

# Spectral line formation

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EINSTEIN COEFFICIENTS  
**LINE PROFILES: NATURAL BROADENING**  
BROADENING OF SPECTRAL LINES  
NATURAL LINE BROADENING  
THERMAL (DOPPLER) BROADENING  
CONVOLUTION OF DIFFERENT BROADENING  
PROCESSES  
PRESSURE BROADENING  
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# Line profiles

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All the spectral lines are not monochromatic but have a finite width and a particular profile. Width and shape of a line depend directly on atomic transitions and plasma environment

Energy levels are **not** infinitely sharp. An unavoidable source of broadening is due to the Heisenberg uncertainty principle:

$$\Delta E \Delta t \sim h/2\pi$$

$\Delta t$  being the timescale of decay (finite lifetime of energy levels).

In each spectral line, photons of **different** frequencies (but close to central frequency  $\nu_0$ ) can be absorbed.

Let us call  $\varphi(\nu)$  the probability that the transition occurs by emitting or absorbing a photon with energy  $h\nu$  (emission or absorption line,  $\int \varphi(\nu) d\nu \equiv 1$ ).

This natural broadening has the form of a **Lorentzian function**.

# Natural Line Width

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- A spectral line of an atom is formed by a transition of electron between two energy levels, whose difference yields the **frequency** of the line.
- The bound-bound absorption problem is **analogous** to the mechanical system of a damped, driven harmonic oscillator.
- In the classical picture of an atom, we can consider the electron as being bound to the atom. Any force trying to remove it will be counteracted by an opposing force. If a force were to pull on the electron and then let go, it would oscillate with eigenfrequencies  $\omega_0 = 2\pi\nu_0$ .
- The **scattering cross-section** for a *classical oscillator* can be written as

$$\sigma = \frac{8\pi}{3} \frac{e^4}{m_e^2 c^4} \left[ \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} \right] \quad \omega = 2\pi\nu$$

where the classical damping constant  $\gamma = 2e^2\omega_0^2 / 3m_e c^3 = (8\pi^2 e^2 / 3m_e c^3) \nu_0^2$

- This is the **Lorentz function** which is sharply peaked around  $\omega = \omega_0$ .

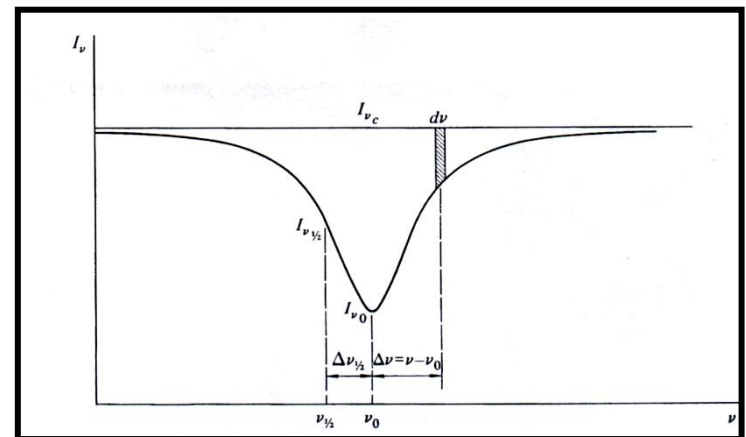
# Lorentz function (1)

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$$\sigma_{\nu} = \frac{8\pi}{3} \frac{e^4}{m_e^2 c^4} \left[ \frac{\omega^4}{(\omega^2 - \omega_0^2)^2 + \gamma^2 \omega^2} \right] \quad \omega = 2\pi\nu \quad \gamma = \frac{8\pi^2 e^2}{3m_e^2 c^3} \nu_0^2$$

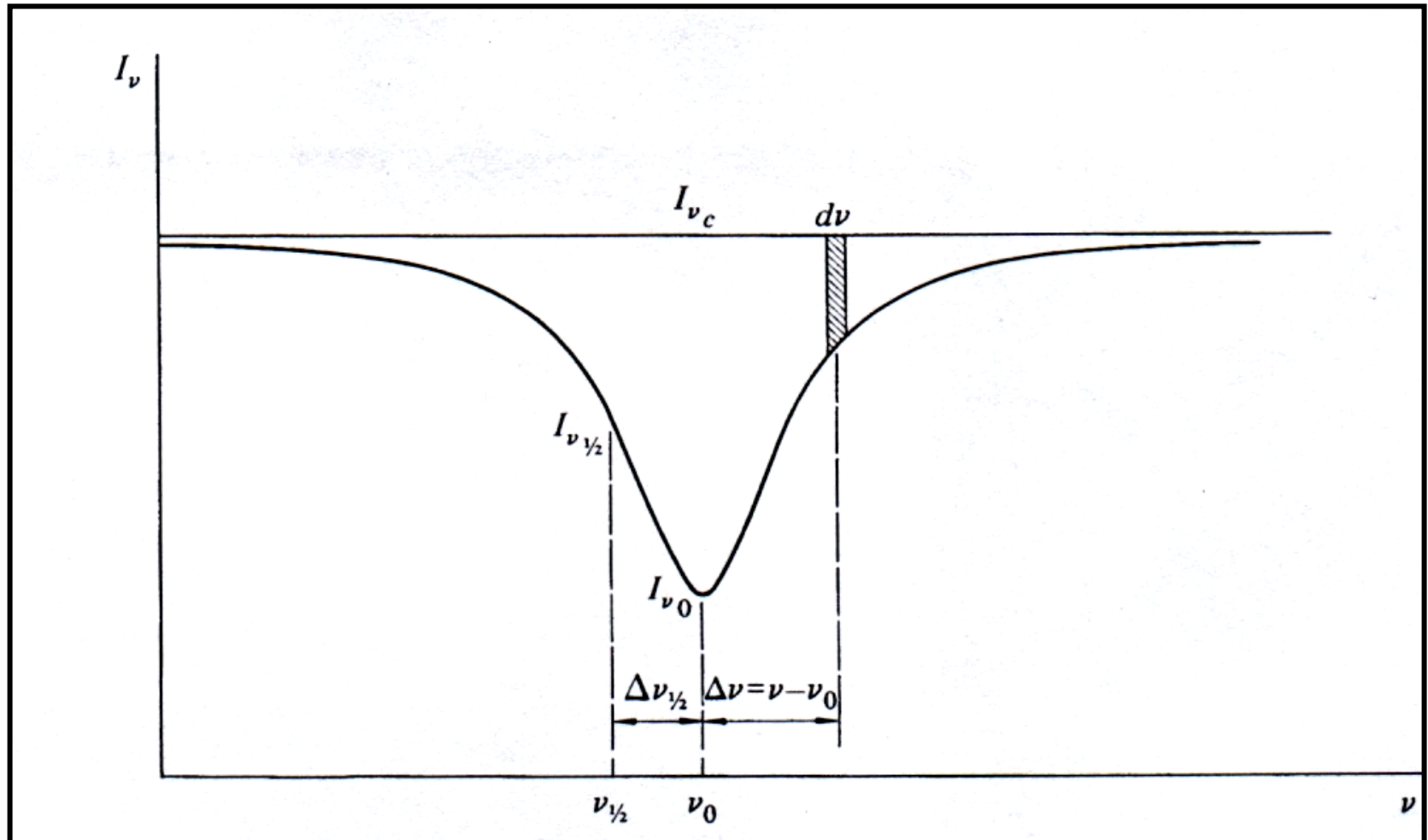
$$\sigma_{\nu} = \frac{e^2}{m_e c} \left[ \frac{\gamma/4\pi}{(\nu^2 - \nu_0^2)^2 + (\gamma/4\pi)^2} \right] = \left[ \frac{A}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2} \right]$$

Note that  $\gamma$  defines the **width** of the line.



# The Classical Damping Line Profile

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# Lorentz function (2)

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$$\gamma = \frac{8\pi^2 e^2}{3m_e^2 c^3} \nu_0^2$$

The Lorentz function  $\varphi(\nu) = \left[ \frac{A}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2} \right]$

is sharply peaked around  $\nu = \nu_0$  with a maximum of  $\varphi(\nu_0) = A/(\gamma/4\pi)^2$ .

To find the full-width at half maximum (FWHM) we find the value of  $\nu_{1/2}$  at which the function is  $1/2$  its maximum, i.e.  $\varphi(\nu_{1/2}) = 1/2 \varphi(\nu_0)$  and then solve for the FWHM =  $\Delta\nu_{1/2} = 2(\nu_{1/2} - \nu_0)$ :

$$\frac{1}{2} \frac{A}{(\gamma/4\pi)^2} = \left[ \frac{A}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2} \right] \quad (\nu - \nu_0)^2 + (\gamma/4\pi)^2 = 2(\gamma/4\pi)^2$$

we obtain  $|\nu_{1/2} - \nu_0| = (\gamma/4\pi)$

$$\Delta\nu_{1/2} = 2(|\nu_{1/2} - \nu_0|) = \gamma/2\pi$$

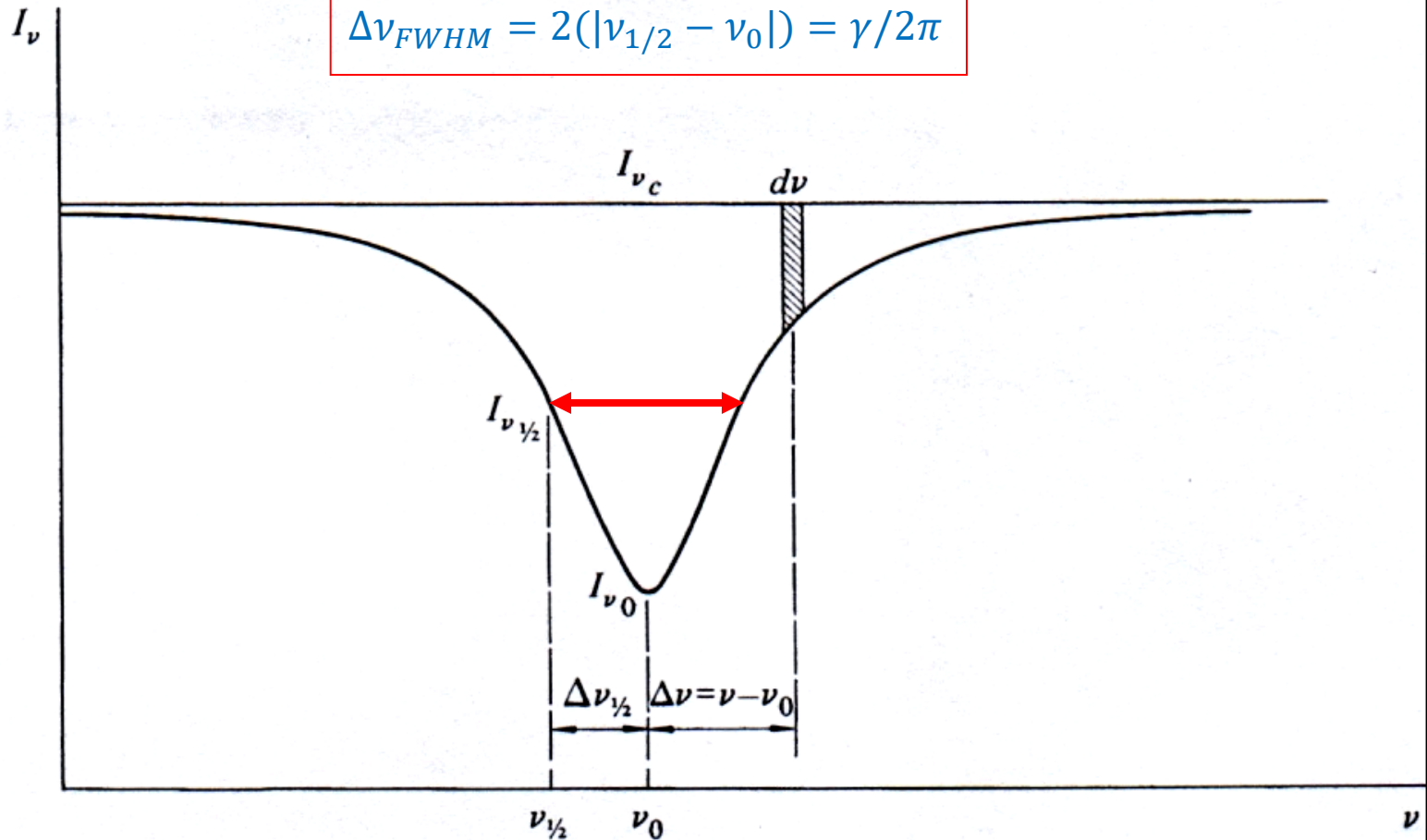
i.e.  $(\Delta\lambda)_{1/2} = \frac{\lambda_0^2}{c} (\Delta\nu)_{1/2} = \frac{\lambda_0^2}{c} \frac{\gamma}{2\pi} = \frac{4\pi e^2}{3mc^2} = \frac{4\pi}{3} r_e = \underline{0.00012 \text{ \AA}}$

Classical electron radius

# FWHM

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$$\Delta\nu_{FWHM} = 2(|\nu_{1/2} - \nu_0|) = \gamma/2\pi$$



# Oscillator Strength

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We obtain the “integrated line scattering cross-section” by integrating over all frequencies

$$\sigma_{total} = \int_0^{\infty} \sigma_{\nu} d\nu = \frac{e^2}{m_e c} \int_0^{\infty} \frac{\gamma/4\pi}{(\nu - \nu_0)^2 + (\gamma/4\pi)^2} d\nu = \frac{\pi e^2}{m_e c}$$

This **classical** result predicts a **unique** scattering relation for **all** transitions.

The **quantum-mechanical** treatment shows that line scattering cross-sections may in fact **differ** greatly. The customary way of writing this result is via

$$\sigma_{total} = \frac{\pi e^2}{m_e c} f_{ij}$$

where  $f_{ij}$  is the (dimensionless) **oscillator strength** of the transition.

Obtained from lab measurements, the Solar spectrum or quantum mechanical calculations (e.g. Opacity Project),  $f_{ij}$  and Einstein  $A$  coefficient are related via:

$$A_{ij} = \frac{6.67 \times 10^{15}}{\lambda_{ij}^2 (\text{\AA})} \frac{g_i}{g_j} f_{ij}$$

# $f_{ij}$ for Lyman and Balmer lines

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Only for the strongest transitions does  $f_{ij}$  approach unity. An electron in the  $n=2$  orbit of H is about 5 times more likely to absorb an H $\alpha$  photon and make a transition to the  $n=3$  orbit, than it is to absorb an H $\beta$  photon and jump to the  $n=4$  orbit.

For **forbidden** lines,  $f_{ij} \ll 1$ .

$\lambda$ (Å)	Line	$f_{lu}$	$g_{low}$	$g_{up}$
1215.7	Ly $\alpha$	0.41	2	8
1025.7	Ly $\beta$	0.07	2	18
972.5	Ly $\gamma$	0.03	2	32
6562.8	H $\alpha$	0.64	8	18
4861.3	H $\beta$	0.12	8	32
4340.5	H $\gamma$	0.04	8	50

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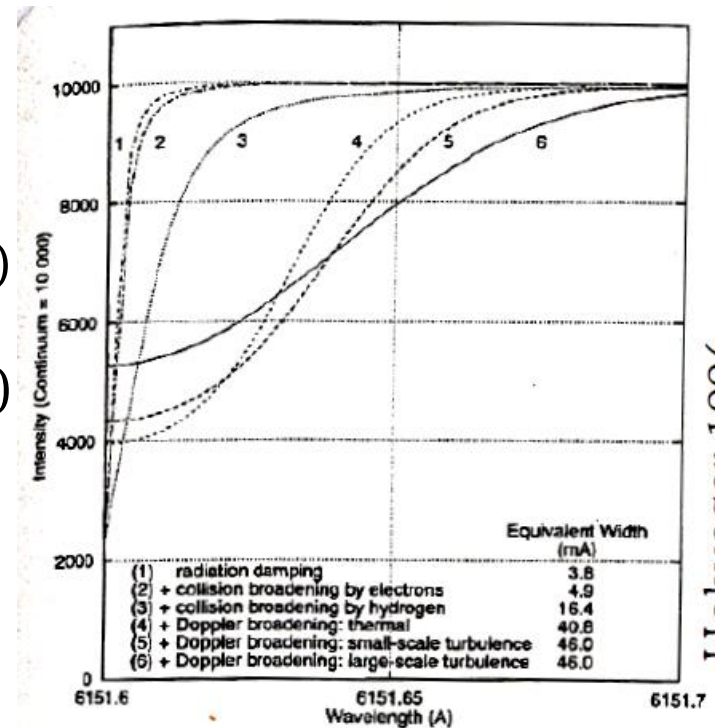
# Broadening of spectral lines

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There are numerous broadening mechanisms which influence the apparent shape of spectral lines:

- microscopic
1. Natural broadening ✓
  2. Thermal broadening ✓
  3. Microturbulence  
(treated like extra thermal broadening)
  4. Collisions (important for strong lines)
  5. Isotopic shift, hyperfine splitting (hfs)  
Zeeman effect

- macro
6. Macroturbulence
  7. Rotation
  8. Instrumental broadening



Holweger 1996

# Natural Line Broadening (1)

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As just noticed above, energy levels of atoms are intrinsically broadened due to the **Heisenberg uncertainty principle**. A decaying state  $j$  does not have a perfectly defined energy  $E_j$ , but rather a superposition of states spread around  $E_j$ .

$$\left. \begin{array}{l} \Delta E \Delta t = h/2\pi \\ E = h\nu = h\omega/2\pi \end{array} \right\} \Rightarrow \Delta\omega \Delta t = 1$$

The longer the atom is in a state ( $dt$  high), the more precisely its energy can be measured ( $dE$  low).

A large transition probability leads to a short life in the state (low  $dt$ ) and a large energy uncertainty (high  $dE$ ).

Thus, the spectral lines are broadened. This type of broadening is called **natural broadening**.

# Natural Line Broadening (2)

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- The resulting absorption coefficients have the same form as the classical case, except that the classical damping coefficient  $\gamma$  is replaced by  $\Gamma$ , the Quantum Mechanical damping constant, the sum of all transition probabilities  $A_{ij}$  for spontaneous emission.

$$\varphi_\nu = \frac{\Gamma/4\pi}{(\nu - \nu_0)^2 + (\Gamma/4\pi)^2}$$

- $\varphi$  is the **natural** or **Lorentz** profile with FWHM (as before)

$$\Delta\lambda_{1/2} = \frac{\lambda_0^2}{c} \Delta\nu_{1/2} = \frac{\lambda_0^2}{c} \frac{\Gamma}{2\pi} \approx f_{ij} \times 7 \times 10^{-4} \text{ \AA}$$

- Still **very small**, since  $f$  is at most of order unity!
- Clearly other line broadening mechanisms should dominate.

# Thermal (Doppler) broadening

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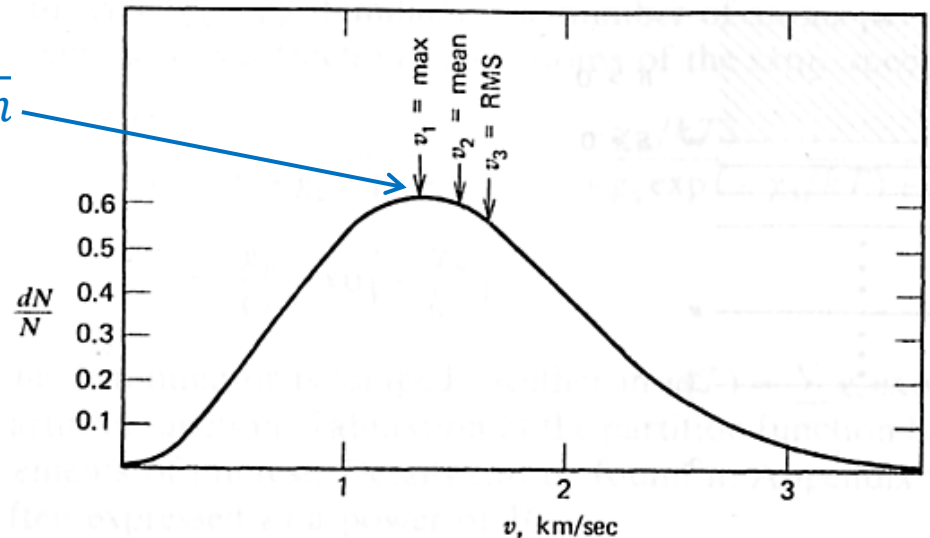
- The light emitting atoms in a stellar atmosphere are not at rest but have a **thermal** motion → Maxwellian velocity distribution.
- Because the particles produce Doppler shifts, the line-of-sight velocities have a distribution that is an important special case for spectroscopy:

$$\frac{dN}{N} = \frac{1}{\sqrt{\pi}} e^{-(v_r/v_{th})^2} \frac{dv_r}{v_{th}}$$

where  $v_r$  is the radial (line of sight) velocity component, and  $v_{th}$  is the most probable velocity  $v_{th} = \sqrt{2kT/m}$

- The frequency (wavelength) shift (linear Doppler effect) is related to  $v_r$ :

$$\frac{\Delta\lambda}{\lambda_0} = \frac{\Delta\nu}{\nu_0} = \frac{v_r}{c}$$



# Doppler broadening

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- The distribution of  $\Delta\lambda$  or  $\Delta\nu$  values gives us the shape of the absorption coefficient.
- Integrating the Maxwell distribution over all velocities, we obtain

$$\phi(\nu) = \frac{\nu_0}{c\sqrt{\pi}\Delta\nu_D} \exp\left[-(\nu - \nu_0)^2/\Delta\nu_D^2\right]$$

substituting  $v_r = \frac{\nu - \nu_0}{\nu_0} c$  and  $\Delta\nu_D = \frac{\nu_0}{c} v_{th} = \frac{\nu_0}{c} \sqrt{\frac{2kT}{m}}$  (the Doppler width)

- With  $\int_0^\infty \phi(\nu) = 1$ , we obtain the **Gaussian** line profile in terms of the Doppler width :

$$\phi(\nu) = \frac{1}{\sqrt{\pi}\Delta\nu_D} e^{-(\nu - \nu_0)^2/\Delta\nu_D^2}$$

Again, the maximum is at  $\nu_0$ .

**Temperature** dependency:  $\Delta\nu_{th} \sim \sqrt{T}$

# Doppler broadening (FWHM)

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- We can again obtain the line **FWHM** via  $\nu = \nu_{1/2}$  where  $\varphi(\nu_{1/2}) = 1/2 \varphi(\nu_0)$  and then solve for the **FWHM**  $= \Delta\nu_{1/2} = 2(\nu_{1/2} - \nu_0)$
- This implies that  $2 = \exp[(\nu_{1/2} - \nu_0)^2 / \Delta\nu_D^2]$  or  $(\nu_{1/2} - \nu_0)^2 = \Delta\nu_D^2 \ln 2$
- Finally,  
$$\Delta\nu_{1/2} = 2(\nu_{1/2} - \nu_0) = 2\Delta\nu_D \sqrt{\ln 2} = 1.67\Delta\nu_D = 2.139 \times 10^{12} \sqrt{(T/\mu)} / \lambda_0(\text{\AA}) \text{ Hz}$$

( $\mu$  is the **atomic mass**)
- In wavelength units  $\Delta\lambda_{1/2} = \frac{\lambda_0^2}{c} \Delta\nu_{1/2} = 7.1 \times 10^{-7} \lambda_0(\text{\AA}) \sqrt{(T/\mu)} \text{\AA}$

# Doppler broadening (example)

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- For the Sun, with  $T \sim 6000\text{K}$  at  $\text{H}\alpha$ :

$$\Delta\nu_{1/2} = 2.139 \times 10^{12} \sqrt{(T/\mu)} / \lambda_0 (\text{\AA}) =$$

$$\mu=1$$

i.e. in wavelength units  $\Delta\lambda_{1/2} = \frac{\lambda_0^2}{c} \Delta\nu_{1/2} = 7.1 \times 10^{-7} \lambda_0 (\text{\AA}) \sqrt{(T/\mu)} \text{\AA} =$

# Doppler broadening (example)

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- For the Sun, with  $T \sim 6000\text{K}$  at  $\text{H}\alpha$ :

$$\Delta\nu_{1/2} = 2.139 \times 10^{12} \sqrt{(T/\mu)/\lambda_0} (\text{\AA}) = 2.139 \times 10^{12} \sqrt{6000/1}/6563 = 25.2 \text{ GHz}$$

i.e. in wavelength units  $\Delta\lambda_{1/2} = \frac{\lambda_0^2}{c} \Delta\nu_{1/2} = \frac{(6563 \times 10^{-8})^2}{3 \times 10^8} 25.2 \times 10^9 = 0.36 \text{ \AA}$

or velocity units:  $\Delta v_{1/2} = c \frac{\Delta\lambda_{1/2}}{\lambda_0} = 3 \times 10^5 \text{ km/s} \frac{0.36}{6562} = 16.5 \text{ km/s}$

- This is **much larger** than the natural damping width of the line ( $10^{-4} \text{ \AA}$ ), but still relatively **small** relative to some pressure broadening mechanisms (will discuss later).
- The atomic mass dependence in the denominator implies **smaller line widths for metallic lines**, e.g. a factor of  $(56)^{1/2}$  smaller for iron lines having wavelengths close to  $\text{H}\alpha$ .

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# Natural and Thermal Broadenings

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From above:

- **Natural** Line Broadening: 
$$\varphi_\nu = \frac{\Gamma/4\pi}{(\nu - \nu_0)^2 + (\Gamma/4\pi)^2} \quad \Gamma = \sum_{i < j} A_{ji}$$

Lorentzian profile with FWHM 
$$\Delta\lambda_{1/2} = \frac{\lambda_0^2}{c} \Delta\nu_{1/2} = \frac{\lambda_0^2}{c} \frac{\Gamma}{2\pi} \approx f_{ij} \times 7 \times 10^{-4} \text{ \AA}$$

- **Doppler** broadening 
$$\varphi(\nu) = \frac{1}{\sqrt{\pi}\Delta\nu_D} e^{-(\nu-\nu_0)^2/\Delta\nu_D^2} \quad \Delta\nu_D = \frac{\nu_0}{c} v_{th} = \frac{\nu_0}{c} \sqrt{\frac{2kT}{m}}$$

Gaussian line profile with FWHM 
$$\Delta\lambda_{1/2} = \frac{\lambda_0^2}{c} \Delta\nu_{1/2} = 7.1 \times 10^{-7} \lambda_0(\text{\AA}) \sqrt{(T/\mu)} \text{ \AA}$$

$$\Delta\nu_{1/2} = 1.67\Delta\nu_D$$

# Comparing broadenings

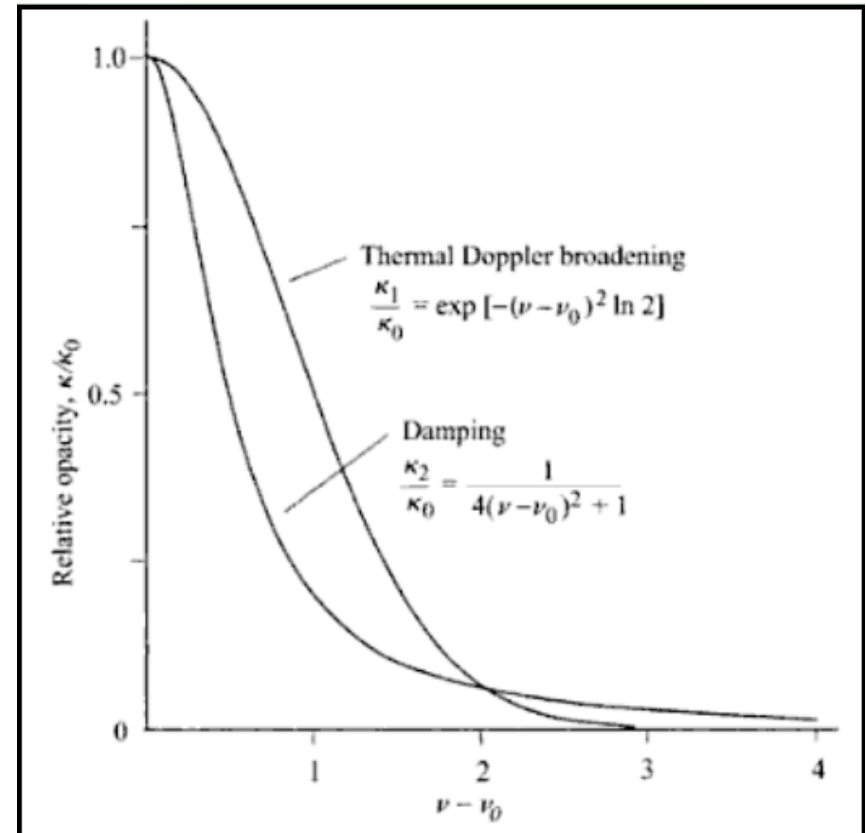
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- Thermal (Doppler):
  - $\Delta \lambda_{\text{th}} = 0.02 \text{ \AA}$  (at  $\lambda_0 = 5000 \text{ \AA}$ ,  $T = 6000 \text{ K}$ , Fe)
  - $\Delta \lambda_{\text{th}} = 0.5 \text{ \AA}$  (at  $\lambda_0 = 5000 \text{ \AA}$ ,  $T = 50000 \text{ K}$ , H)
- Radiation damping:
  - $\Delta \lambda_{\text{FWHM}} = \text{a few} \times 10^{-4} \text{ \AA}$
- **But:** decline of Gauss profile in wings is much steeper than for Lorentz profile:
$$\begin{array}{l} \text{Gauss } (10\Delta\lambda_{\text{th}}) \quad : \quad e^{-10^2} \approx 10^{-43} \\ \approx \\ \text{Lorentz } (1000\Delta\lambda_{\text{rad}}) : \quad 1/1000^2 \approx 10^{-6} \end{array}$$
- In the line **wings** the **Lorentz** profile is **dominant**

# Broadening mechanisms profiles

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- Different broadening mechanisms have the form of
  - A **Lorentzian** function (natural profile and broadening, some pressure broadenings)
  - A **Gaussian** function (thermal broadening, instrumental broadening, etc.)
  - **Other functions** are possible (e.g., Linear Stark broadening)
- Generally, we have to consider both (all) types of profiles. For example, the pressure damping profile is negligible in the line core, but the Doppler profile decreases very steeply in the wings, whilst the damping profile decreases only as  $1/\Delta\lambda^2$
- **The Gaussian dominates the line core** (or is confined to it), while the **Lorentzian profile dominates in the line wings** out to several times the FWHM.



# Joint effect of different mechanisms

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Mathematically: **convolution**

$$(f_A * f_B)(x) = \int_{-\infty}^{\infty} f_A(y) f_B(x-y) dy$$

Properties:

- commutative:

$$f_A * f_B = f_B * f_A$$

- Fourier transformation:  $F(f_A * f_B) = \text{normfactor} \cdot F(f_A) \cdot F(f_B)$   
where  $F$  denotes the Fourier transform of  $f$ .

i.e., in Fourier space the convolution  
is a multiplication

# Application to profile functions

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## Convolution of two Gaussian profiles

$$G_A(x) = \frac{1}{A\sqrt{\pi}} e^{-\frac{x^2}{A^2}} \quad G_B(x) = \frac{1}{B\sqrt{\pi}} e^{-\frac{x^2}{B^2}}$$

$$G_C(x) = G_A(x) * G_B(x) = \frac{1}{C\sqrt{\pi}} e^{-\frac{x^2}{C^2}} \quad \text{with} \quad C^2 = A^2 + B^2$$

Result: Gauss profile with **quadratic summation** of half-widths.

## Convolution of two Lorentzian profiles (e.g., radiation + collisional damping)

$$L_A(x) = \frac{A/\pi}{x^2 + A^2} \quad L_B(x) = \frac{B/\pi}{x^2 + B^2}$$

$$L_C(x) = L_A(x) * L_B(x) = \frac{C/\pi}{x^2 + C^2} \quad \text{with} \quad C = A + B$$

Result: Lorentz profile with **sum** of half-widths

# Voigt profile

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Convolving Gauss and Lorentz profile

(e.g. thermal + natural broadening)

$$G(\nu) = \frac{1}{\Delta\nu_D\sqrt{\pi}} e^{-\frac{(\nu-\nu_0)^2}{\Delta\nu_D^2}} \quad L(\nu) = \frac{\gamma/4\pi^2}{(\nu-\nu_0)^2 + (\gamma/4\pi)^2}$$

$$V = G * L \quad \text{depends on } \nu, \Delta\nu, \gamma, \Delta\nu_D: \quad V(\nu) = \int_{-\infty}^{\infty} G(\nu') L(\nu - \nu') d\nu'$$

$$\text{Transformation: } \nu: = \frac{(\nu - \nu_0)}{\Delta\nu_D} \quad a: = \gamma/(4\pi\Delta\nu_D) \quad y: = \frac{(\nu' - \nu_0)}{\Delta\nu_D}$$

$$G(y) = \frac{1}{\Delta\nu_D\sqrt{\pi}} e^{-y^2} \quad L(y) = \frac{a/\Delta\nu_D\pi}{y^2 + a^2} \quad V = \frac{1}{\Delta\nu_D\sqrt{\pi}} \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(\nu - y)^2 + a^2} dy$$

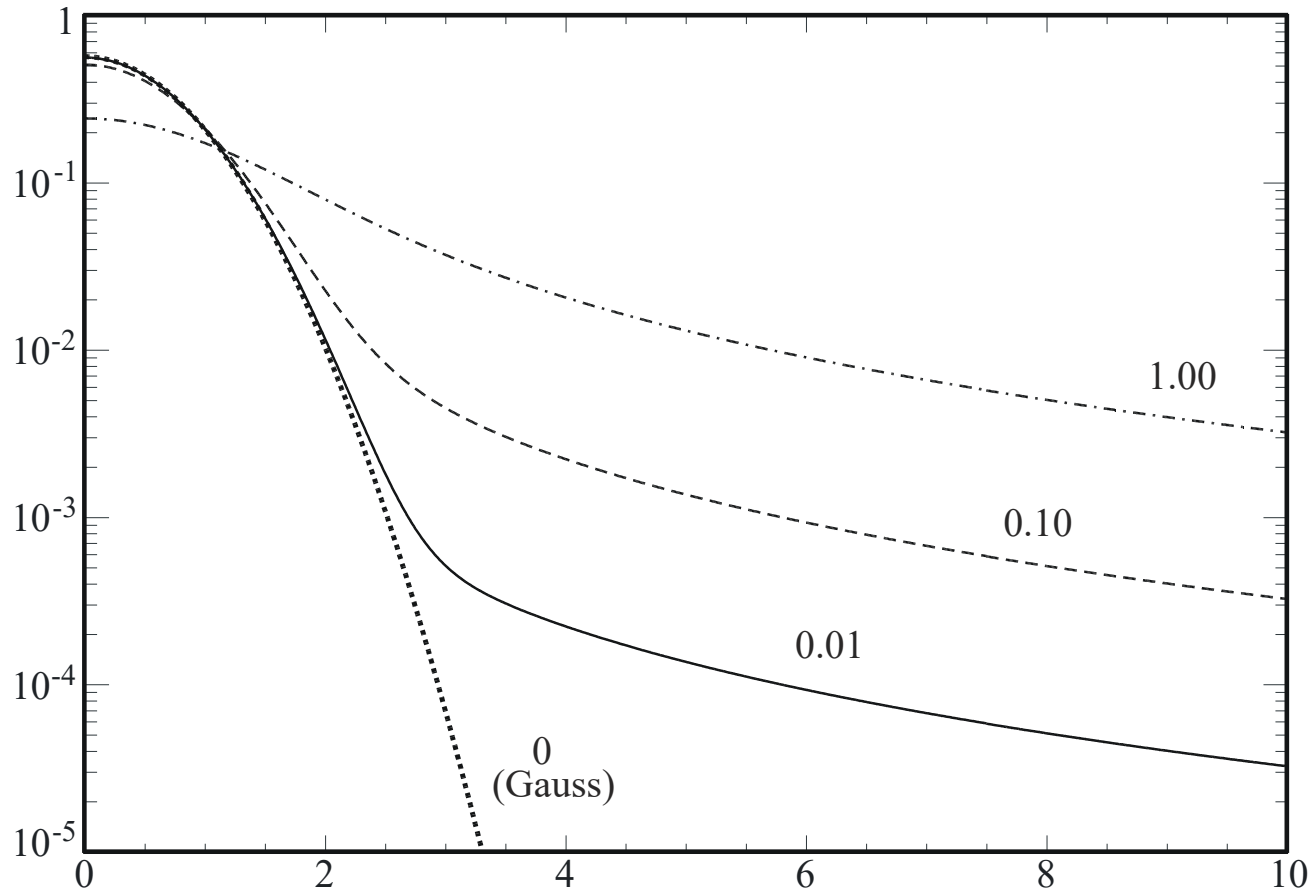
$$\text{Def: } V = \frac{1}{\Delta\nu_D\sqrt{\pi}} H(a, \nu) \quad \text{with } H(a, \nu) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{e^{-y^2}}{(\nu - y)^2 + a^2} dy$$

**Voigt function**, no analytical representation possible.  
(approximate formulae or numerical evaluation)

$$\text{Normalization: } \int_{-\infty}^{\infty} H(a, \nu) d\nu = \sqrt{\pi}$$

# The Voigt func for various $a$ (1)

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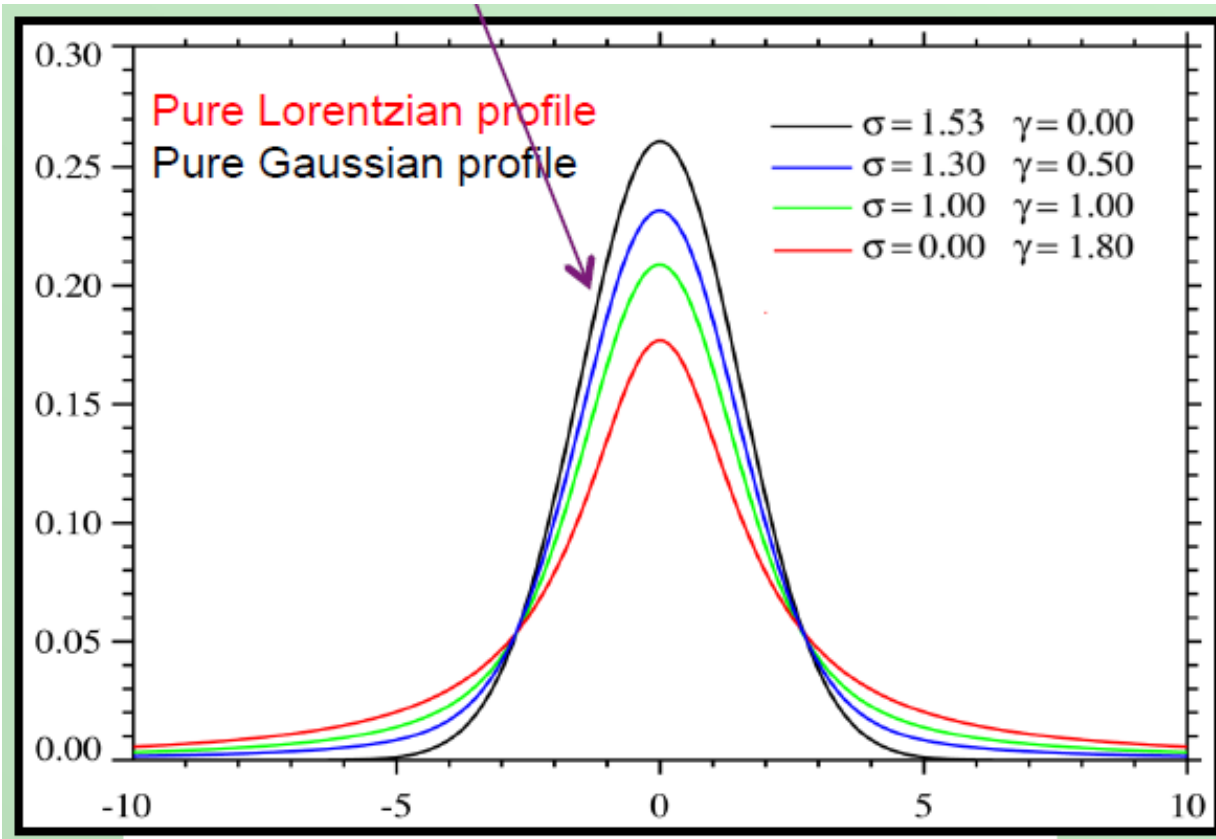
The final form of the combined **Voigt** profile depends on  $\alpha = 2\pi a = \gamma/2\Delta\nu_D$ , the ratio of the damping widths  $\gamma/2$  to the Doppler width  $\Delta\nu_D$

As **a rule of thumb**, the damping wings start to contribute a distance  $-(\log \alpha)\Delta\lambda_D$  from the line centre

# The Voigt func for various $a$ (2)

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Voigt profiles



The final form of the combined **Voigt** profile depends on  $\alpha = 2\pi a = \gamma/2\Delta v_D$ , the ratio of the damping widths  $\gamma/2$  to the Doppler width  $\Delta v_D$

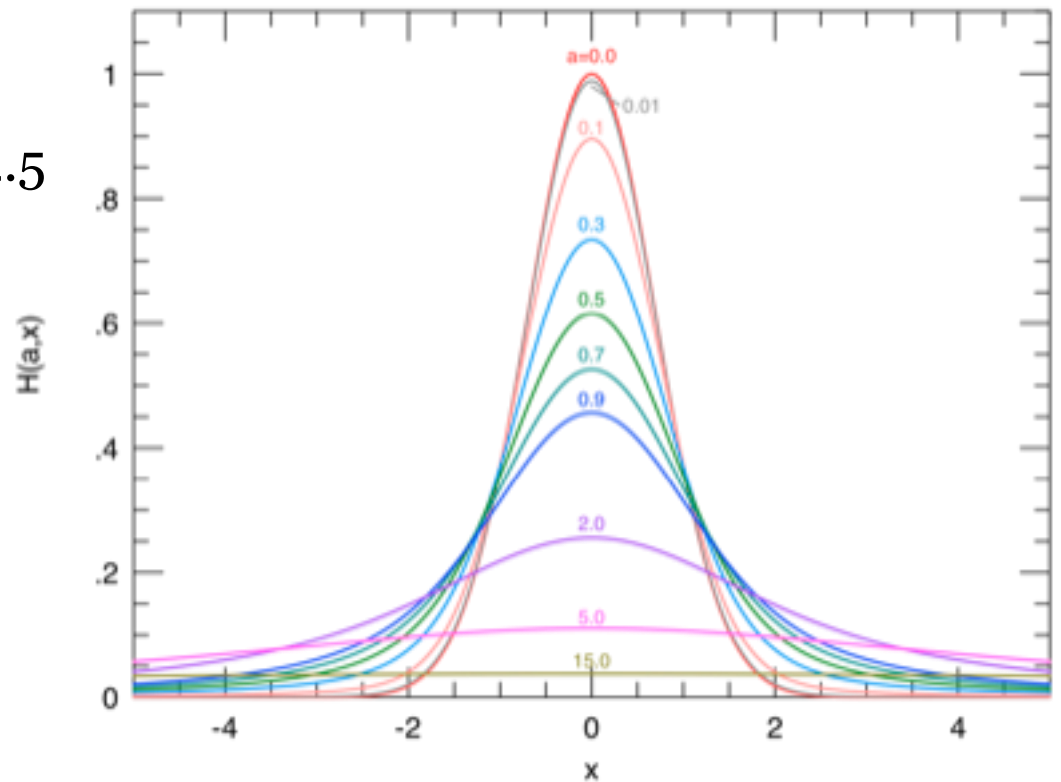
As **a rule of thumb**, the damping wings start to contribute a distance  $-(\log \alpha)\Delta\lambda_D$  from the line centre

# Calculation of a Voigt profile

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No analytical representation is possible, but...

- IDL:  
IDL> u=findgen(201)/40.-2.5  
IDL> v=voigt(0.5,u)  
IDL> plot,u,v
- Python



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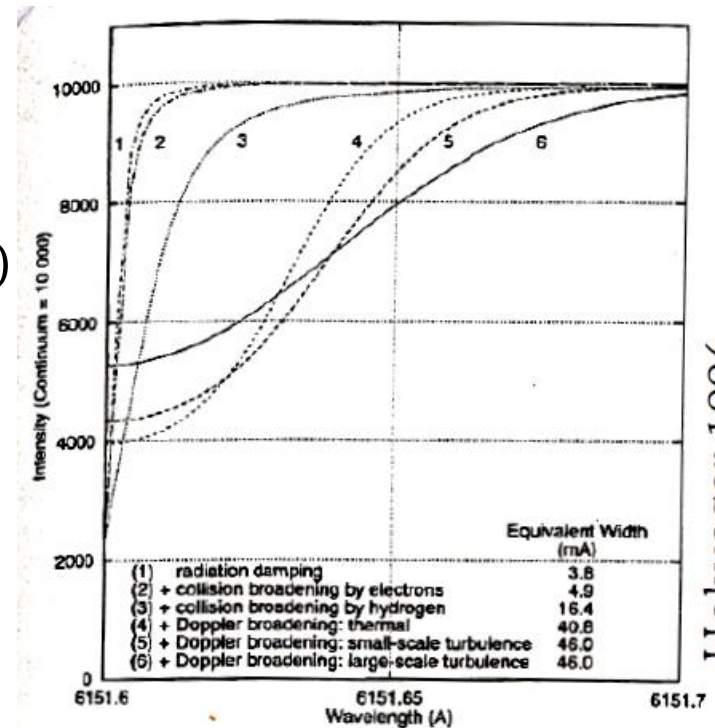
# Other broadening mechanisms

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1. Natural broadening ✓
  2. Thermal broadening ✓
  3. Microturbulence (treated like extra thermal broadening) ✓
  4. Collisions (important for strong lines)
  5. Isotopic shift, *hfs*, Zeeman effect

- macro
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Holweger 1996

# Collisional and Pressure broadening

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- The orbitals of an atom can be perturbed in a collision with a neutral atom (**collisional** broadening) or encounter with the electric field of an ion (**pressure** broadening).

# Direct collisions?

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- Collisions in the gas de-excite atoms before they naturally decay, shortening its lifetime.
- The resulting line profile is **Lorentzian** (as with natural broadening) with a width of  $\nu_{1/2} = 1/(\pi t)$  where  $t$  is the time between collisions.
- The number of collisions (per second) is the number of perturbers in the volume swept out by the atom, i.e.  $N\sigma v$ . Since  $\frac{1}{2}mv^2 = \frac{3}{2}kT$ , the time between collisions is

$$t \approx 1/(N\sigma\sqrt{3kT/m})$$

- So, the FWHM in terms of pressure ( $P=NkT$ ) is:

$$\Delta\nu_{1/2}(\text{Hz}) = P\sigma/\pi\sqrt{3/kTm} = 3.6 \times 10^{19} P\sigma/\sqrt{mT/m_H}$$

- For the Sun ( $T=5800\text{K}$ ,  $P=10^5$  dyne/cm<sup>2</sup>),  
H atom *direct* collisions ( $\sigma=\pi a_0^2=8\times 10^{-17}$  cm<sup>2</sup>) cause  $\Delta\nu_{1/2} = 4$  MHz

i.e. **less than the natural width**

$$\Delta\lambda_{1/2} = \frac{\lambda_0^2}{c} \Delta\nu_{1/2} = 5 \times 10^{-5} \text{ \AA}$$

# Impact broadening

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- Nevertheless, the **impact** approximation can be used for **some** broadening mechanisms, which are important since atoms can interact without direct collision.
- The change in energy induced by the collision is a function of the separation  $r$  between the absorber and perturbing particle, and can be approximated by a power law of the form  $\Delta E \sim \text{Constant} \times r^{-n}$  where  $n$  is an integer, such that the change in frequency is  $\Delta \nu = \Delta E/h = C_n r^{-n}$

Constants  $C_n$  are determined by laboratory measurements, or calculations.

# Pressure broadening (1)

$$\Delta\nu = \frac{C_n}{r^n}$$

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n =	name	interaction of
2	linear Stark effect	hydrogen-like ions + p, e
3	resonance broadening	neutral atoms with each other, H+H
4	quadratic Stark effect	ions + e, p
6	van der Waals broadening	metals + H

Two approximations exist – impact broadening for  $n > 2$  ( $n=3$  resonance,  $n=4$  quadratic Stark effect,  $n=6$  van der Waals) and a quasi-static approximation (i.e. surrounding particles are nearly at rest; for linear Stark broadening,  $n=2$ ).

# Pressure broadenings...

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The orbitals of an atom can be perturbed in a collision with a neutral atom or encounter with the electric field of an ion.

n =	name	interaction of
2	linear Stark effect	hydrogen-like ions + p, e
3	resonance broadening	neutral atoms with each other, H+H
4	quadratic Stark effect	ions + e, p
6	van der Waals broadening	metals + H

resonance broadening (n=3)  
 quadratic Stark effect (n=4)  
 van der Waals broadening (n=6)

impact broadening approximations

**Lorentz profile**

$$\Delta \nu = \frac{C_n}{r^n}$$

(Ansatz): constants  $C_n$  are determined by laboratory measurements, or calculations

→ Let's discuss in a bit more detail

Additional material for self-study

# Collisional Broadening

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- Frequency of collisions =  $1/T_0$
- Suppose collisions occur if particles pass within distance = impact parameter  $\rho_0$

$$\frac{1}{T_0} = N\pi\rho_0^2v$$

$N = \text{\#perturbers/cm}^3$ ,  $v = \text{relative velocity cm/s}$

- Then damping parameter is

$$\Gamma = 2N\pi\rho_0^2v$$

We used  $\sigma = \pi a_0^2$  for direct collisions

# Weisskopf approximation (1)

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- perturber is a classical particle
- path is a straight line
- no transitions caused in atom
- interaction creates a phase shift or frequency shift given by

$$\Delta\omega = \frac{C_p}{r^p}$$

- $p$  exponents of astronomical interest: 3,4,6

# Weisskopf approximation (2)

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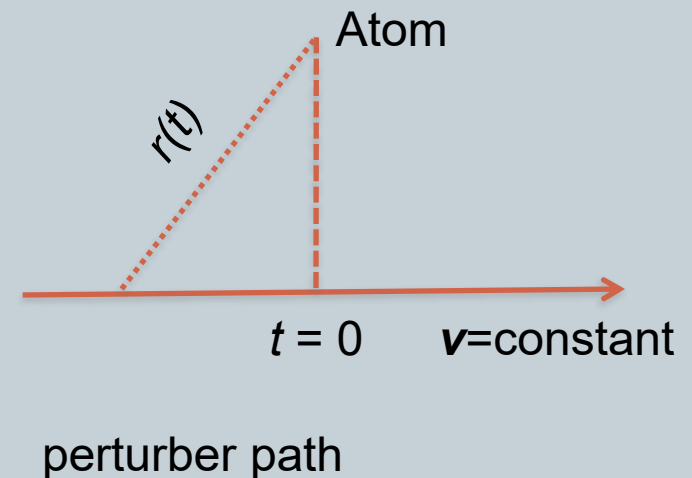
## Total phase shift

$$\eta(\rho) = C_p \int_{-\infty}^{+\infty} \frac{dt}{r^p} = C_p \int_{-\infty}^{+\infty} \frac{dt}{[\rho^2 + v^2 t^2]^{p/2}} = \frac{C_p}{v \rho^{p-1}} \psi_p$$

$$\psi_p = \sqrt{\pi} \frac{\Gamma[(p-1)/2]}{\Gamma[p/2]}$$

$p$	$\psi_p$
2	$\pi$
3	2
4	$\pi/2$
6	$3\pi/8$

$$r(t) = [\rho_0^2 + v^2 t^2]^{1/2}$$



# Weisskopf approximation (3)

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- Assume that only collisions that produce a phase shift  $> \eta_0$  are effective in broadening:  
then impact parameter is

$$\rho_0 = \left( \frac{C_p \psi_p}{\eta_0 v} \right)^{\frac{1}{p-1}}$$

- Weisskopf assumed  $\eta_0 = 1$ , yields damping

$$\Gamma_W = 2\pi N v \left( \frac{C_p \psi_p}{v} \right)^{\frac{2}{p-1}}$$

depends on  $\rho$ ,  $T$

- Ignores weak collisions  $\eta < \eta_0$

# Better Impact Model: Lindholm-Foley

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- Includes effects of multiple weak collisions, which introduce a phase shift  $\Delta\omega_0$ ;  $\Gamma_{LF} > \Gamma_W$

$$I(\omega) = \frac{\Gamma / (2\pi)}{(\omega - \omega_0 - \Delta\omega_0)^2 + (\Gamma / 2)^2}$$

$\rho$	3	4	6
$\Gamma$	$2\pi^2 C_3 N$	$11.37 C_4^{2/3} v^{1/3} N$	$8.08 C_6^{2/5} v^{3/5} N$
$\Delta\omega_0$	0	$9.85 C_4^{2/3} v^{1/3} N$	$2.94 C_6^{2/5} v^{3/5} N$

- Impact theory fails for small  $\rho$

Additional material for self-study

# Impact broadenings (n=3,4,6)

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- **Resonance Broadening (n=3)** occurs between **identical species**, restricted to upper/lower level having an electric dipole transition to ground state (resonance line):

$$\Delta\lambda_{1/2} = 8.6 \times 10^{-30} (g_i / g_k)^{1/2} \lambda^2 \lambda_{res} f_{res} N_i$$

- **Quadratic Stark broadening (n=4)**: Interaction of electron or proton with a system without dipole moment. The frequency shift depends on the square of the local electric field generated by passing **electrons**. With  $C_4$  a constant obtained from laboratory data,

$$\log \gamma_4 = 19 + \frac{2}{3} \log C_4 + \log P_e - \frac{5}{6} \log T$$

- **Van der Waals broadening (n=6)**: A momentary dipole on one neutral atom induces a change in lifetime, by inducing a dipole on the other. Because of its overwhelming abundance, **neutral H** acts as a perturber. For  $C_6$  a constant (excitation and ionization dependent),

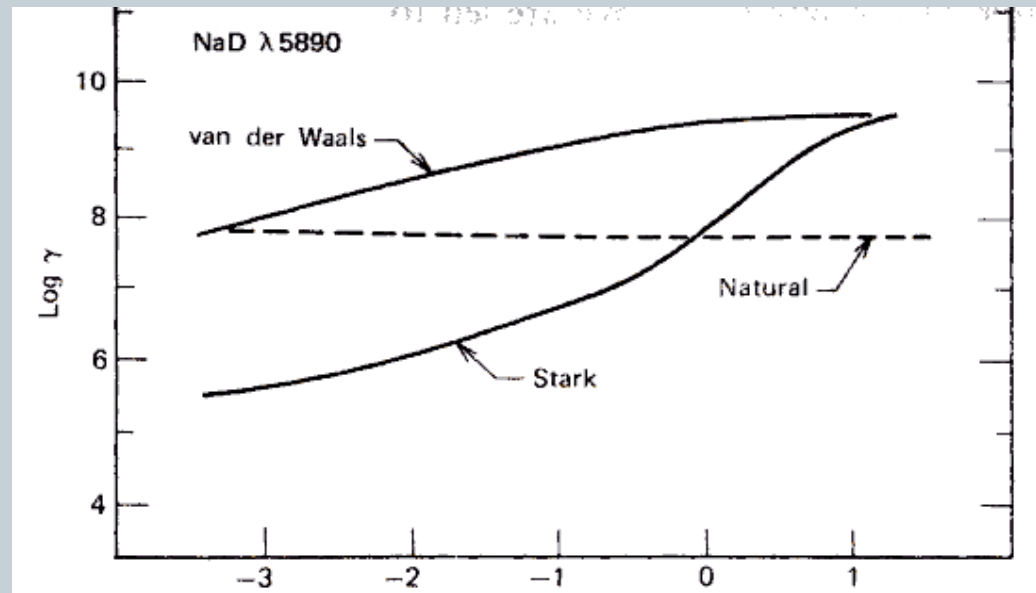
$$\log \gamma_6 = 19.6 + \frac{2}{5} \log C_6 + \log P_g - \frac{7}{10} \log T$$

# Example

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A comparison of quadratic Stark and van der Waals broadening for the Na I 5890 line at various optical depths in the Sun.

The latter dominates here, and greatly exceeds the natural width by a factor of about 30.



In general,

- **Quadratic Stark broadening** ( $n=4$ ) affects most lines in **hot stars** since **electron** pressure approaches gas pressure.
- **Van der Waals broadening** ( $n=6$ ) affects most lines in **cool stars** since this involves interactions between **neutral atoms**

Additional material for self-study

# Linear Stark broadening (n=2)

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- Atoms do **not** generally have permanent electric dipole moments. If there were such a moment, the Stark effect would be *linear*. Such a moment can occur only for **two or more levels of the same energy (they are degenerate) but different orbital quantum numbers**. This happens only for **single electron atoms** (H, He<sup>+</sup>, Li<sup>2+</sup>, ...).
- The frequency shift depends on the the local electric field generated by passing **electrons**.
- Unfortunately, **impact theory** is no longer satisfactory and we have to consider the distribution of electric fields. In the star there is no a uniform field – there is an average field distribution felt by an average atom (**statistical** Stark effect). This distribution is called the Holtsmark distribution.

# Holtmark Statistical Theory

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- Ensemble of perturbers instead of single
- more particles, more chances for strong field
- e- attracted to ions, reduce perturbation by Debye shielding
- in stellar atmospheres density is low, number of perturbers is large, and Holtmark distribution is valid

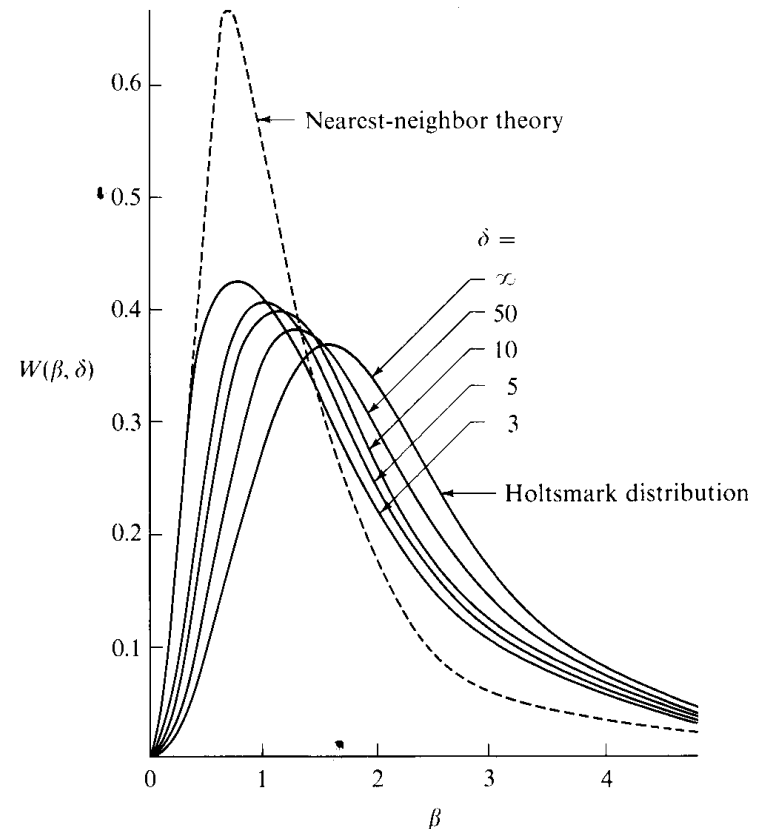


FIGURE 9-1  
Probability distribution of field strength at a test point, including shielding effects;  $\delta$  is the number of charged particles within the Debye sphere. From (205), by permission.

# Hydrogen: Linear Stark Effect

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- Each level degenerate with  $2n^2$  sublevels.
- Perturbing field will separate sublevels.
- Observed profile is a **superposition** of components **weighted** by relative intensities and shifted by field probability function.
- $\Delta\lambda_{1/2} \approx 2.5 \times 10^{-9} \alpha_{1/2} N_e^{2/3}$  where  $\alpha_{1/2}$  is a half-width parameter widely used for plasma diagnostics (NIST).
- For H $\alpha$  ( $n=2$  to 3) in the Sun ( $P_e=20$  dyne/cm<sup>2</sup>,  $T=5800$ K),  $\Delta\lambda_{\text{FWHM}}=0.5 \text{ \AA}$ , i.e. a width 1000 times the natural width.
- Hot stars have very **high** electron pressures, so the Linear Stark effect greatly affects H I lines in hot stars (including white dwarfs), and is also relevant for hydrogenic ions (e.g. He II lines) in O stars.

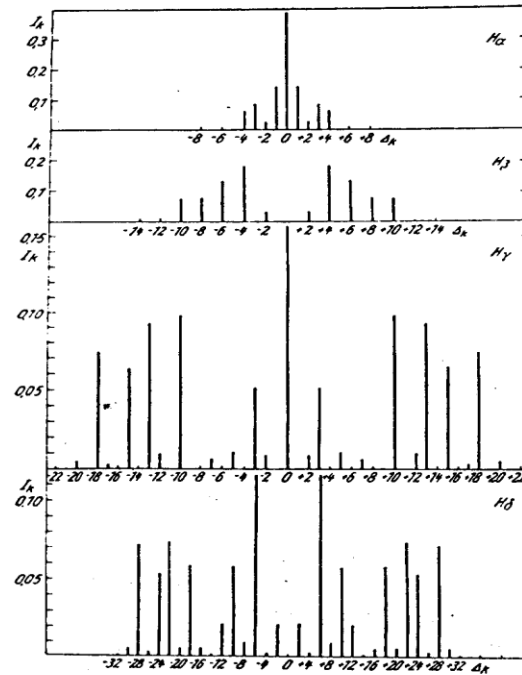
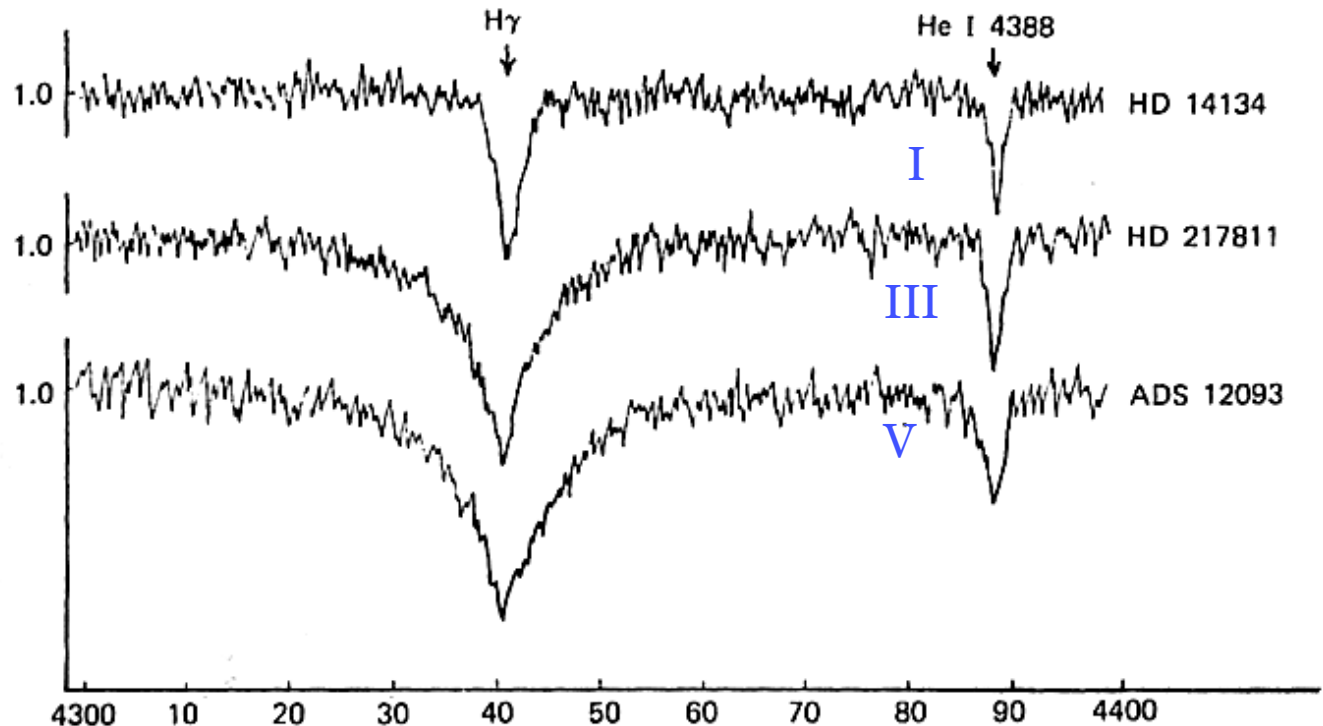


Fig. 11.1. The Stark effect splitting of the different Balmer lines (according to Unsöld, 1955, p. 320.)

# Linear Stark broadening: examples (1)

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Example of linear Stark broadening in early B stars – increased  $H\gamma$  line width for increased pressure (this effect becomes significant for  $T_{\text{eff}} > 7500$  K).



# Linear Stark broadening: examples (2)

*Vidal, Cooper & Smith (1973):*

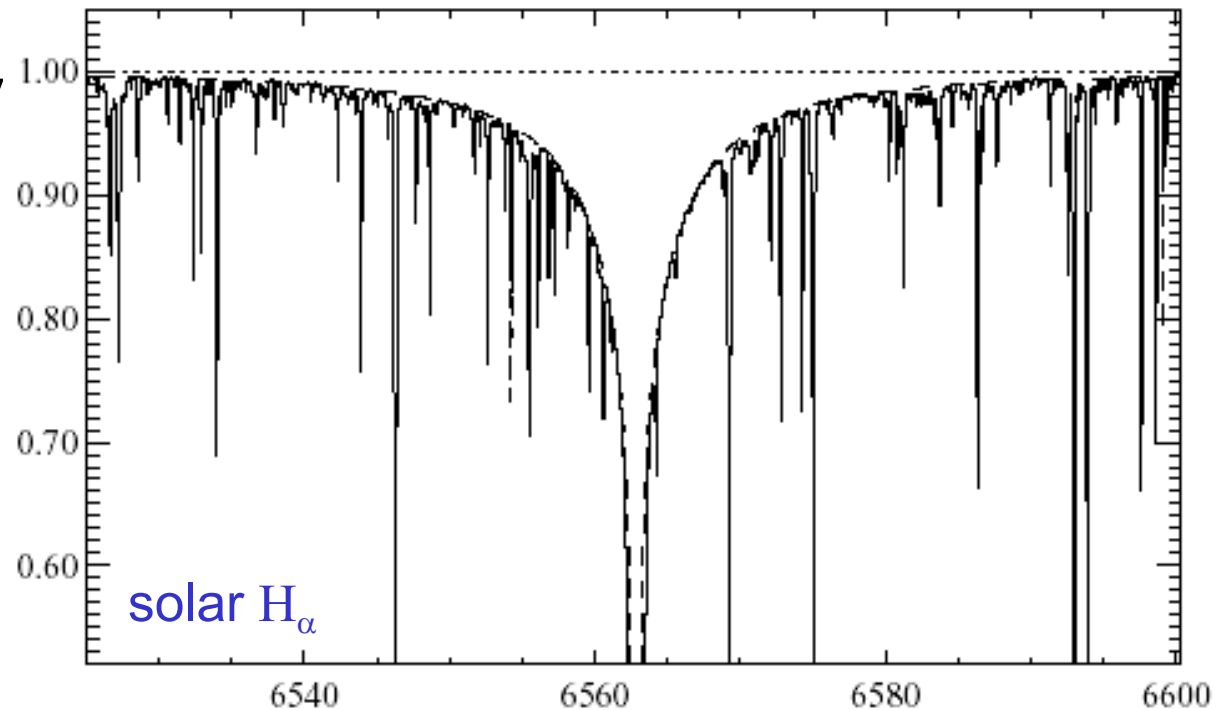
H I + p      quasistatic approach;

H I + e      collisional approximation – in a core  
                 quasistatic approach      - in wings

*Observations*

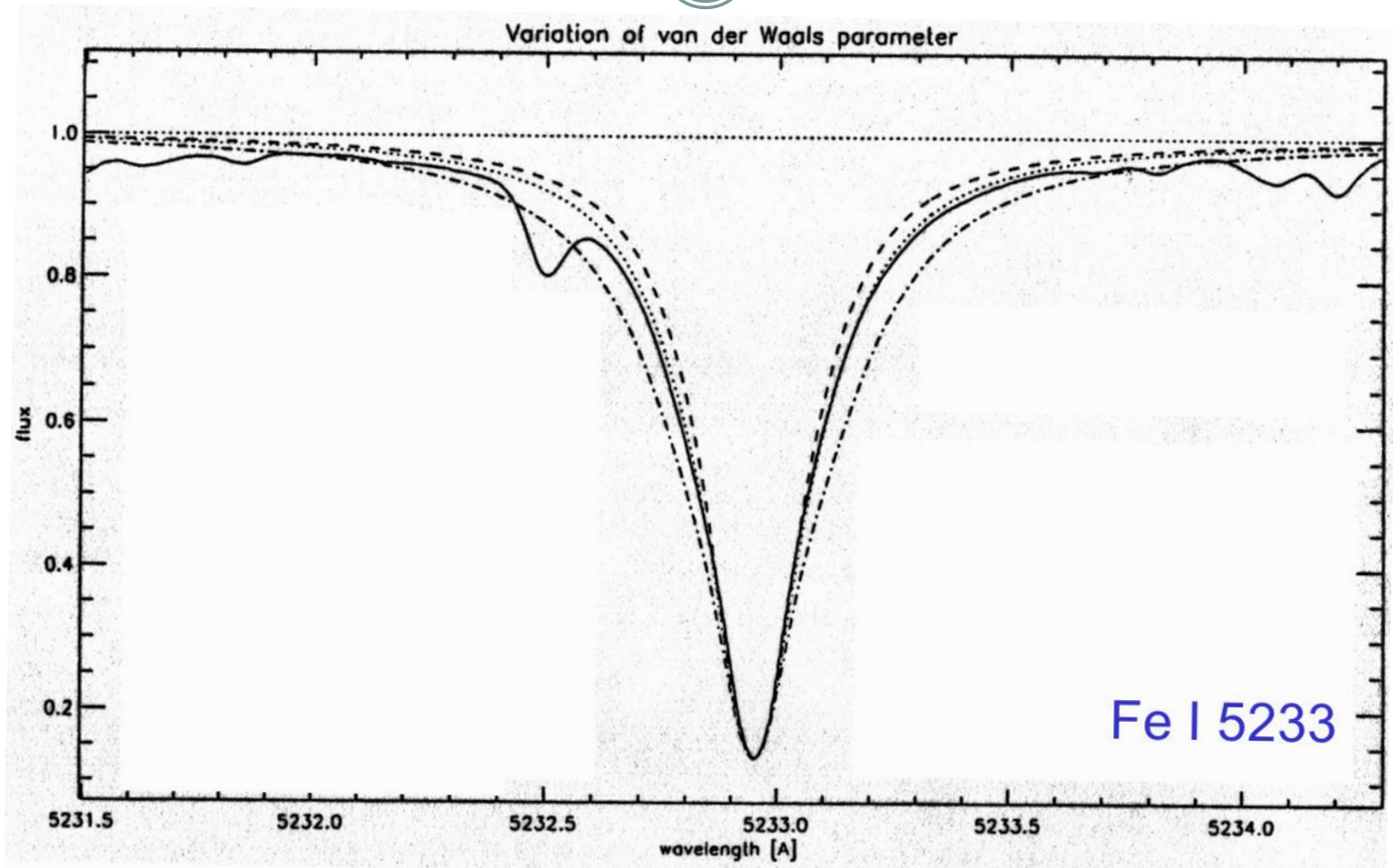
(solid line) and theory

(dash line)



# van der Waals broadening: example

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**Example:**  $\log C_6$  varies from -31.40 (top), -31.10 (middle), to -30.50 (bottom)

# New Developments in the Theory of Pressure-Broadening

## ◆ Linear Stark broadening

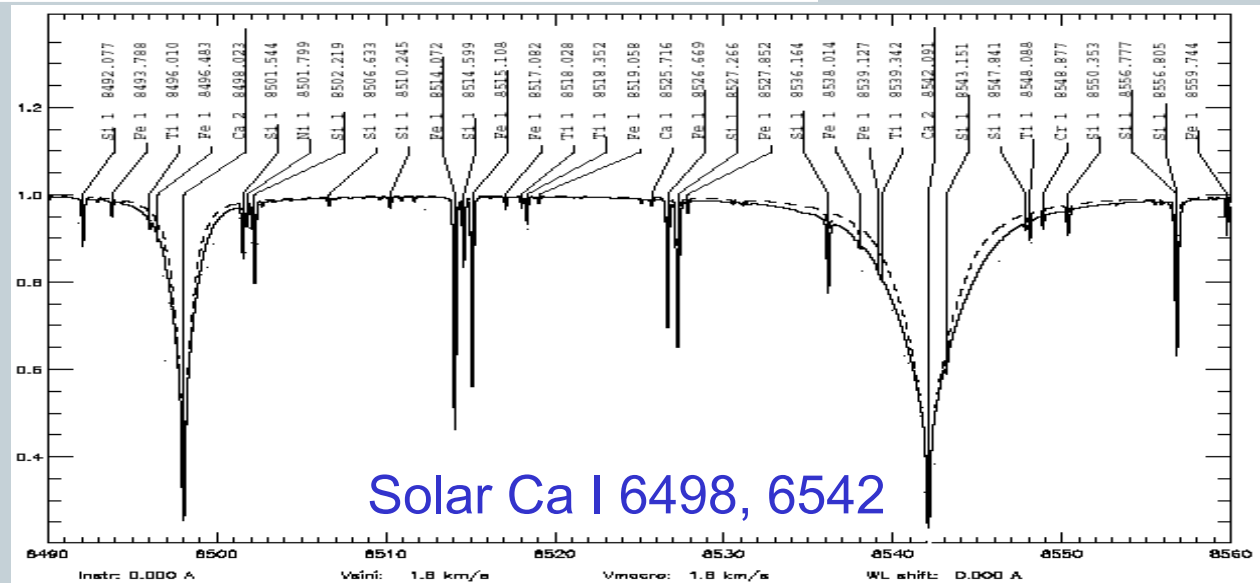
*Stehle & Hutcheon* (1999, *A&AS*, 140, 93) – tables of Stark profiles

## ◆ van der Waals broadening

*Anstee & O'Mara* (1995, *MNRAS*, 276, 859) and following papers

$$\gamma_6 / 4\pi = N_H (4/\pi)^{\alpha/2} \Gamma((4-\alpha)/2) v \sigma_0 (v/v_0)^{-\alpha}$$

$\sigma_0$ ,  $\alpha$  - tabulated parameters



Solar Ca I 6498, 6542

Additional material  
for self-study

Observations and Theory of *Anstee & O'Mara* are consistent!  
Dash line – approximation of *Unsold* (1955)

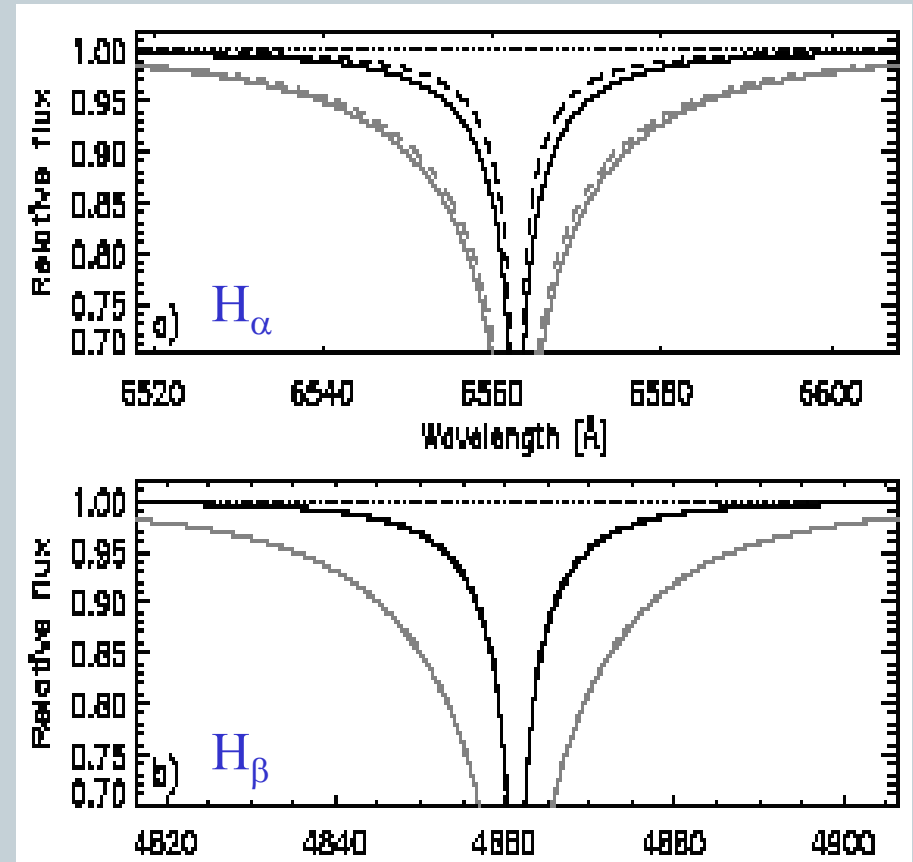
## ◆ Resonance Broadening

*Barklem et al. (2000, A&A, 363, 1091)*

Influence of resonance broadening on the line profiles of  $H_\alpha$  and  $H_\beta$

## ◆ Quadratic Stark broadening

Papers by *Dimitrijevic et al.*



$T_{\text{eff}} = 5780 \text{ K}$ ,  $\log g = 4.44$

$T_{\text{eff}} = 7000 \text{ K}$  (grey)

Dash line:

Without resonance broadening

Additional material for self-study

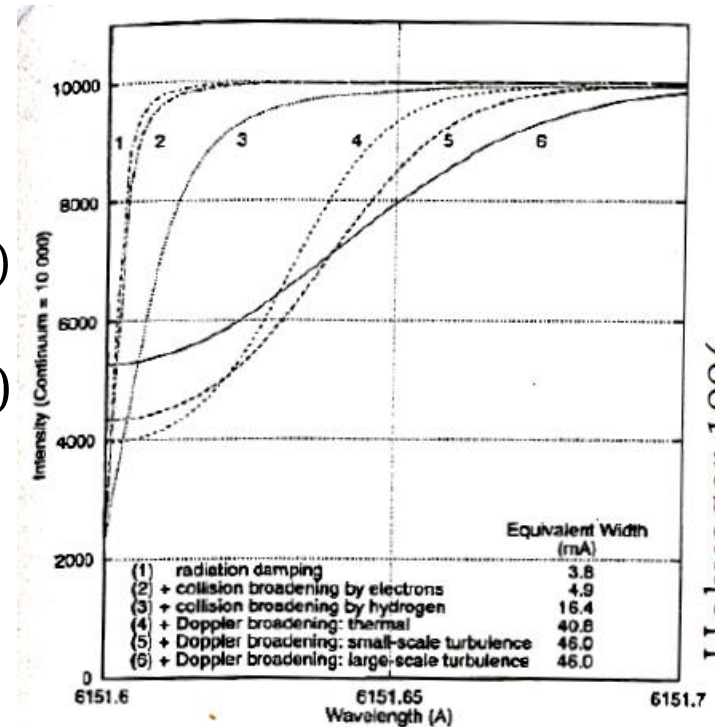
# Broadening of spectral lines

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There are numerous broadening mechanisms which influence the apparent shape of spectral lines:

- microscopic
1. Natural broadening
  2. Thermal broadening
  3. Microturbulence  
(treated like extra thermal broadening)
  4. Collisions (important for strong lines)
  5. Isotopic shift, hyperfine splitting (hfs)  
Zeeman effect

- macro
6. Macroturbulence
  7. Rotation
  8. Instrumental broadening



Holweger 1996

# Other broadening mechanisms

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- **Turbulent Broadening:** In addition to microscopic (thermal) and macroscopic (rotation) motions, there are other motions in stellar atmospheres which are introduced, operating on microscopic (**microturbulence**) and macroscopic (**macroturbulence**) scales, via convolutions with Gaussian velocity distribution
- **Isotope splitting:** Different isotopes have different nuclear mass and so slightly different term energies – the effect is greatest for hydrogen (e.g. deuterium vs hydrogen).
- **Zeeman splitting:** Magnetic fields split magnetically sensitive lines – at optical wavelengths the splitting is seen as line broadening, towards the IR the splitting becomes more noticeable since it increases as  $\lambda^2$  versus  $\lambda$  for Doppler broadening.

# Turbulent broadening

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- Added to thermal broadening in quadrature. The Gaussian line profile (normalized to unity) remains. Recall the convolution of two Gaussian profiles!

$$\phi(v) = \frac{1}{\sqrt{\pi} \sqrt{\Delta v_D^2 + \xi_t^2}} \exp[-(v - v_0)^2 / (\Delta v_D^2 + \xi_t^2)]$$

where  $\xi_t$  is a microturbulence velocity.

- Note that the broadening because of microturbulence does **not** depend on the mass of an atom!