

For a star of mass M and radius R, find the value of e_G for two cases:

(a) a uniform density
(b) a density profile as in Exercise 1.1.

Solution:

$e_G = \int_0^1 \frac{q}{x} dq \quad q = m/M,\, x = r/R$

$$e_G = \int_0^M \frac{mR}{M^2 r} dm = \frac{R}{M^2}\int_0^M \frac{m}{r} dm$$

a) From Problem 1.2a:

$$m(r) = \frac{Mr^3}{R^3} \quad \rightarrow \quad r = R\left(\frac{m}{M}\right)^{1/3}$$

$$e_G = \frac{R}{M^2}\int_0^M \frac{m}{R}\left(\frac{M}{m}\right)^{1/3} dm = M^{-5/3}\int_0^M m^{2/3} dm = M^{-5/3}\frac{3}{5}M^{5/3} = \frac{3}{5}$$

b) From Problem 1.1:

$$\rho_0 = \frac{15M}{8\pi R^3}$$

$$m(r) = 4\pi\rho_0\left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right] = \frac{15M}{2R^3}\left[\frac{r^3}{3} - \frac{r^5}{5R^2}\right]$$

$$e_G = \frac{R}{M^2}\int_0^M \frac{m}{r} dm$$

$$= \frac{R}{M^2}\frac{15M}{2R^3}\int_0^R \left[\frac{r^2}{3} - \frac{r^4}{5R^2}\right] d\left[\frac{15M}{2R^3}\left(\frac{r^3}{3} - \frac{r^5}{5R^2}\right)\right] =$$

$$= \frac{15}{2MR^2}\frac{15M}{2R^3}\int_0^R \left[\frac{r^2}{3} - \frac{r^4}{5R^2}\right]\left[r^2 - \frac{r^4}{R^2}\right] dr = \frac{15^2}{4R^5}\int_0^R \left[\frac{r^4}{3} - \frac{8r^6}{15R^2} + \frac{r^8}{5R^4}\right] dr$$

$$= \frac{15^2}{4R^5}\left[\frac{r^5}{15} - \frac{8r^7}{105R^2} + \frac{r^9}{45R^4}\right]_0^R = \frac{15^2}{4R^5}\left(\frac{7R^5}{105} - \frac{8R^5}{105} + \frac{R^5}{45}\right) = \frac{15}{4}\left(-\frac{1}{7} + \frac{1}{3}\right) = \frac{15}{21}$$

$$= \frac{5}{7}$$

Problem 1.4:

Assume that the whole mass of a star is concentrated in the centre. Please show that the time for collapse from radius R to 0 is

$$t_{dyn} = \frac{\pi}{2\sqrt{2}}\left(\frac{R^3}{GM}\right)^{1/2}$$

Solution:
We can simply use Kepler's Third law:

$$\frac{a^3}{T^2} = \frac{G(M+m)}{4\pi^2} \implies T^2 = \frac{4\pi^2}{G(M+m)}a^3$$

Assuming eccentricity $e \rightarrow 1$ and m $\rightarrow 0$, the free-fall time will be $\tau_{ff} = T/2$ and $a = R/2$:

$$\tau_{ff} = t_{dyn} = \frac{\pi}{2\sqrt{2}}\left(\frac{R^3}{GM}\right)^{1/2}$$

There are other ways to solve this problem.

Problem 1.5 (from lecture 4):

How much mass per time must the Sun accrete in order for its luminosity to equal that of observed? Show the answer in Solar masses. Discuss how this process of accretion will affect Earth?

Solution:
$L_\odot = 3.85 \cdot 10^{33}$ erg/s

The accreted gas must lose the gravitational potential energy liberated as it falls toward the mass gaining star. If this energy is radiated, luminosity is:

$$L = \frac{GM\,dm/dt}{R} = \frac{GM\dot{M}}{R} \quad \rightarrow \quad \dot{M} = \frac{LR}{GM} = \frac{3.85 \cdot 10^{33} \times 6.96 \cdot 10^{10}}{6.67 \cdot 10^{-8} \times 1.99 \cdot 10^{33}} = 2.0 \times 10^{18} \text{ g/s}$$

$$\dot{M} = 3.2 \times 10^{-8}\ M_\odot \text{ / yr}$$

On a relatively short time scale (~30 million years) the solar mass would be doubled which will result in changing of the Earth year duration and the distance from Earth to the Sun.