[Theoretical] Astrophysics (765649S)

VITALY NEUSTROEV

SPACE PHYSICS AND ASTRONOMY RESEARCH UNIT

UNIVERSITY OF OULU

2026

Contact details

- **Lecturer:** Vitaly Neustroev
- **Location:** MA 308
- **Email:** vitaly@neustroev.net;
  vitaly.neustroev@oulu.fi
- **Assistent:** Paula Kvist
  paula.kvist@oulu.fi
- **Web-site:**
  https://vitaly.neustroev.net/teach/spring-2026/

Aim of the Course

Develop physical understanding of:

- How astronomical objects produce radiation, radiation transport
- Formation, structure, evolution and death of stars
- Stellar atmospheres
- Spectral line formation
- Interstellar medium

This course is a “descendant” of previously taught courses “Stellar structure and evolution”, “Interstellar matter”, and “Stellar atmospheres”.

Syllabus

1. **Introduction:** What is astrophysics and theoretical astrophysics? Astronomical units.
2. **Radiation processes:** fluxes and magnitudes; bolometric flux and bolometric correction; spectral types; luminosity classes; effective temperature; specific intensity; optical depth; source function; equation of radiative transfer; optical depth; blackbody radiation; radiation from atoms; spectral lines; bremsstrahlung; synchrotron radiation.
3. **Stars:** basic assumptions and observations; measuring masses; hydrostatic equilibrium; virial theorem; characteristic timescales; gas and radiation pressure; degeneracy pressure; Eddington limit; energy transport by radiation; nuclear reactions; neutrino oscillations; convection.
4. **Stellar evolution:** star formation; Young Stellar Objects; binary formation; low mass stars; white dwarfs; mass transfer binaries; Type II supernovae; neutron stars; black holes.
5. **Stellar photospheres:** Stellar types, spectra, temperatures. Continuous and line spectra. Spectral analysis. Theory of line formation.
6. **Interstellar Medium:** Cooling and heating of the gas and dust. Multiphase interstellar medium. Basics of gas dynamics. Physics of HII regions. Shock waves. Evolution of photoionized nebulae. Stellar winds. Supernovae explosions.

Schedule

1. Officially, lectures should take place on Wednesday, and Thursday, while exercise sessions on Monday, in one of the lecture halls M203 or M204.
2. However, I will **usually** give lectures on **Monday** and **Wednesday**, and exercise and practical sessions sessions on **Thursday**.
3. **Important!** Classes on **Thursday** will only occur when I announce them!
4. Also, a few lectures will be skipped due to classes overscheduling.

Check the course web-page for announcements!

https://vitaly.neustroev.net/teach/spring-2026/

Recommended Textbooks

Textbook choice for this course is largely a matter of personal taste. Here is a list of recommended books (check the course web-page!). Study them in parallel with the lectures:

- D. Prialnik: An introduction to the theory of stellar structure and evolution*
- R. Kippenhahn, A. Weigert: Stellar structure and evolution
- J. E. Dyson, D. A. Williams: The physics of the interstellar medium, 2nd ed., Institute of Physics Publishing, 2003
- E. Böhm-Vitense: Stellar astrophysics, vol. 2 & 3, Cambridge Univ. Press, 1992*
- David F. Gray, The Observation and Analysis of Stellar Photospheres, 3rd Edition, 2005, Cambridge University Press, ISBN: 9780521066815**
- Lecture notes (**not enough!**)

* Available in the Pegasus Library

** eBook (through the Pegasus Library)

Assessment

Astrophysics is a difficult course.

To pass it, students must attend class on a regular basis, complete homework assignments, present an essay, and to pass written intermediate exams and at the end of the course.

Your grade will be based on:

- 50% Exams: an intermediate and the final
- 30% Homeworks: at least 5 sets of compulsory problems (return by the deadline)
- 20% Essay: 5 pages + 15 min presentation (compulsory)
- Active participation will be taken into account

Introduction

WHAT IS ASTROPHYSICS?
THEORETICAL ASTROPHYSICS
ASTRONOMICAL UNITS

Astrophysics

Application of the laws of physics to understand the nature and behaviour of astronomical objects, and to predict new phenomena that could be observed.

Observational Astrophysics

deals with collecting useful data through observations of astronomical objects using different scientific instruments.

Observational astronomy (765640S, 5 ECTS)

Theoretical Astrophysics

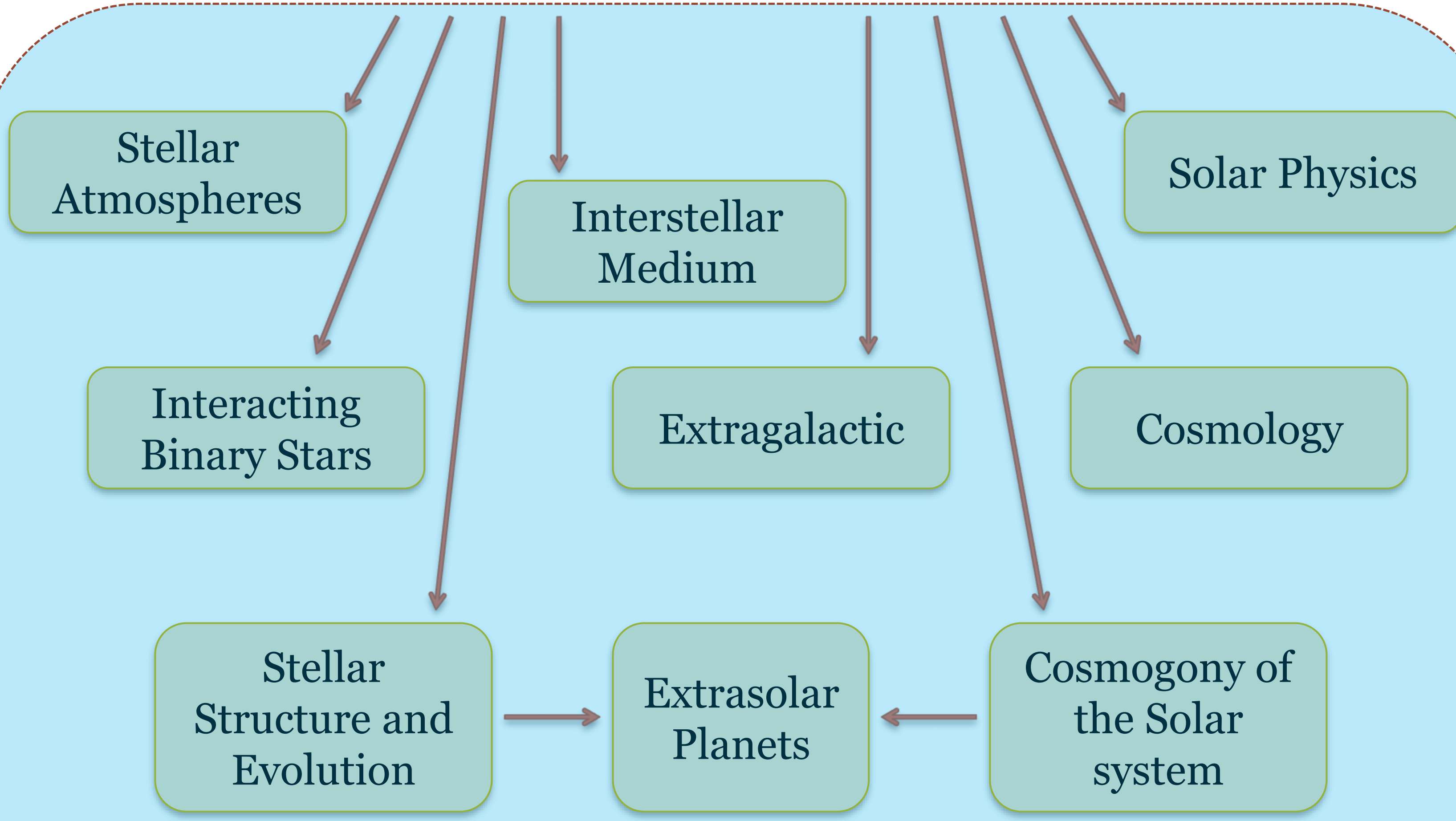
uses physics to interpret observational data and construct physical models. Theory connects all the data together into a full understanding and makes predictions about phenomena we haven't observed yet.

This course

Astrophysics

- Most branches of physics find some application in astronomy.
- Main difference between astrophysics and other branches of physics: controlled experiments are (almost) never possible.
- This means:
  - If many different physical effects are operating at the same time in a complex system, we can't isolate them one by one.
  - Knowledge of rare events is limited - nearest examples will be distant. For example, no supernova has exploded within the Milky Way since telescopes were invented.
  - Need to make best use of all the information available - many advances have come from opening up new regions of the electromagnetic spectrum.
  - Statistical arguments play a greater role than in many areas of lab physics.

Theoretical Astrophysics



Most of astronomical objects are detected via electromagnetic radiation

Physical conditions

Physical conditions in astronomical objects are very diverse:

- Temperatures:
  - 3 K (microwave background radiation);
  - 10 K (molecular gas in star forming region);
  - 10^{12} K (gas near a black hole).
- Densities as high as 10^{15} g cm^{-3} in neutron stars
- Velocities above 0.99 c (speed of light)

Examples: The Sun

Physical conditions in the center of the Sun are:

- Temperature 1.5×10^7 K
- Density 150 g cm^{-3}

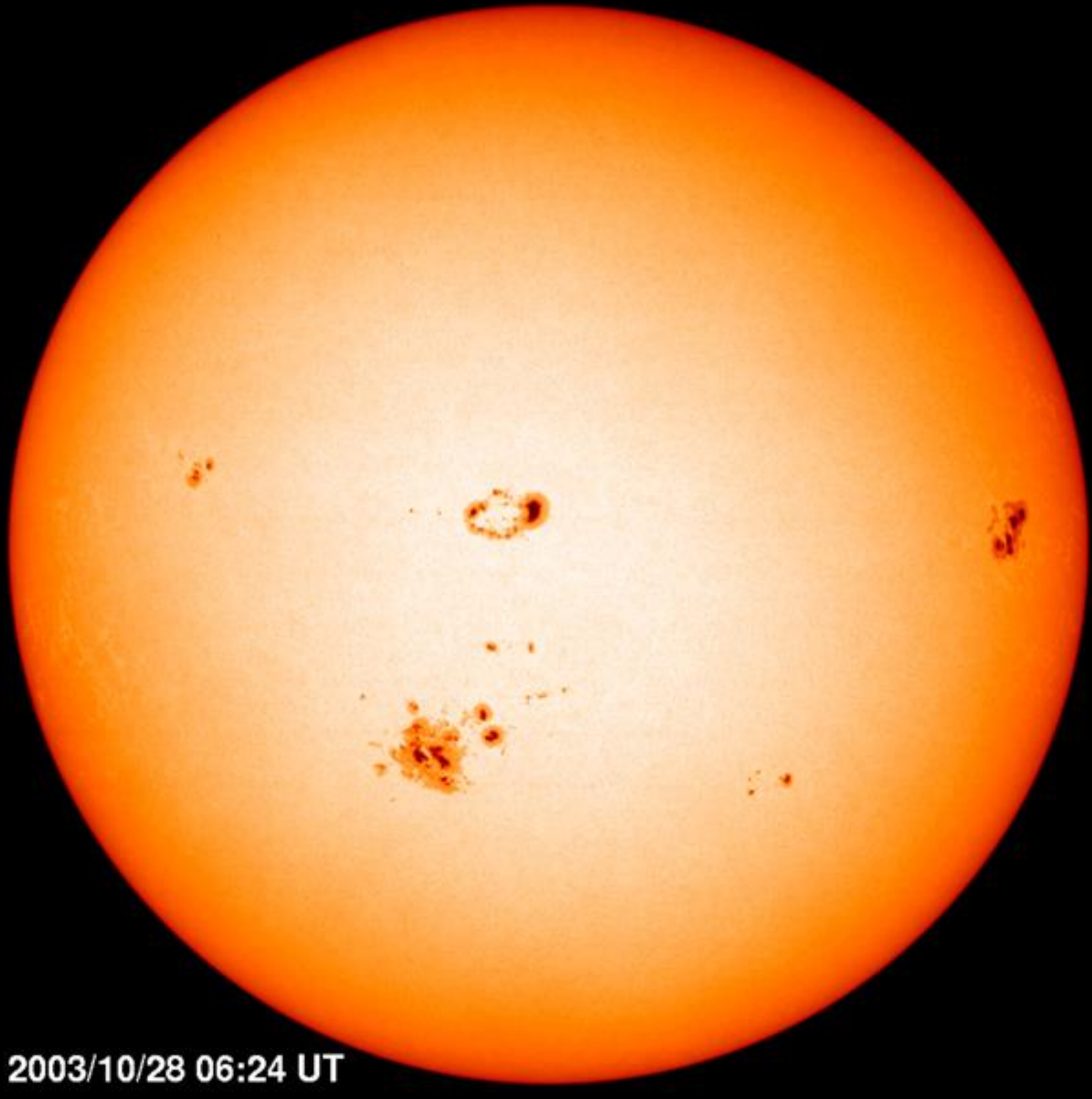
Temperature (and to a lesser extent density) is well within range accessible to lab experiments.

→ High confidence that we understand most of the basic physics (i.e. what are the laws) pretty well.

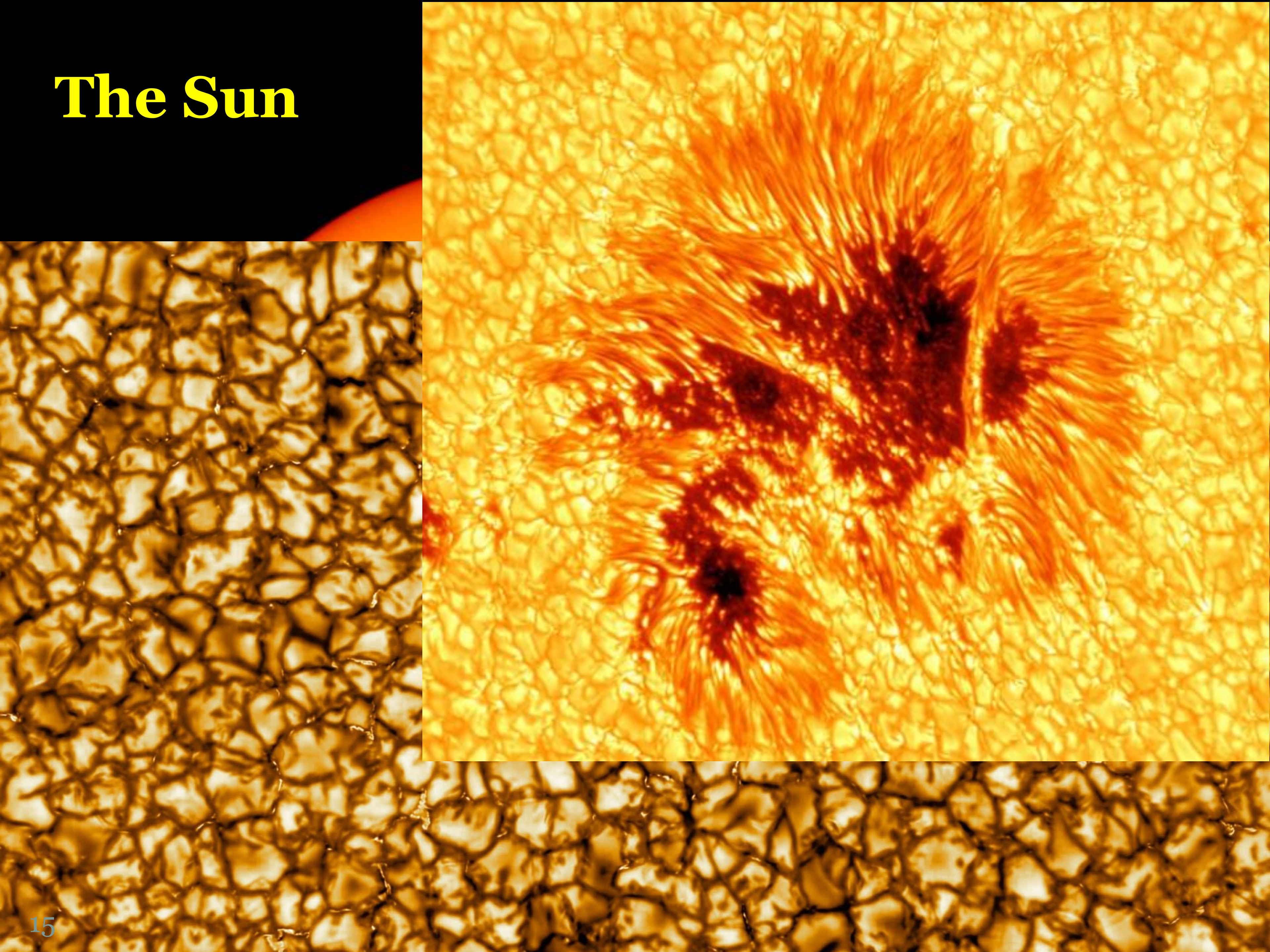
Numerous excellent observations also available.

Detailed comparison between theory and observations is possible.

The Sun



The Sun



Examples: Stellar remnants

Neutron stars can have magnetic fields >> lab conditions:

- Earth's magnetic field ~10^{-4} Tesla (1 Gauss)
- Strongest magnets ~40 T (4 x 10^5 G)
- Magnetar ~10^{11} T (10^{15} G)

Black holes produce gravitational fields enormously stronger than any found in the Solar System.

In both cases we think we know the physics needed to understand these objects:

- matter in superstrong magnetic fields (Quantum electro-dynamics)
- Black Holes (General Relativity)

But untested in the lab - ideally would hope observations could test theory in new regimes.

First-ever Image of a Black Hole

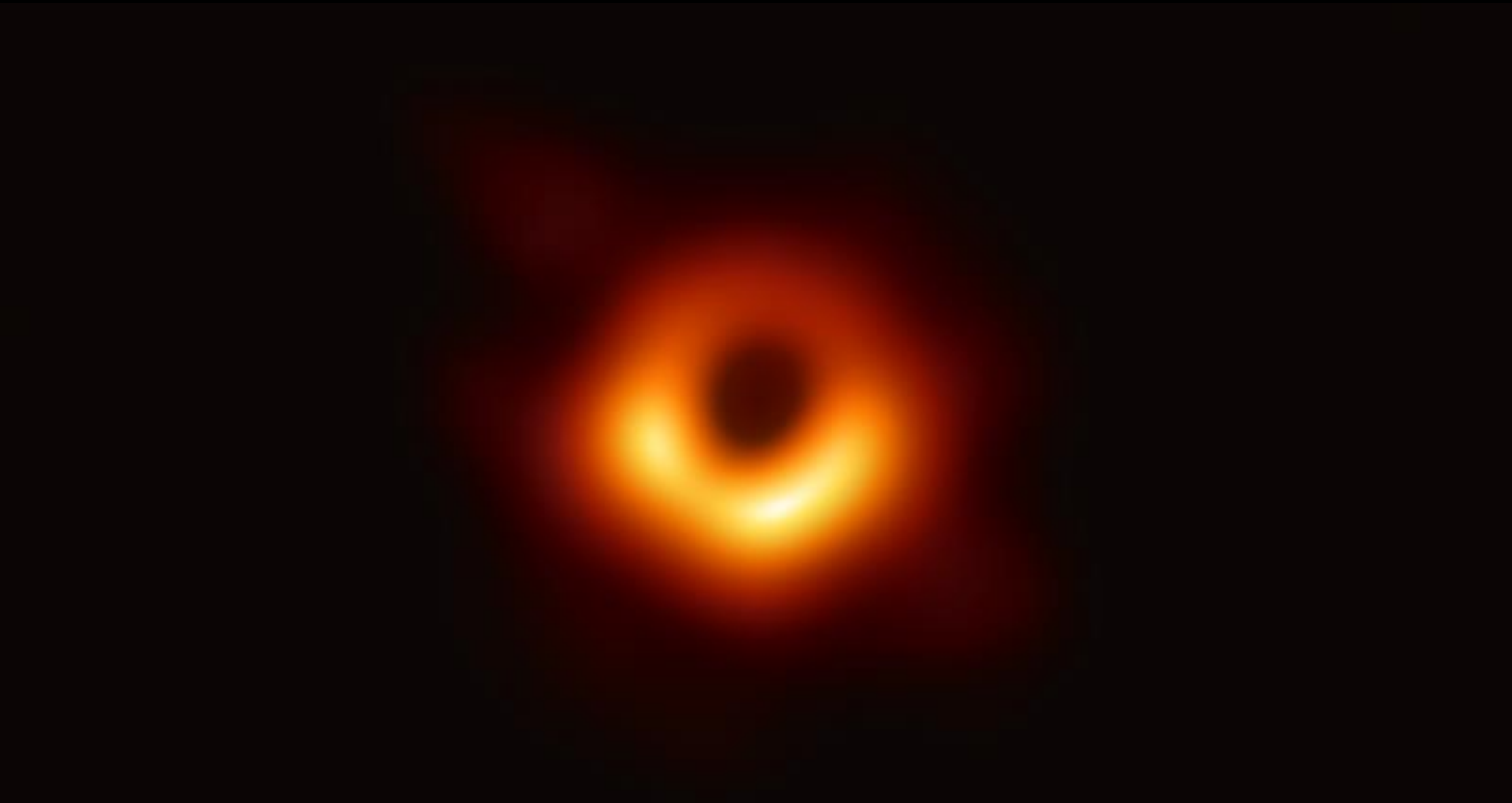


Image credit: the Event Horizon Telescope Collaboration

Black Hole

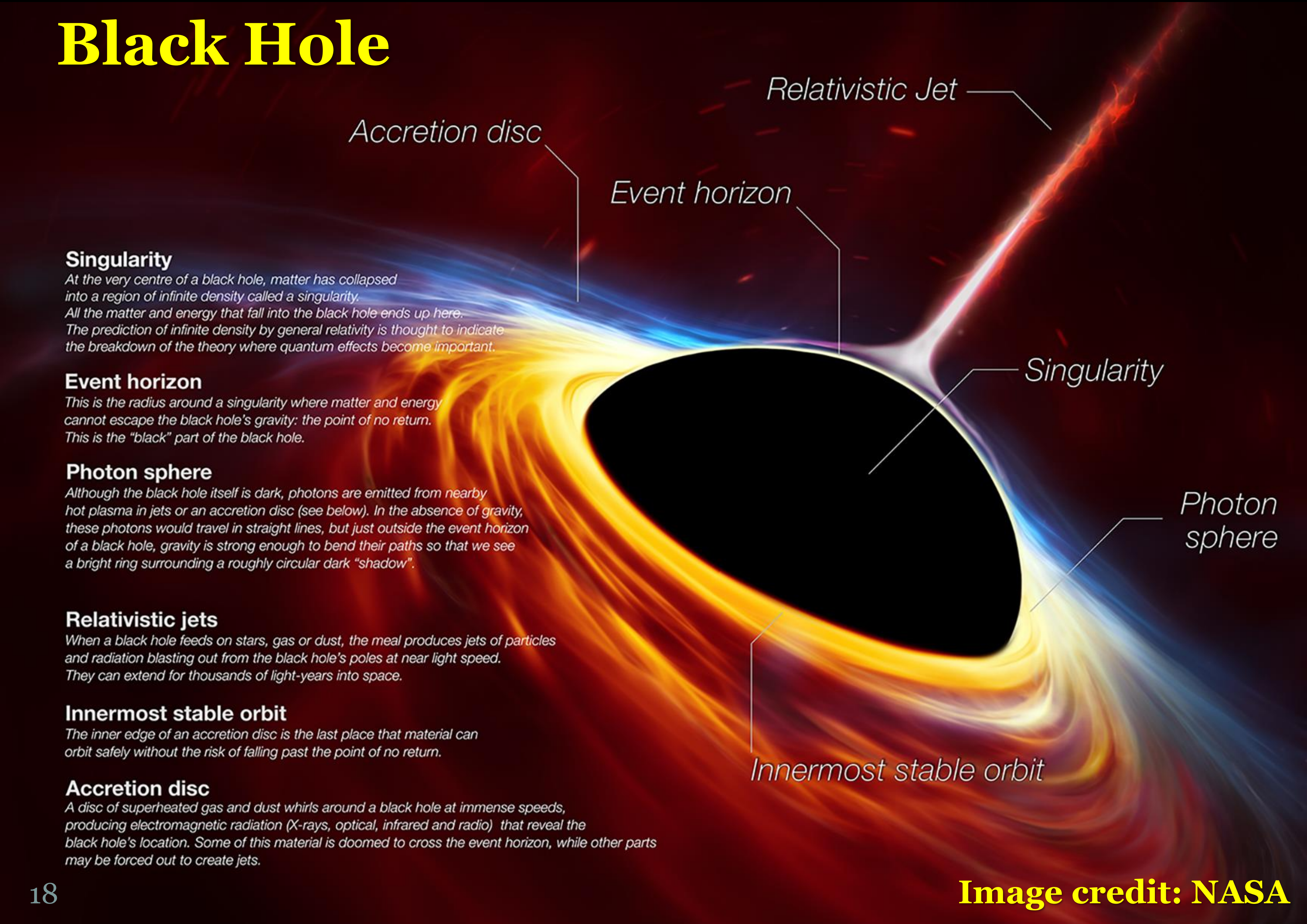


Image credit: NASA

Examples: Cosmology

Observations of the rotation curves of spiral galaxies suggest presence of **dark matter** - probably in the form of an unknown elementary particle.

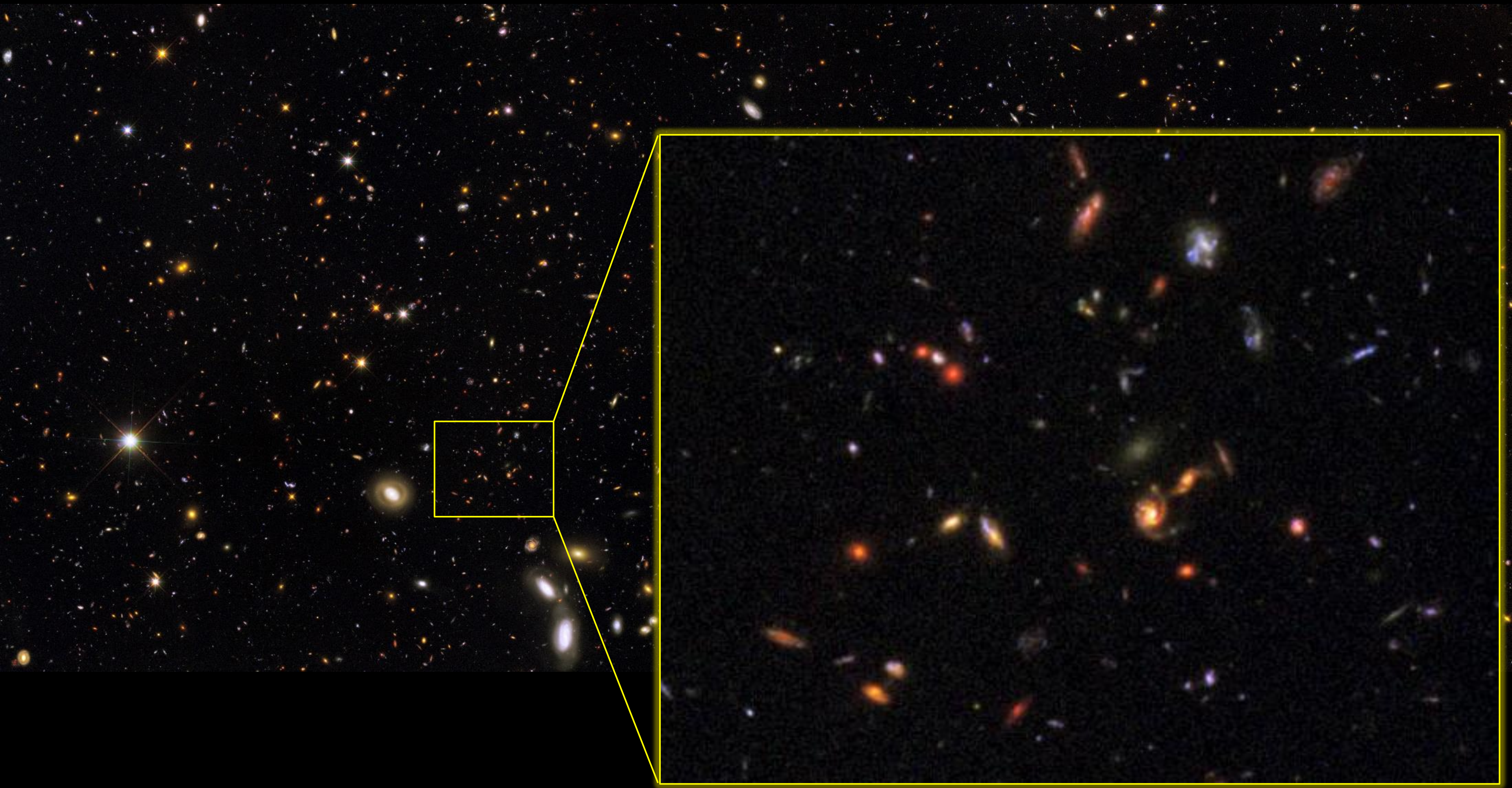
Observations of the brightness of distant supernovae suggest presence of **dark energy** - not understood at all.

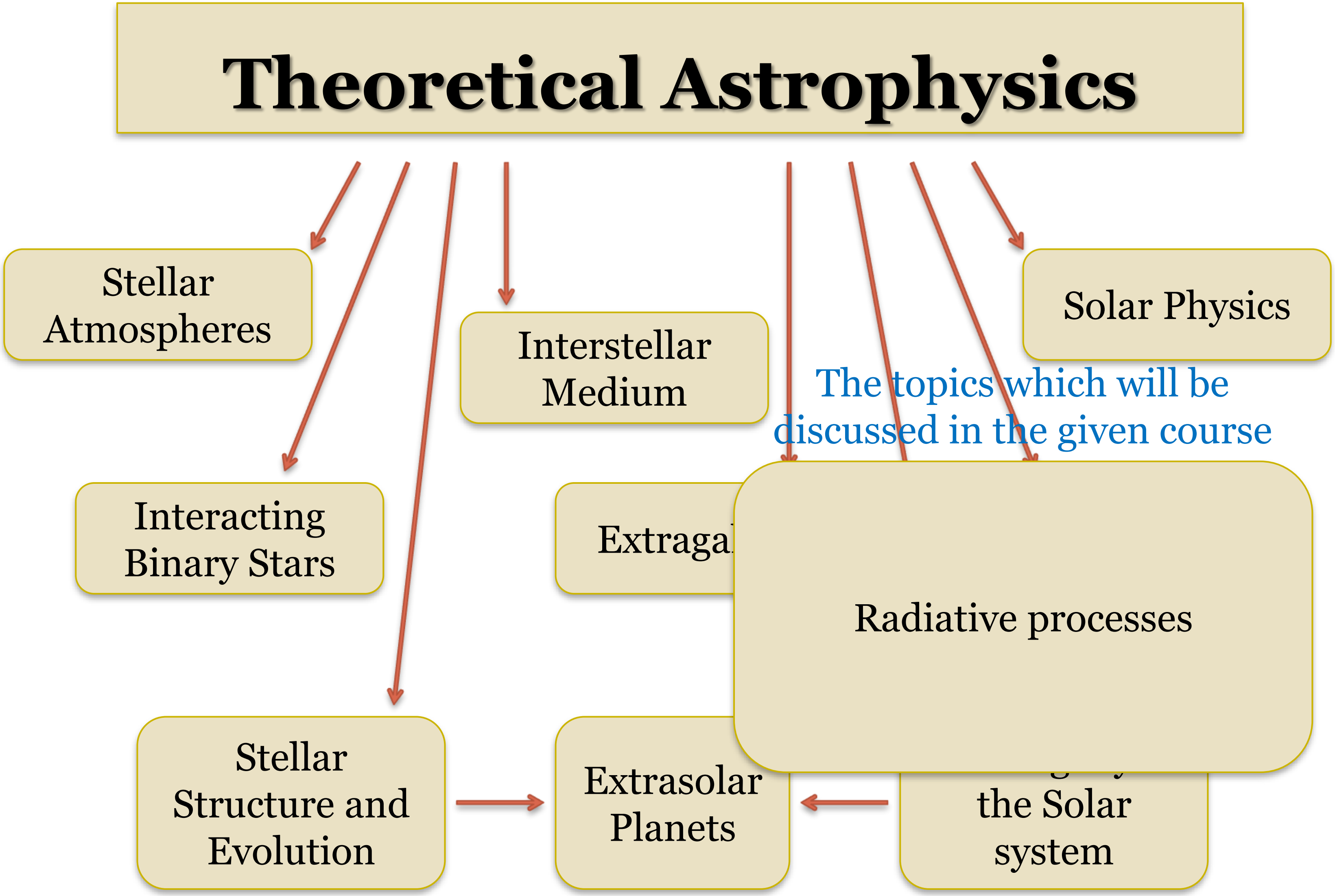
Astronomical observations hint at presence of new physics, which may be testable in the lab in the future.

One of the deepest views of the Universe

Great Observatories Origins Deep survey data

One of the deepest views of the Universe





Most of astronomical objects are detected via electromagnetic radiation

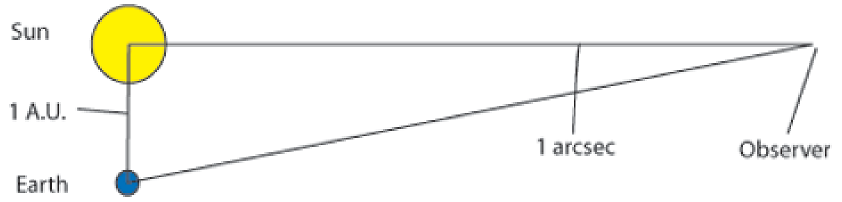
Astronomical units (1)

- Distance: an astronomical unit (AU or au) is the mean distance between the Earth and the Sun (technically the radius of a circular orbit with same period as the Earth).

$$1 \text{ au} = 1.496 \times 10^{13} \text{ cm}$$

- A parsec (pc) is defined as the distance at which 1 au subtends an angle of 1 arcsecond.

$$1 \text{ arcsec} = 4.85\times10^{-6} \text{ radians}$$



Astronomical units (2)

$$1'' = \frac{1\ \text{au}}{1\ \text{pc}} \qquad 1\ \text{pc} = \frac{1.496\times10^{13}\text{cm}}{4.85\times10^{-6}} = 3.086 \times 10^{18}\ \text{cm}$$

1 pc = 3.26 light years - roughly the distance to the nearest stars. Convenient unit for stellar astronomy.
Sizes of galaxies usually measured in kpc (galaxy scales are 10-100 kpc).
Cosmological distances are hundreds of Mpc to Gpc.
Observable Universe is a few Gpc across.

Astronomical units (3)

- Wavelengths in the optical range are usually measured in Angstroms, but may show plots in nm:

$$1\ \text{Å} = 10^{-8}\ \text{cm} = 10^{-10}\ \text{m} = 0.1\ \text{nm}$$

- Other common units are the Solar mass, Solar radius, and Solar luminosity:

  $R_\odot = 6.955\text{x}10^{10}$ cm,
  $L_\odot = 3.845\text{x}10^{33}$ erg/s,
  $M_\odot = 1.989\text{x}10^{33}$ g

cgs (cm/gram/second) units

Fundamental:

Gravitational constant, $G=6.679x10^{-8}$ $cm^3/g/s^2$
Stefan-Boltzmann: $\sigma=5.6705x10^{-5}$ $erg/cm^2/s/K^4$
Speed of light, $c=2.99792x10^{10}cm$,
Electron mass: $m_e=9.109x10^{-28}$ g
Planck constant $h=6.626x10^{-27}$ erg s
Electron charge: $4.803x10^{-10}$ e.s.u.
Boltzmann const: $k=1.380x10^{-16}$ erg/s or $8.617x10^{-5}$ eV/K
Gas constant $R=8.314x10^{7}$ erg/mol K

Solar:

Radius $R_\odot=6.955x10^{10}$ cm,
Luminosity $L_\odot= 3.845x10^{33}$ erg/s,
Mass $M_\odot=1.989x10^{33}$ g

Astronomical:

Parsec, pc: $3.086x10^{18}cm$
Astronomical Unit, AU: $1.496x10^{13}$ cm
Angstrom, 1 Å = $10^{-8}cm$

Miscellaneous

Energy of 1 $eV=1.602x10^{-26}$ erg

Stars

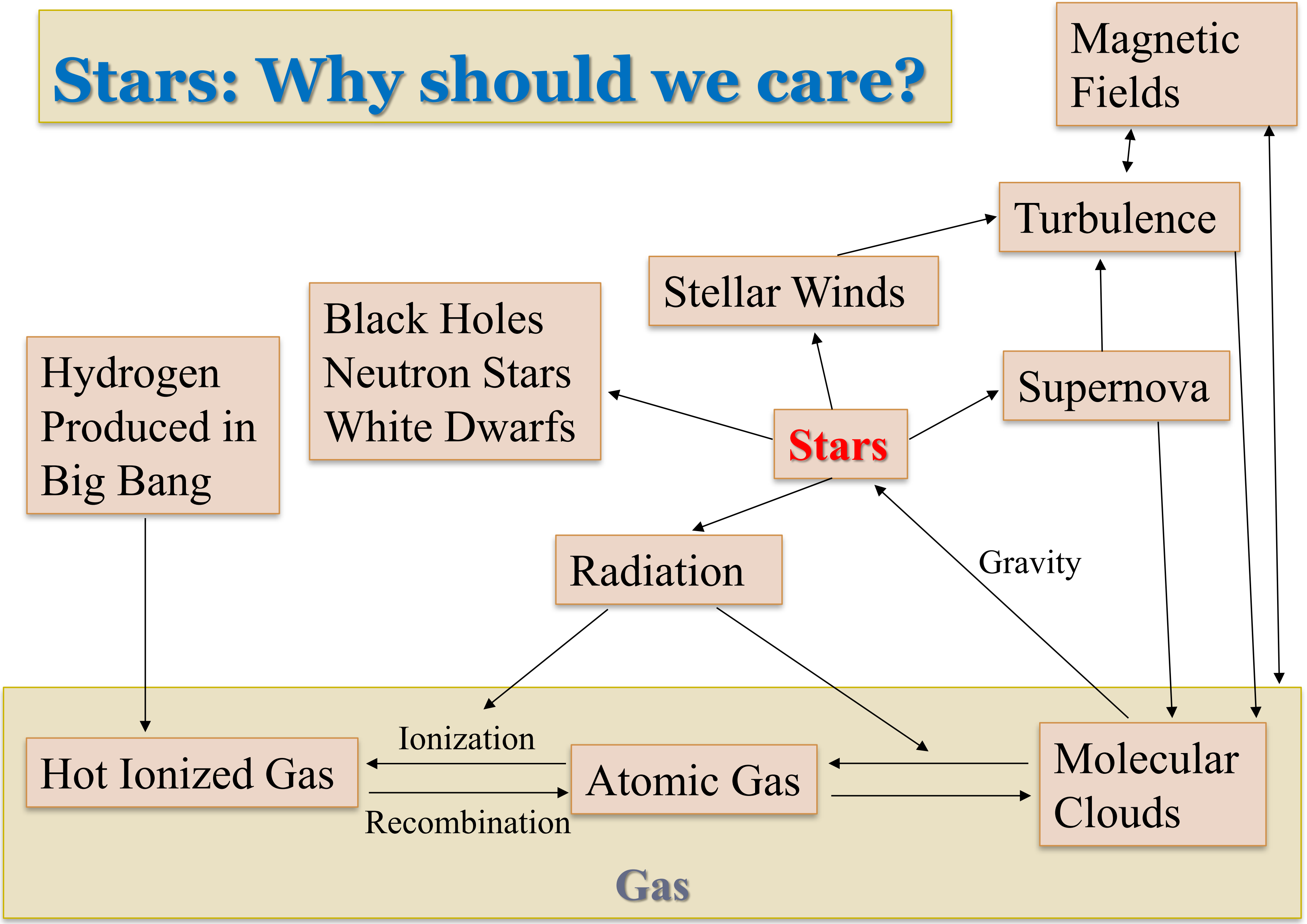
ROLE OF STARS
DEFINITION
WHAT CAN WE LEARN FROM OBSERVATIONS?
PROPERTIES OF STARS
STELLAR TIMELINE

Star Field as seen through the Hubble Space Telescope

Role of stars

- Astronomy (Greek :
  αστρονομία = ἄστρον + νόμος,
  astronomia = astron + nomos,
  literally, “law of the stars”).
- Most of the visible **matter** in the Universe (97% in our Galaxy) is contained within stars.
- Stars produce most of the **energy** in the Universe in the present time.
- The creation of the **chemical elements** (nucleosynthesis) mostly occurs in stars at present.
- In the Milky Way about 400 billion ($4x10^{11}$) stars.

Stars: Why should we care?



What is a star?

Elementary definition:

- Star is a huge glowing self-luminous gaseous sphere

Size

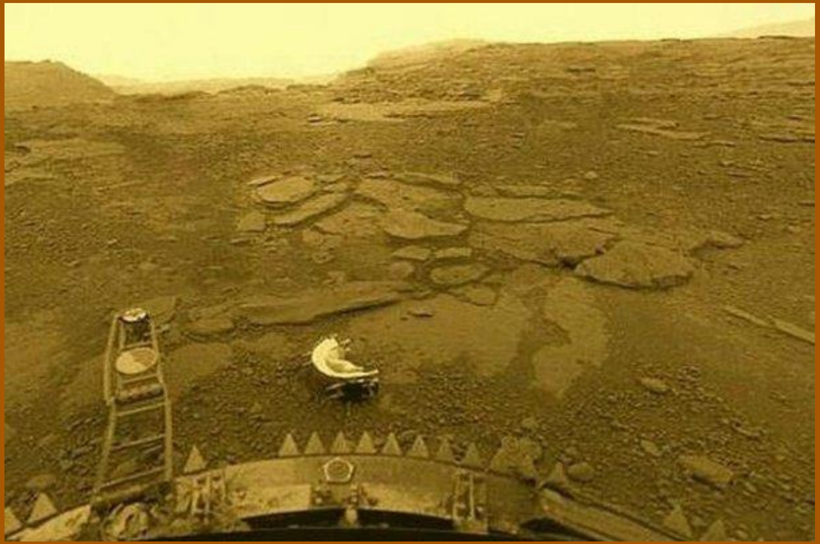
Star is a **huge** glowing self-luminous gaseous sphere

- Sun 700 000 km= 7×10^{10} cm
- Earth 6500 km = 6.5×10^{8} cm
- Compact stars:
  - White dwarfs ~ 10^4 km ~ 10^9 cm
  - Neutron stars ~10 km ~ 10^6 cm

Temperature

Star is a huge **glowing** self-luminous gaseous sphere

- By temperature, the temperature of the surface is usually meant.
- However, normal stars have no solid or liquid surface. Therefore, the photosphere (a star's outer shell from which light is radiated) is typically used to describe the star's visual surface.
  - Sun T = 5777 K
  - Very low-mass stars T = 800-1000 K
  - Compare with Venus surface T = 700 K



Energy production

Star is a huge glowing **self-luminous** gaseous sphere

- Jupiter (a planet, not a star) radiates about twice as much energy as it receives from the Sun, so it produces energy and can also be treated as self-luminous object

State of matter

Star is a huge glowing self-luminous **gaseous** sphere

- Sun:
  - Average density: 1.4 g/cm^3
  - Centre density: 162 g/cm^3
  - Photosphere density: 2×10^{-7} g/cm^3

- Densities of compact stars are extremely high
  - White dwarfs: 10^6 g/cm^3
  - Neutron stars: 10^{15} g/cm^3
  - Neutron stars have solid surface (crust) with cracks and starquakes

Form

Star is a huge glowing self-luminous gaseous **sphere**

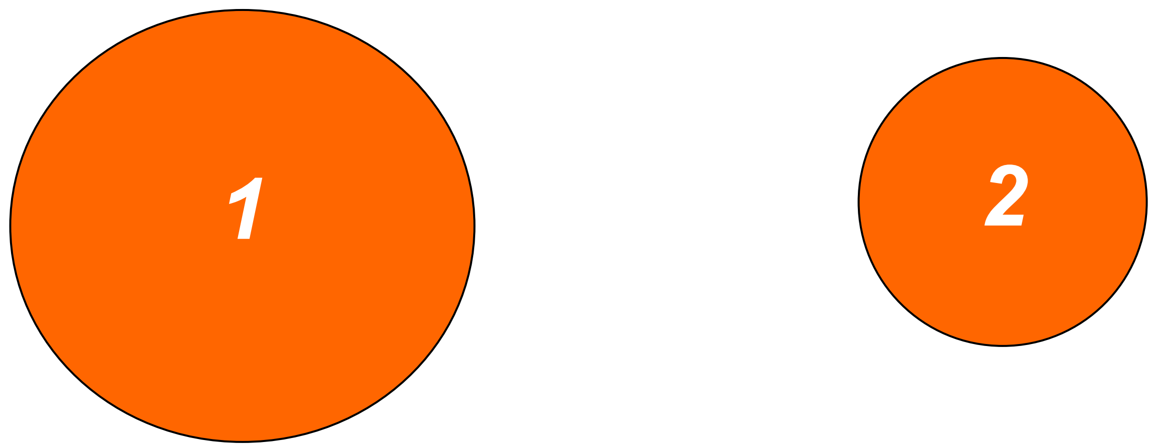
- Rotating stars are oblate due to centrifugal forces.
  - Example: α Eri (Achernar) V=0.5 mag
    - Rotational velocity at equator is 225 km/s
    - Equatorial radius is 11.8 $R_\odot$
    - Polar radius is 7.6 $R_\odot$
    - Flatness is 2:3
  - Sun:
    - Flatness is 9×10^{-6}
- Stars in double systems are elongated due to gravitational attraction

What is a star?

Definition:

- Stars are self-gravitating objects where thermonuclear reactions, which convert hydrogen into helium, operate or occurred in the past.

L vs R and T_{eff}

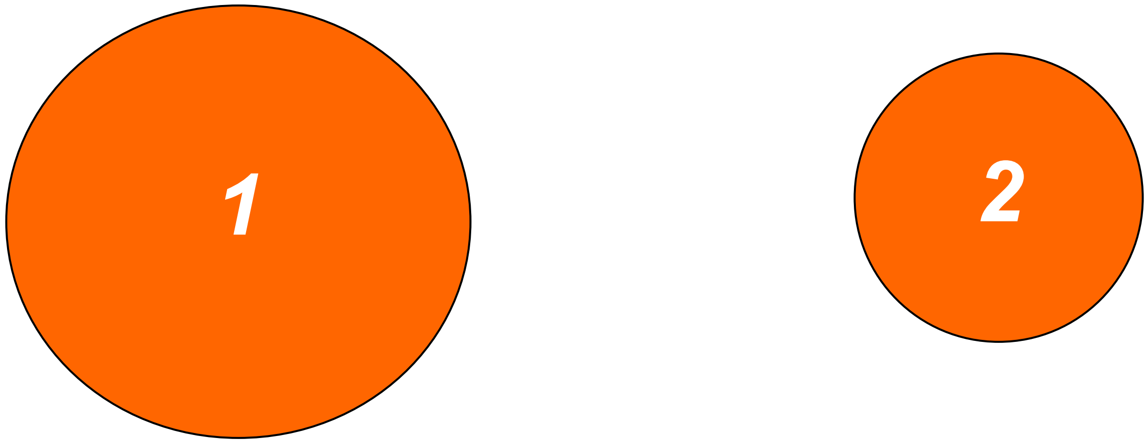


Star 1: $R = 2R_{\odot}$, $T = T_{\odot}$

Star 2: $R = R_{\odot}$, $T = T_{\odot}$

Which star is more luminous?

L vs R and T_{eff}



$R = 2R_{\odot}$ $\quad$ $R = R_{\odot}$

$T = T_{\odot}$ $\quad$ $T = T_{\odot}$

Each cm^2 of each surface emits the same amount of radiation.

The larger stars emits more radiation because it has a larger surface.
It emits 4 times as much radiation.

L vs R and T_{eff}

$R = R_\odot$ $\qquad$ $R = R_\odot$

$T = T_\odot$ $\qquad$ $T = 2\,T_\odot$

Which star is more luminous?

L vs R and T_{eff}

$R = R_\odot$ $\quad$ $R = R_\odot$

$T = T_\odot$ $\quad$ $T = 2\,T_\odot$

The hotter star is more luminous.

Luminosity varies as T^4 (Stefan-Boltzmann Law)

Luminosity Law

$$L \propto R^2\, T^4$$

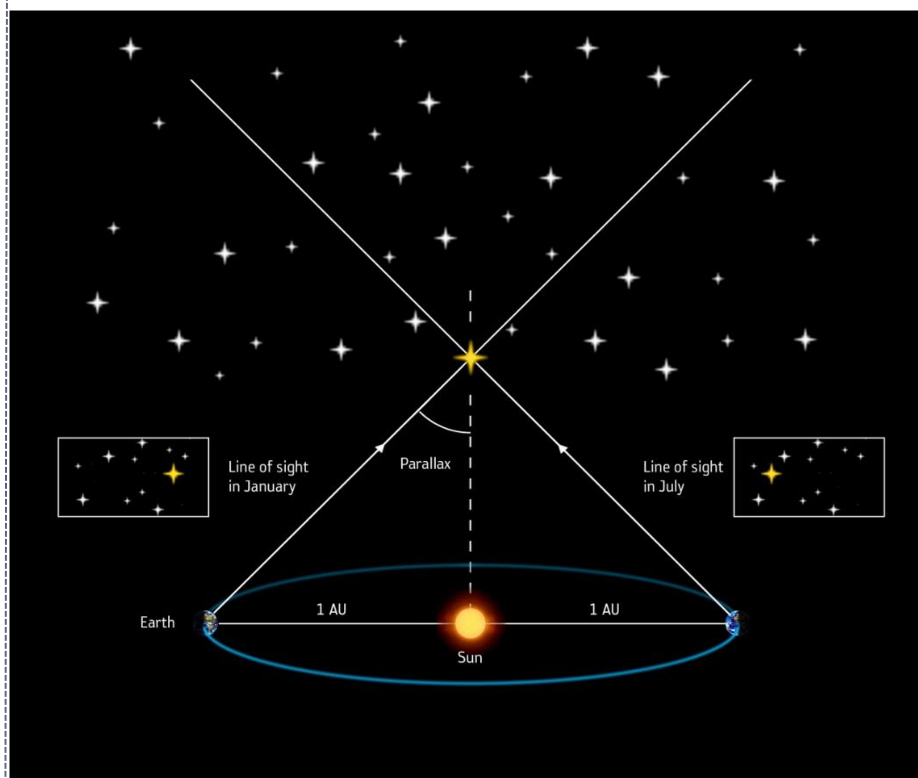
Luminosity $\propto$ Surface Area $\propto$ (Radius)^2

Luminosity $\propto$ (Temperature)^4

If star 1 is 2 times as hot as star 2, and the same radius, then it will be $2^4 = 16$ times as luminous.

What can we learn from observations?

- Stellar Luminosities come from distance measurements. The best way to perform such measurements is through parallax:
  - Ground-based measurements (difficult)
  - Hipparcos (1989-1993, >100 000 stars measured)
  - Gaia (launched in 2013, goal – to measure one billion sources, 1% of the Galaxy's population).



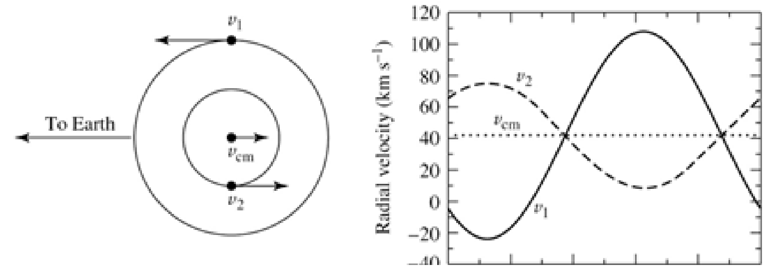
What can we learn from observations?

- Stellar Sizes and Masses primarily come from Binary Stars:

Spectroscopic Binaries (4)

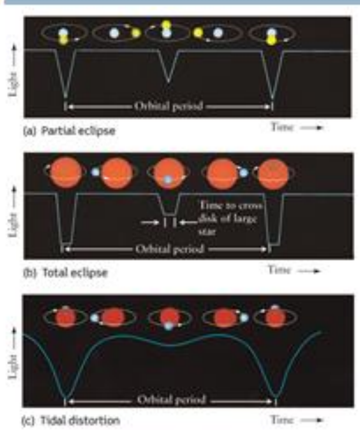
21

- Radial Velocity curve for Double-lined SB in **a Circular Orbit**:



Eclipsing (Photometric) Binaries

24

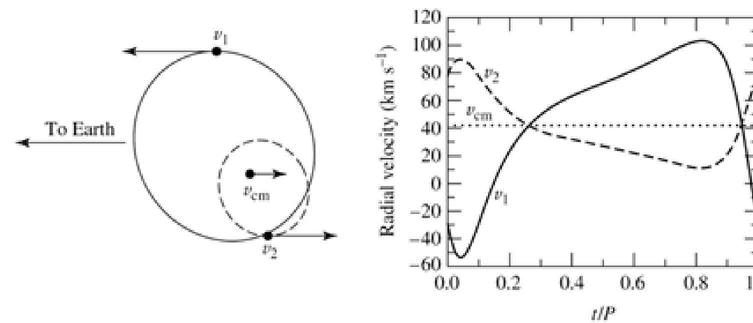


- By studying the shape of the eclipses, in conjunction with a knowledge of their radial velocity curves, it is possible to determine the masses and radii of the stars in the binary.
- Eclipsing binaries are hence extremely useful systems.

Spectroscopic Binaries (5)

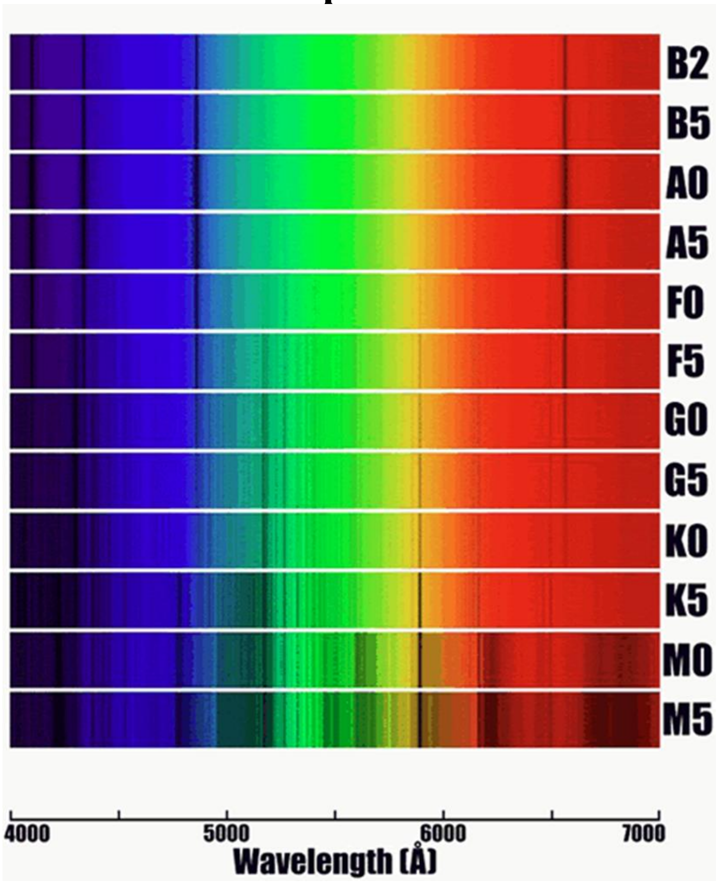
22

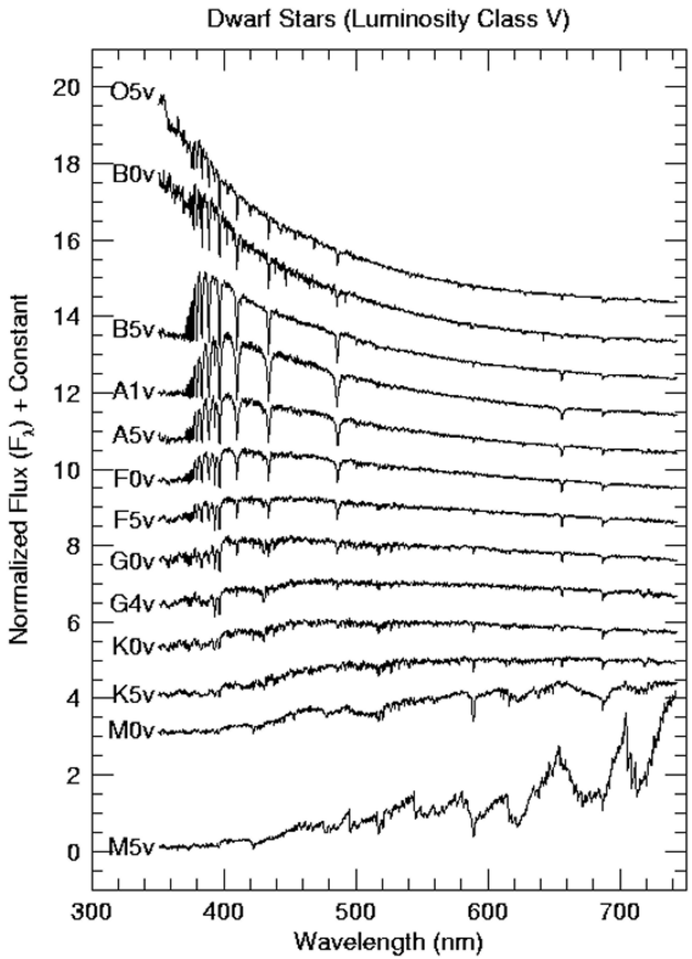
- Radial Velocity curve for Double-lined SB in **an Elliptical Orbit (e=0.4)**:



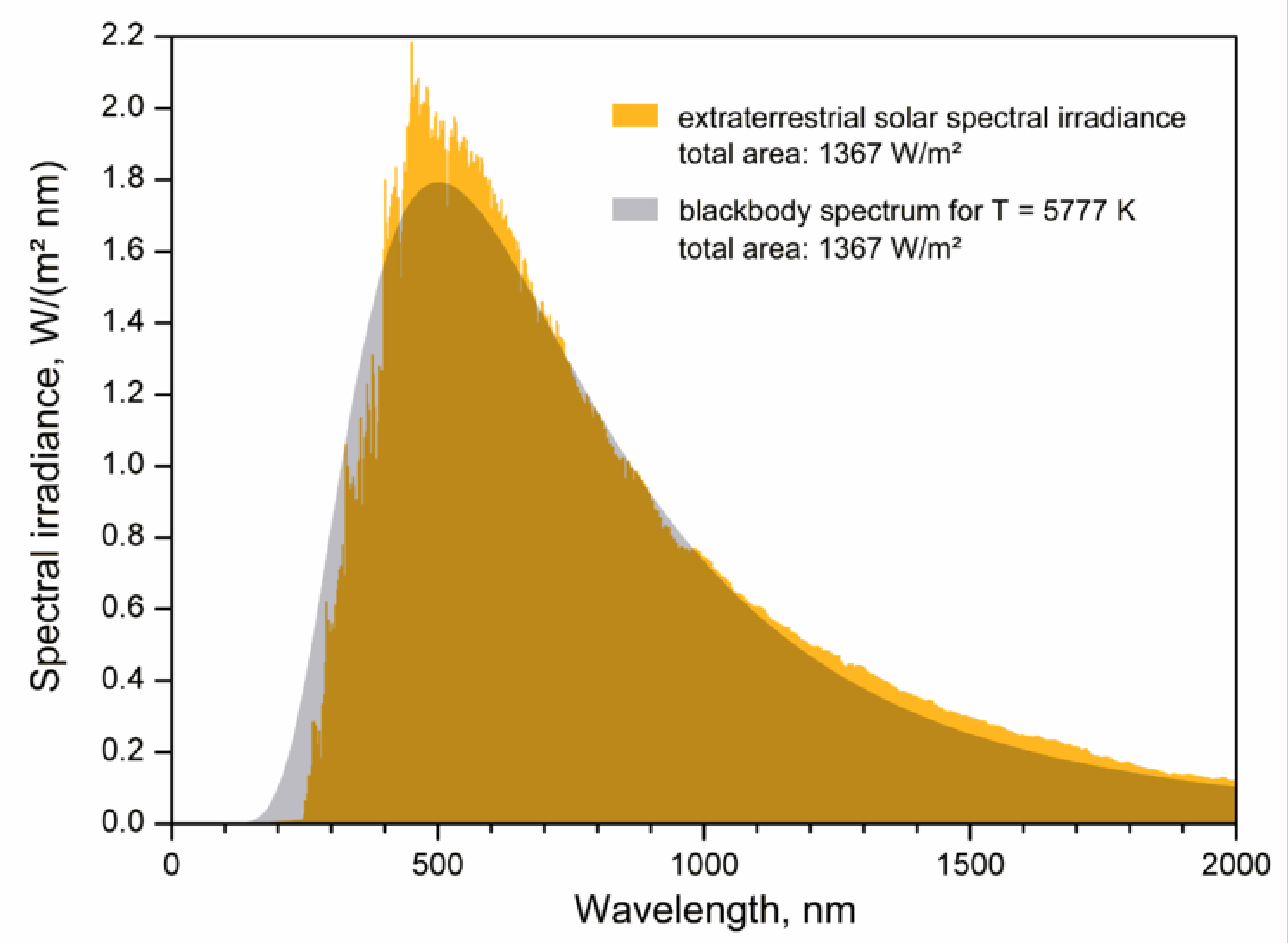
What can we learn from observations?

- Stellar Temperatures come from spectra:





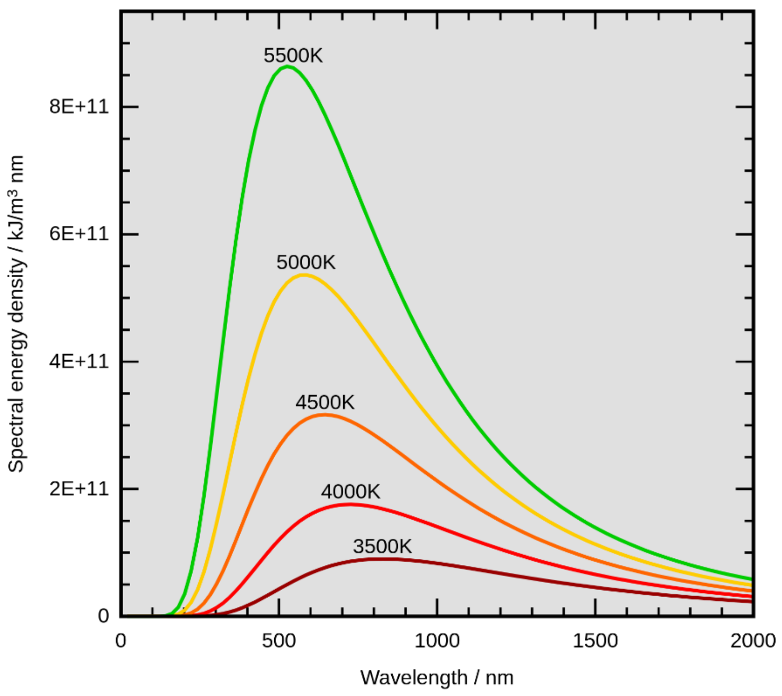
Stars share properties of black-bodies



Properties of the Planck law

- For increasing temperatures, the black body intensity increases for all wavelengths. The maximum in the energy distribution shifts to shorter λ (longer ν) for higher temperatures.
- $\lambda_{max} T = 2.98978 \times 10^{7}$ Å K

  is Wien's displacement law for the maximum I_λ providing an estimate of the peak emission (λ_{max} =5175 Å for the Sun).



Stefan – Boltzmann Law

Blackbody radiation is continuous and isotropic whose intensity varies only with wavelength and temperature.

Following empirical (Josef Stefan in 1879) and theoretical (Ludwig Boltzmann in 1884) studies of black bodies, there is a well known relation between Flux and Temperature known as Stefan-Boltzmann law:

$$F=\sigma T^4$$

with $\sigma=5.6705 \times 10^{-5}$ erg cm^{-2} s^{-1} K^{-4}

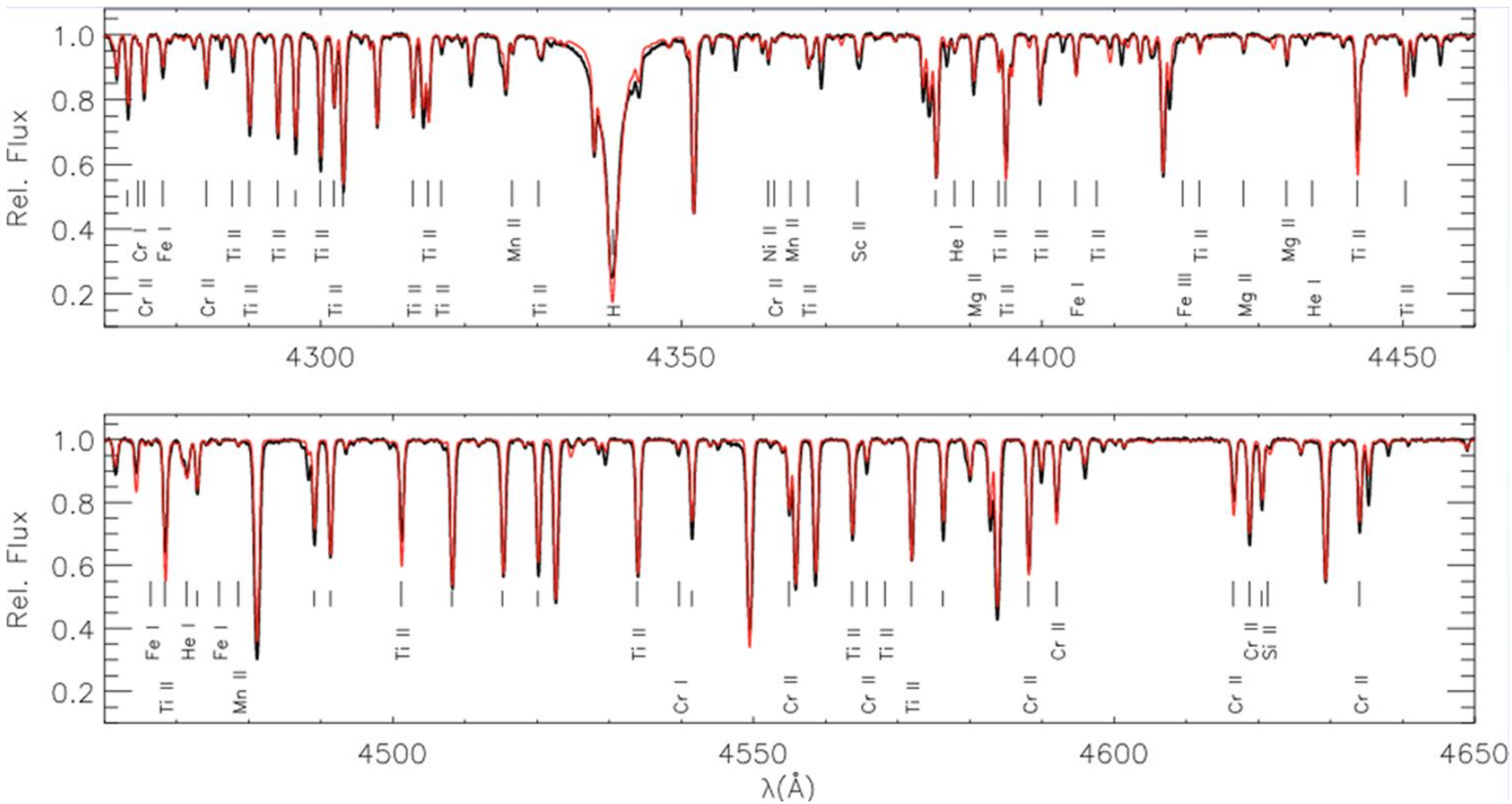
Effective temperatures of stars

- The Stefan-Boltzmann law, $F = \sigma T^4$, or alternatively $\frac{L}{4\pi R^2} = \sigma T_{\mathrm{eff}}^4$ defines the "effective temperature" of a star, i.e. the temperature which a black body would need to radiate the same amount of energy as the star.

- T_{eff} is 5777K for the Sun.

What can we learn from observations?

Stellar Abundances also come from spectra:



Primary star parameters (T_{eff}, log g)

- **Effective temperature** (in K) is defined by $L=4\pi R^2\sigma T_{eff}^4$

  (here L - luminosity, R - stellar radius), related to *ionization*.

- **Surface gravity** (cm/s^2), $g = GM/R^2$, related to *pressure*.

- The Sun has T_{eff}=5777K, log g=4.44 – its atmosphere is only a few hundred km deep, <0.1% of the stellar radius.
- A red giant has log g~1 (extended atmosphere), whilst a white dwarf has log g~8 (effectively zero atmosphere), and neutron stars have log g~14-15

Our Star, the Sun

A very ordinary star!

table 18-1 | Sun Data

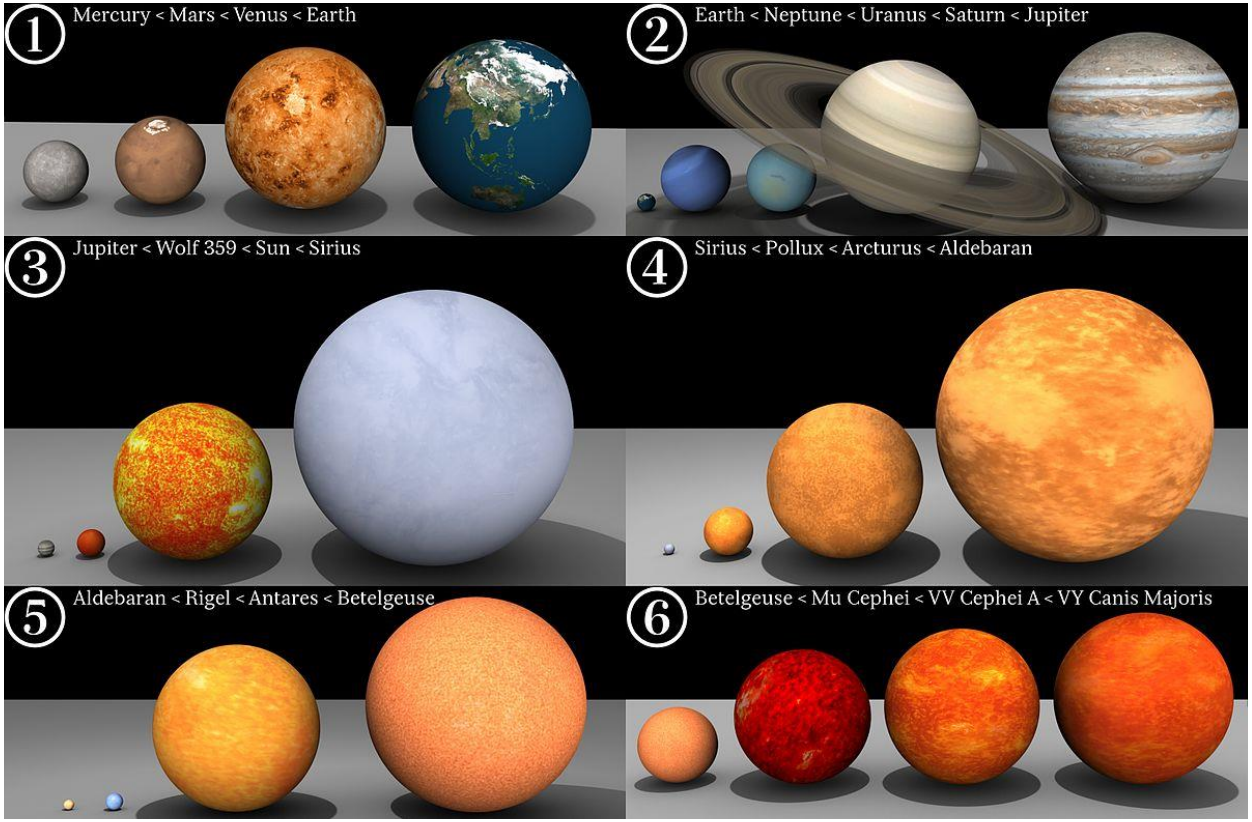
Distance from the Earth:	Mean: 1 AU = 149,598,000 km
	Maximum: 152,000,000 km
	Minimum: 147,000,000 km
Light travel time to the Earth:	8.32 min
Mean angular diameter:	32 arcmin
Radius:	696,000 km = 109 Earth radii
Mass:	1.9891×10^{30} kg = 3.33×10^{5} Earth masses
Composition (by mass):	74% hydrogen, 25% helium,
	1% other elements
Composition (by number of atoms):	92.1% hydrogen, 7.8% helium,
	0.1% other elements
Mean density:	1410 kg/m^3
Mean temperatures:	Surface: 5800 K; Center: 1.55×10^{7} K
Luminosity:	3.86×10^{26} W
Distance from center of Galaxy:	8000 pc = 26,000 ly
Orbital period around center of Galaxy:	220 million years
Orbital speed around center of Galaxy:	220 km/s

Are all the stars the same?

- **No!**

- Luminosity
  $10^{-6} L_\odot < L < 10^{6} L_\odot$:
  factor of 10^{12} in L
- Radius
  $10^{-5} R_\odot < R < 10^{3} R_\odot$:
  factor of 10^{8} in R
- Mass
  $10^{-2} M_\odot < M < 10^{2} M_\odot$:
  factor of 10^{4} in M
- Temperature
  10^{3} K $< T_{eff} < 10^{5}$ K:
  factor of 10^{2} in T_{eff}

  (but note that in neutron stars T_{eff} can be higher than 10^{7} K)



Spectral Lines

- Spectral lines originate in a stellar atmosphere – a thin, tenuous transition zone between (invisible) stellar interior and (essentially vacuum) exterior.

- Stellar interiors are effectively **invisible** to external observers (apart for e.g. astroseismology) so all the information we receive from stars originates from their atmospheres. Understanding how radiation interacts with matter affecting the emergent line and continuous spectrum is a part of this course. We will discuss it later.

Spectral Types

Morgan-Keenan (M-K) classification scheme orders stars via "OBAFGKM" spectral classes using ratios of spectral line strength.

O-types have the highest T_{eff}'s. OBA stars are "early-type" star, whilst cooler stars are "late-type".

Spectral classes are each subdivided into (up to) ten divisions – e.g. O2 .. O9, B0, B1 .. B9, A0, A1 .. etc

Table 15.1. *MK spectral classes.*

MK spectral class	Class characteristics
O	Hot stars with He II absorption
B	He I absorption; H developing later
A	Very strong H, decreasing later; Ca II increasing
F	Ca II stronger; H weaker; metals developing
G	Ca II strong; Fe and other metals strong; H weaker
K	Strong metallic lines; CH and CN bands developing
M	Very red; TiO bands developing strongly

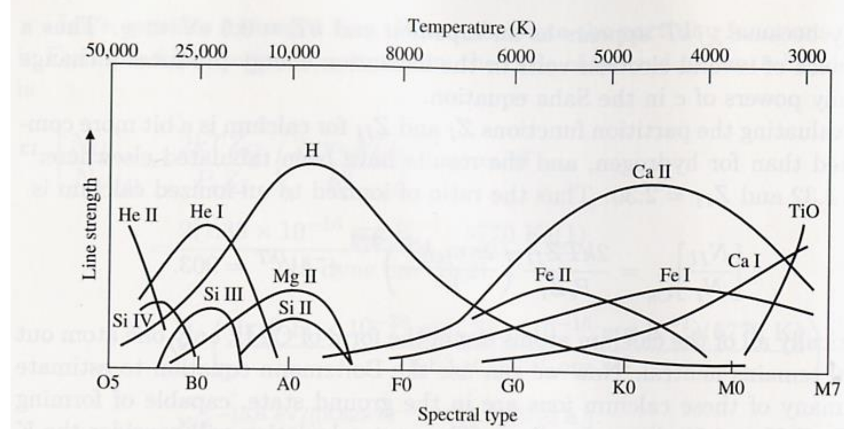


Figure 8.9 The dependence of spectral line strengths on temperature.

Luminosity Class classification

- Luminosity class information is often added, based upon spectral line widths:

Ia	Most luminous supergiants
Ib	Less luminous supergiants
II	Luminous giant
III	Normal giants
IV	Subgiants
V	Main sequence stars (dwarfs)
VI	Subdwarfs
VII	White dwarfs

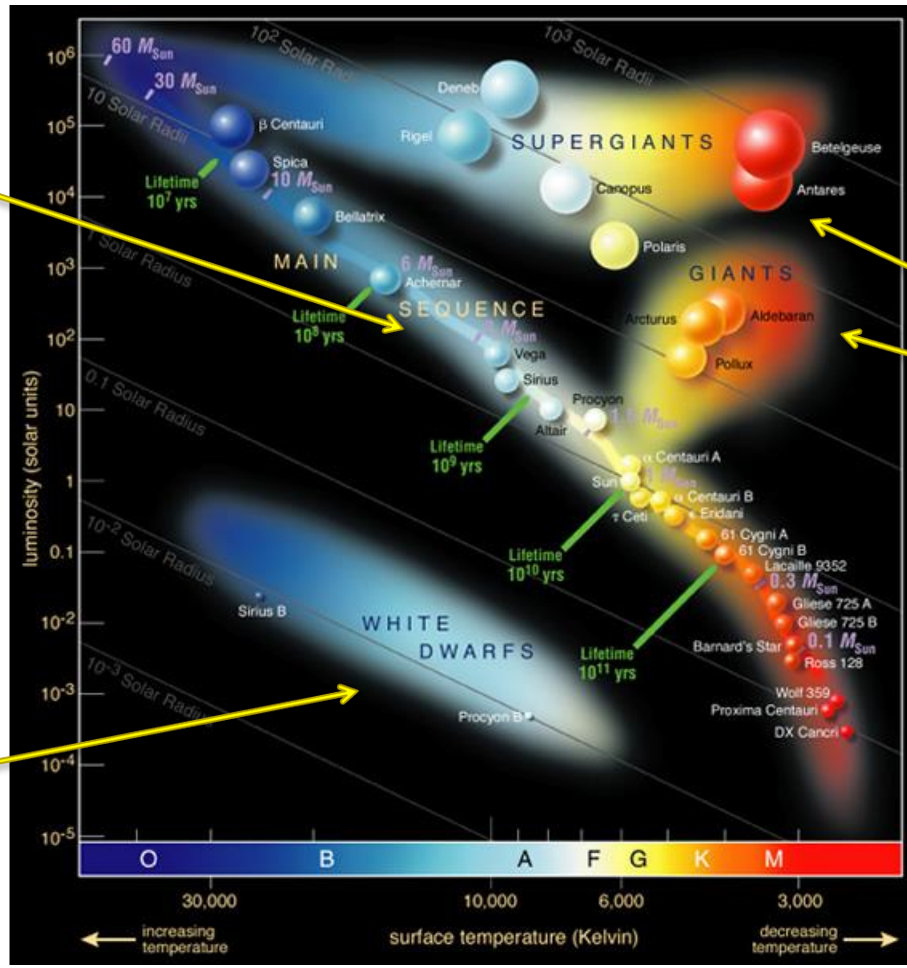
- Dwarfs have high pressures (large line widths) and supergiants have lower pressures (smaller line widths).

Some properties of representative stars

Star	Sp	T_{eff} (K)	$L/L_{\odot}$	$R/R_{\odot}$	$M/M_{\odot}$	ρ (g/cm^3)	Remarks
α Sco A (Antares)	M0 Ib	3300	34000	530	20	$1.7 \cdot 10^{-7}$	Supergiant
α Boo (Arcturus)	K2 III	3970	130	26	4	$3.2 \cdot 10^{-4}$	Giant
η Ori	B1 V	23000	13000	7	14	0.052	
α CMa A (Sirius)	A1 V	9700	60	2.4	3.3	0.81	
Sun	G2 V	5800	1	1	1	1.4	Main sequence
Barnard's Star	M5 V	3000	0.015	0.5	0.38	4.3	
α CMa B (Sirius B)	A5 VII	8200	0.003	0.03	0.96	$7.7 \cdot 10^{4}$	White dwarf

Hertzsprung-Russell (HR) diagram

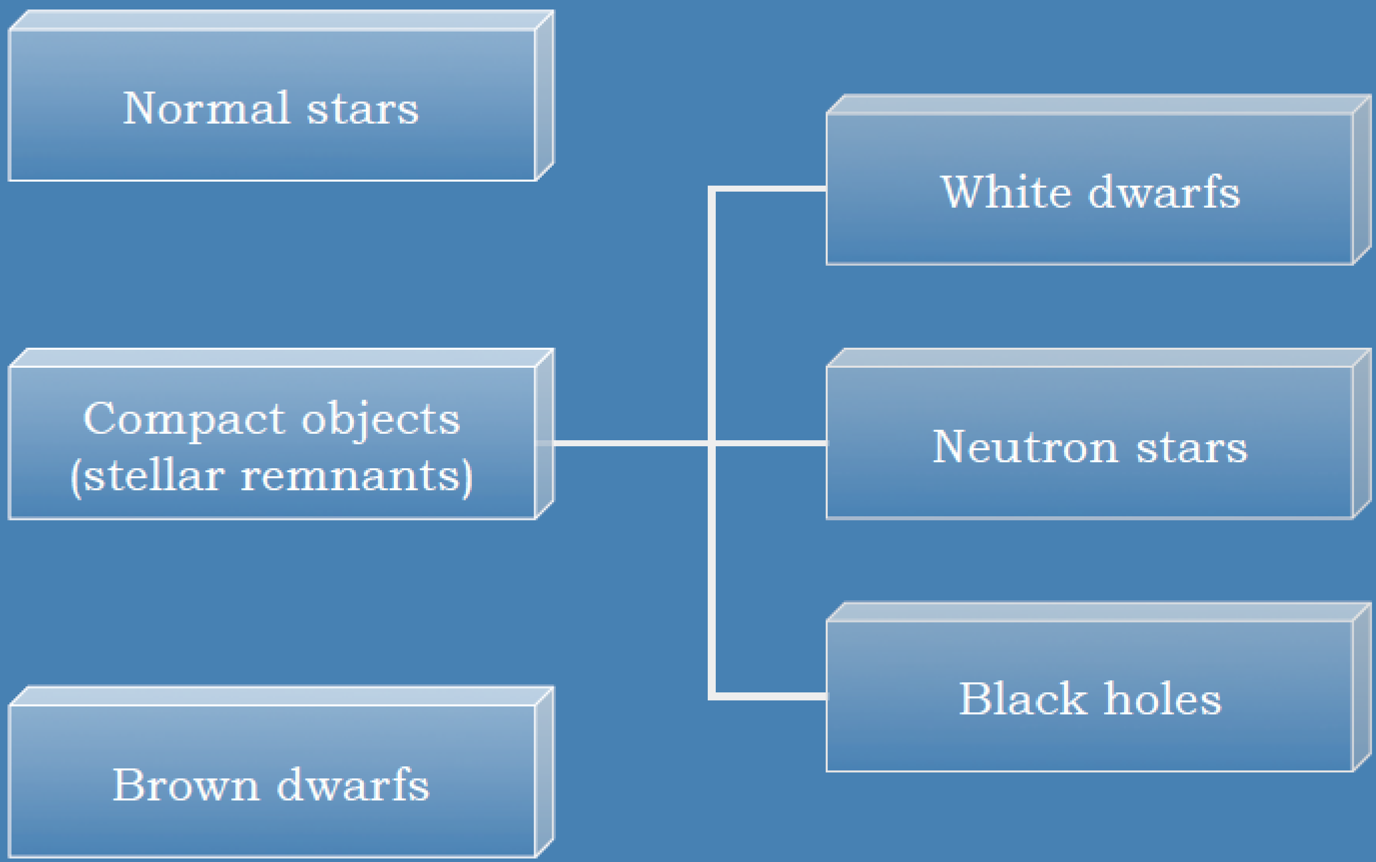
Most of the stars lie on the Main Sequence, with increasing L as T increases



A relatively cool star can be quite luminous if it has a large enough radius (10-100 $R_{\odot}$): Red Giants and Supergiants

A relatively hot star can have very low luminosity, if its radius is very small (0.01 $R_{\odot}$): White Dwarfs

Physical classification of stars



Brown dwarfs

- **Definition**
  Brown dwarfs are objects having insufficient mass to sustain normal hydrogen burning

- Masses of the brown dwarfs are ranging from 13 to 70 M_{Jup} (0.01 to 0.07 $M_{\odot}$)
- The brown dwarfs are cold ($T_{eff} < 1400$ K) and dim objects, which can be detected by infrared observation
- There are many uncertainties both theoretical and observational concerning these objects

White dwarfs

- White dwarfs consist of degenerate matter and may be treated as graveyard for the stars with initial masses $\leq 8\ M_{\odot}$
- Typical values:
  - $R \sim 10^{-2}\ R_{\odot} \sim R_{\oplus}$
  - $M \sim (0.3 \div 1)\ M_{\odot}$
  - $L \sim (10^{-2} \div 10^{-3})\ L_{\odot}$
  - $\rho \sim (10^{5} \div 10^{6})\ g/cm^{3}$

Limiting mass
(Chandrasekhar limit)
$M_{WD} < 1.4\ M_{\odot}$

Neutron stars

- Neutron stars consist of neutron "fluid" and originate from the evolution of massive stars with $M > 8\ M_\odot$
- Typical values:
  - $R \sim 10\text{-}15$ km
  - $M \sim (1 \div 2)\ M_\odot$
  - $\rho \sim 10^{15}\ g/cm^3$

Limiting mass
(Oppenheimer-Volkoff limit)
$M_{NS} < (2 \div 3)\ M_\odot$

Black holes

- Schwarzschild (gravitational) radius: $r_g = \frac{2GM}{c^2}$

- Values of Schwarzschild radius:
  - for Earth is 0.9 cm
  - for the Sun is 3 km

- Supermassive black holes (not stars) are found in the centres of many galaxies as well as in the centre of our own Galaxy, Milky Way.
  Typical values:
  - $M \sim (10^6 \div 10^9)\, M_\odot$
  - $R \sim (10^{11} \div 10^{14})$ cm $\sim (0.01 \div 10)$ AU
    So radii range approximately from $R_\odot$ to the Saturn distance

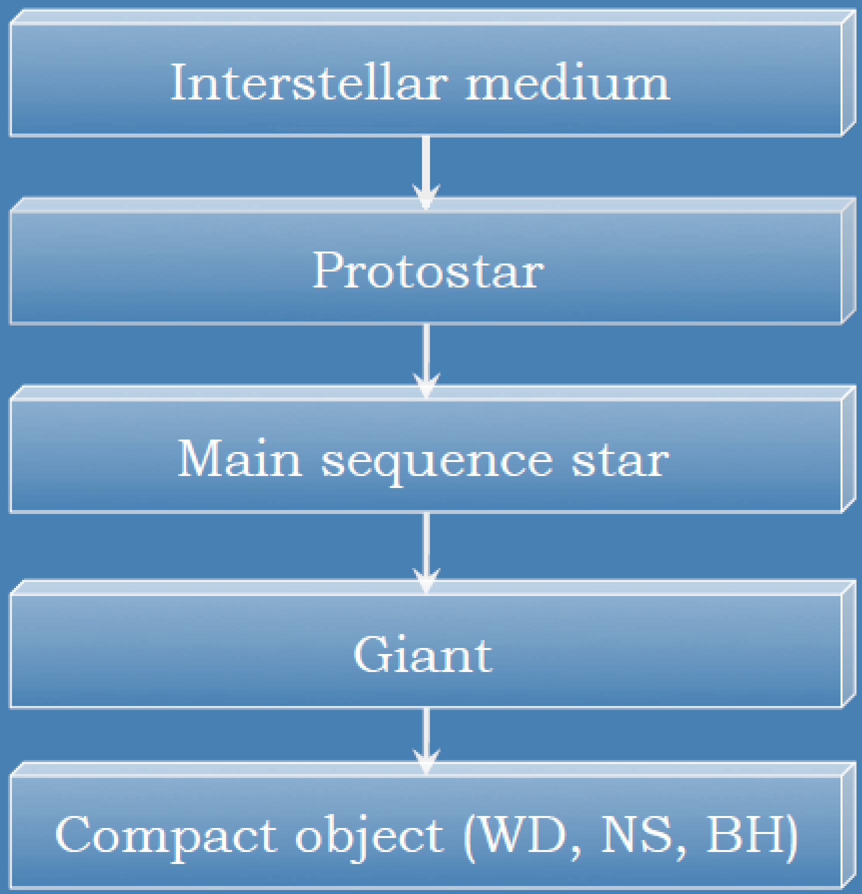
Physics involved

- The subject of stellar interiors covers a very broad area of physics and is expanding even further into fields that were previously not considered as a part of the subject matter of stellar structure.

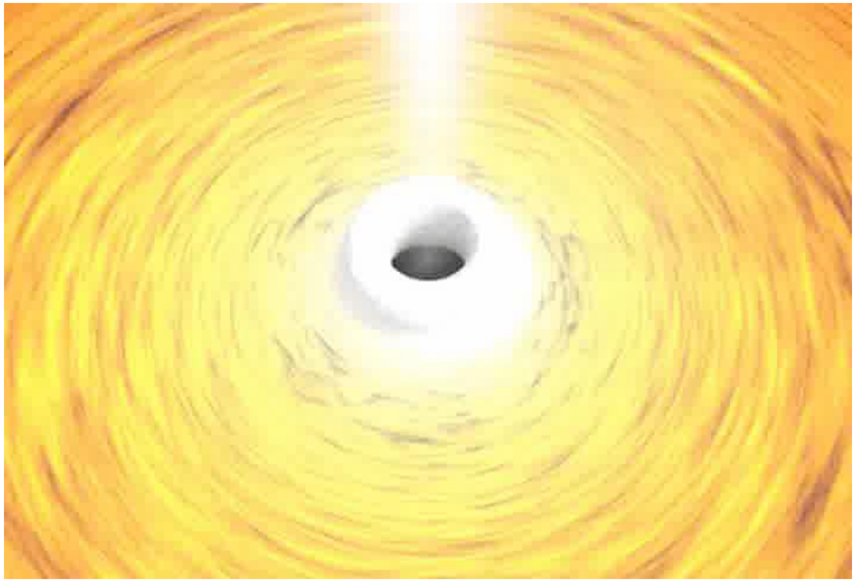
- Mechanics
- Thermodynamics
- Radiation theory
- Relativity
- Atomic physics
- Nuclear physics
- Hydrodynamics
- Solid state physics

Stellar timeline

Stars are like people in that they are born, grow up, mature, and die.

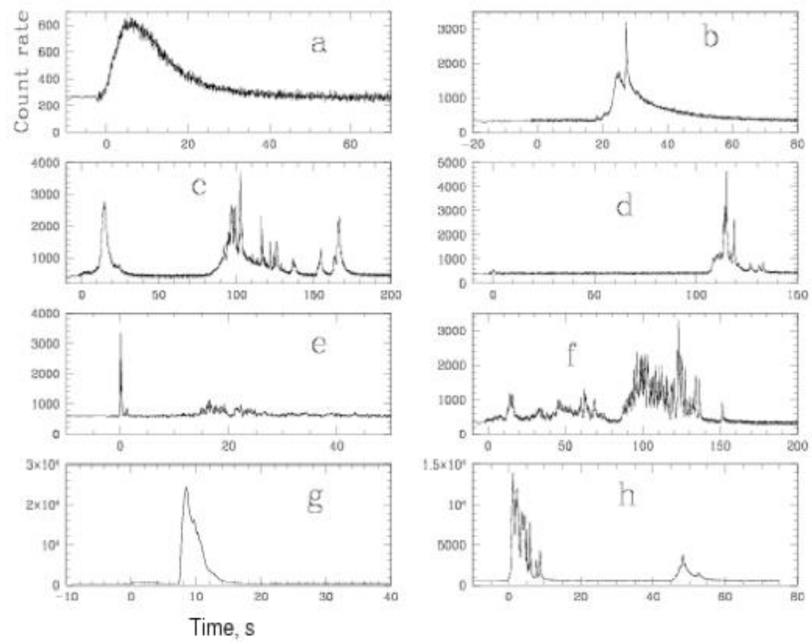


Stellar formation

Standard	Exotic
	
In interstellar gas clouds	In a gaseous disc around a supermassive black hole

Stellar collapse

- Evolution of massive stars, with $M > 20\,M_\odot$, ends up in huge eruption observed as supernova or hypernova.
- The core of the star collapses directly into a compact object and two extremely energetic jets of plasma are emitted from its rotational poles.
- In a supernova, the collapse results in a neutron star.
- In a hypernova, a black hole is formed, and a gamma-ray burst might be produced.



Equilibrium in stellar interiors

BASIC ASSUMPTIONS
MASS CONSERVATION
HYDROSTATIC EQUILIBRIUM
VIRIAL THEOREM
STELLAR TIME-SCALES

Introduction and recap (1)

Definition of **a star** as an object:

- Bound by self-gravity
- Radiates energy that is primarily released by nuclear fusion reactions in the stellar interior

Other energy sources are dominant during star formation and stellar death:

- **Star formation** - before the interior is hot enough for significant fusion, gravitational potential energy is radiated as the radius of the forming star contracts. *Protostellar or pre-main-sequence evolution*.
- **Stellar death** - remnants of stars (white dwarfs and neutron stars) radiate stored thermal energy and slowly cool down. Sometimes refer to these objects as stars but more frequently as *stellar remnants*.

Introduction and recap (2)

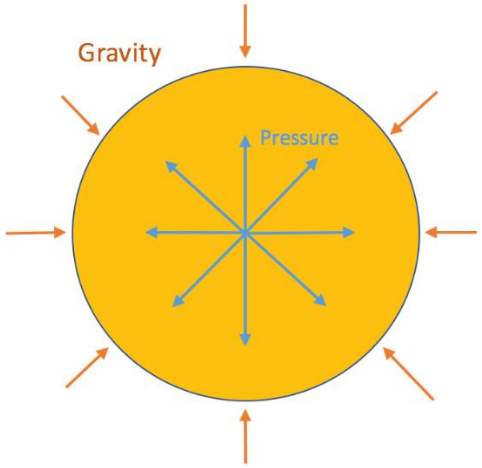
With this definition:

- **planets** are not stars - no nuclear fusion.
- objects in which release of gravitational potential energy is always greater than fusion are not stars either – these are called **brown dwarfs.**

Distinction between brown dwarfs and planets is less clear, most people reserve "planet" to mean very low mass bodies in orbit around a star.

Irrespective of what we call them, physics of stars, planets, stellar remnants is similar.
Balance between:

- **Gravity**
- **Pressure**



Basic assumptions (1)

What are the **main** physical processes which **determine** the structure of **stars**?

- Stars are **held together** by **gravitation** – attraction exerted on each part of the star by all other parts
- Collapse is **resisted** by internal thermal **pressure**.
- These two forces play the principal role in determining stellar structure – they must be (at least almost) in balance. If they are not, the star will explode or collapse on very short (dynamical) time-scale. Since stars do seem to be rather stable on time-scale of millennium, the balance is good.
- Stars **continually radiating** energy into space. As they do not seem to cool dramatically on the civilization lifetime-scale, an energy source must exist (we will see later that thermal energy is not enough).
- Theory must describe - **origin of energy** and **transport to surface**.

Basic assumptions (2)

We make two fundamental assumptions :

1. Neglect the rate of change of properties – assume constant with time.
2. All stars are spherical and symmetric about their centres. Thus, all quantities (e.g., density, temperature, pressure) depend only on the distance from the centre of the star - radius r.

Density as function of radius is $\rho\,(r)$.

If m is the mass interior to r, then: $m(r) = \int_0^r 4\pi r^2 \rho(r)\, dr$

Differential form of this equation is: $dm = 4\pi r^2 \rho dr$

Two equivalent ways of describing the star:

- Properties as $f(r)$: e.g. temperature $T(r)$
- Properties as $f(m)$: e.g. $T(m)$

Second way often more convenient: over its lifetime, a star's radius will change by many orders of magnitude, while its mass will remain relatively constant. Moreover, the amount of nuclear reactions occuring inside a star depends on ρ and T, not where it is in the star. Thus, a better and more natural way to treat stellar structure is to write radius as a function of mass, i.e.

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

We will start with these assumptions and later reconsider their validity.

Stellar structure

For our stars – which are isolated, static, and spherically symmetric – there are **four** basic equations to describe structure. All physical quantities depend on the distance from the centre of the star alone

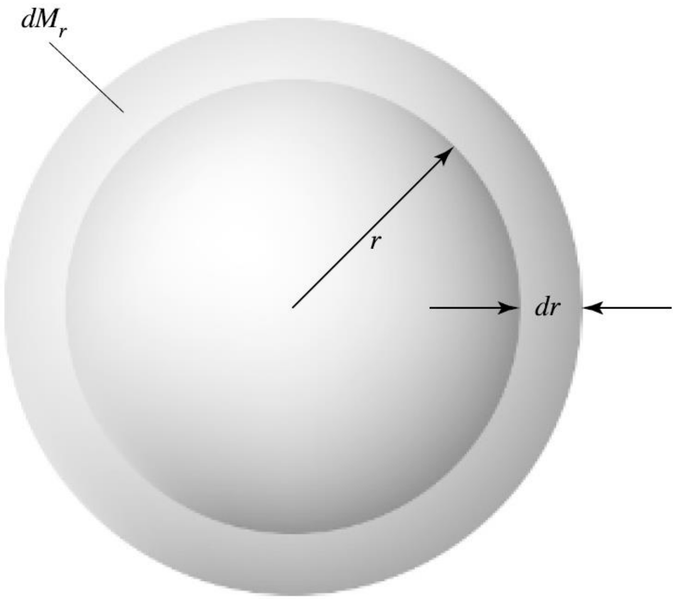
- **Conservation of mass**
- **Equation of hydrostatic equilibrium**: at each radius, forces due to pressure differences balance gravity
- **Conservation of energy**: at each radius, the change in the energy flux equals the local rate of energy release
- **Equation of energy transport**: relation between the energy flux and the local gradient of temperature

These basic equations supplemented with

- **Equation of state** (pressure of a gas as a function of its density and temperature)
- **Opacity** (how opaque the gas is to the radiation field)
- Nuclear energy generation rate as $f(\rho,T)$.

Equation of mass conservation

Mass $m(r)$ contained within a star of radius r is determined by the density of the gas $\rho(r)$.



Consider a thin shell inside the star with radius r and outer radius $r+dr$:

$$dV = 4\pi r^2 dr$$

$$dM = dV\rho(r) = 4\pi r^2 \rho(r) dr$$

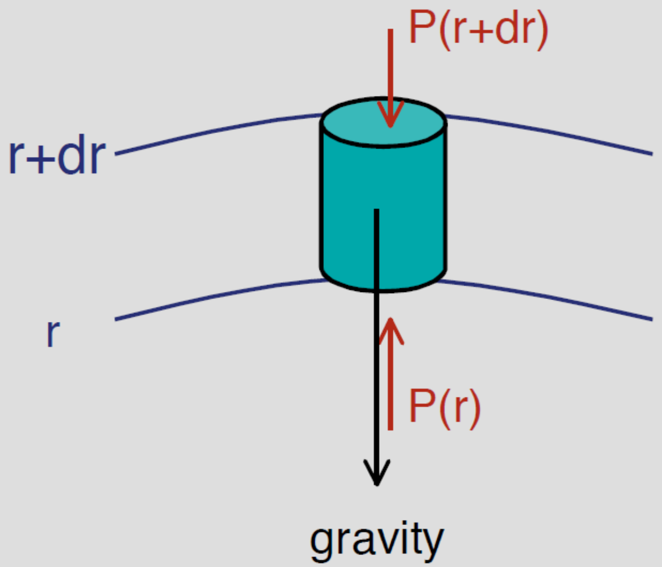
In the limit where $dr \rightarrow 0$:

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

This is the **equation of mass conservation**.

Hydrostatic equilibrium (1)

- Balance between gravity and gradient of internal pressure is known as hydrostatic equilibrium.



- Consider a small cylindrical element between radius r and radius $r + dr$ in the star.
  - Its surface area = ds
  - Mass of the element: $dm = \rho(r)\, ds\, dr$
  - Mass of gas in the star at smaller radii: m = m(r)

Hydrostatic equilibrium (2)

Consider forces acting in radial direction:

- Outward force: pressure exerted by stellar material on the bottom face:

$$F_{P,b} = P(r)ds$$

- Inward forces:
  - Gravity (gravitational attraction of all stellar material lying within r):

$$F_g = \frac{Gm}{r^2} dm = \frac{Gm}{r^2} \rho(r)\, ds\, dr$$

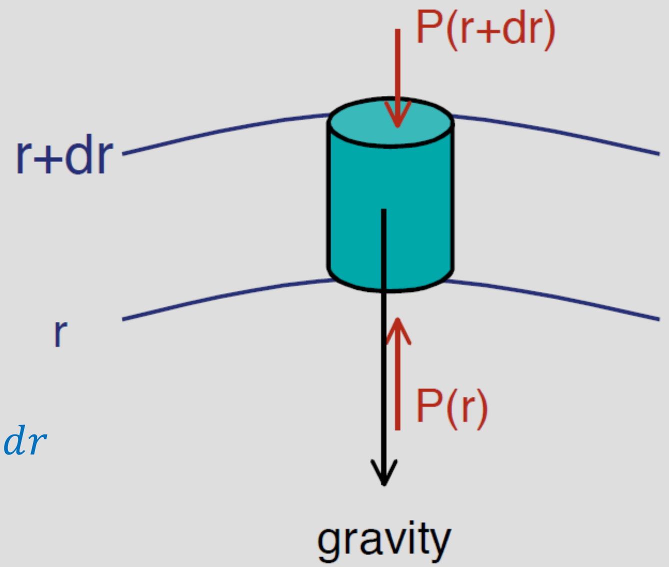
  - Pressure exerted by stellar material on the top face:

$$F_{P,t} = P(r + dr)ds$$

- In hydrostatic equilibrium:

$$F_{P,b} = F_{P,t} + F_g$$

$$P(r)ds = P(r + dr)ds + \frac{Gm}{r^2} \rho(r)\, ds\, dr$$



Hydrostatic equilibrium (3)

$$P(r)ds = P(r+dr)ds + \frac{Gm}{r^2}\rho(r)\,ds\,dr$$

$$\Rightarrow \quad P(r+dr) - P(r) = -\frac{Gm}{r^2}\rho(r)\,dr$$

If we consider an infinitesimal element, we write for $dr \rightarrow 0$

$$\frac{P(r+dr) - P(r)}{dr} = \frac{dP(r)}{dr}$$

Hence rearranging above, we get **the equation of hydrostatic equilibrium:**

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

Hydrostatic equilibrium (4)

The equation of hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

Combining it with the equation of mass conservation, we obtain an **alternate** form of hydrostatic equilibrium equation, in which enclosed mass m is used as the dependent variable:

$$\frac{dP(r)}{dm} = \frac{dP(r)}{dr} \times \frac{dr}{dm} = -\frac{Gm}{r^2}\rho \times \frac{1}{4\pi r^2 \rho}$$

$$\frac{dP(r)}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$$

Hydrostatic equilibrium (5)

Properties of the equation of hydrostatic equilibrium: $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$

1) Pressure always **decreases** outward
2) Pressure gradient vanishes at $r = 0$
3) Condition at surface of star: $P = 0$ (to a good first approximation)

(2) and (3) are **boundary conditions** for the hydrostatic equilibrium equation.

Accuracy of hydrostatic assumption (1)

We have assumed that the gravity and pressure forces are balanced – how valid is that ?

Consider the case where the outward and inward forces are not equal,
there will be a resultant force acting on the element which will give rise to an acceleration *a:*

$$P(r+dr)ds + \frac{Gm}{r^2}\rho(r)\,ds\,dr - P(r)ds = dm \times a = a\,\rho(r)\,ds\,dr$$

[Applying Newton's second law (*F*=*ma*) to the cylinder]

acceleration = 0 everywhere if star **static**

$$\frac{dP(r)}{dr} + \frac{Gm}{r^2}\rho(r) = a\,\rho(r)$$

Now acceleration due to gravity is $g = \frac{Gm}{r^2}$

$$\frac{dP(r)}{dr} + g\rho(r) = a\rho(r)$$

This is a generalized form of the equation of hydrostatic support.

Accuracy of hydrostatic assumption (2)

$$\frac{dP(r)}{dr} + g\rho(r) = a\rho(r)$$

Now suppose there is a resultant force on the element (LHS≠0).

Suppose their sum is small fraction of gravitational term (β): $\beta g\rho(r) = a\rho(r)$

Hence there is an inward acceleration of

$$a = \beta g$$

Assuming it begins at rest, the spatial displacement d after a time t is

$$d = \frac{1}{2}at^2 = \frac{1}{2}\beta g t^2$$

Calculate!

Accuracy of spherical symmetry assumption

Stars are rotating gaseous bodies – to what extent are they flattened at the poles?

If so, departures from spherical symmetry must be accounted for.

Consider mass m near the surface of a star of mass M and radius r.

Element will be acted on by centrifugal force $F_c = m\omega^2 r$, where ω= angular velocity of the star.

There will be **no** departure from spherical symmetry provided that

$$\frac{F_c}{F_g} = m\omega^2 r \Big/ \frac{GMm}{r^2} \ll 1 \quad \text{or} \quad \omega^2 \ll \frac{GM}{r^3}$$

Solar rotation period is about P ≈ 27 days.

Angular velocity $\omega = 2\pi/P \approx 2.7\times10^{-6}\ s^{-1}$ $\qquad F_c/F_g \sim 2\times10^{-5}$

…even rotation rates much faster than that of the Sun are negligibly small to influence star's structure.

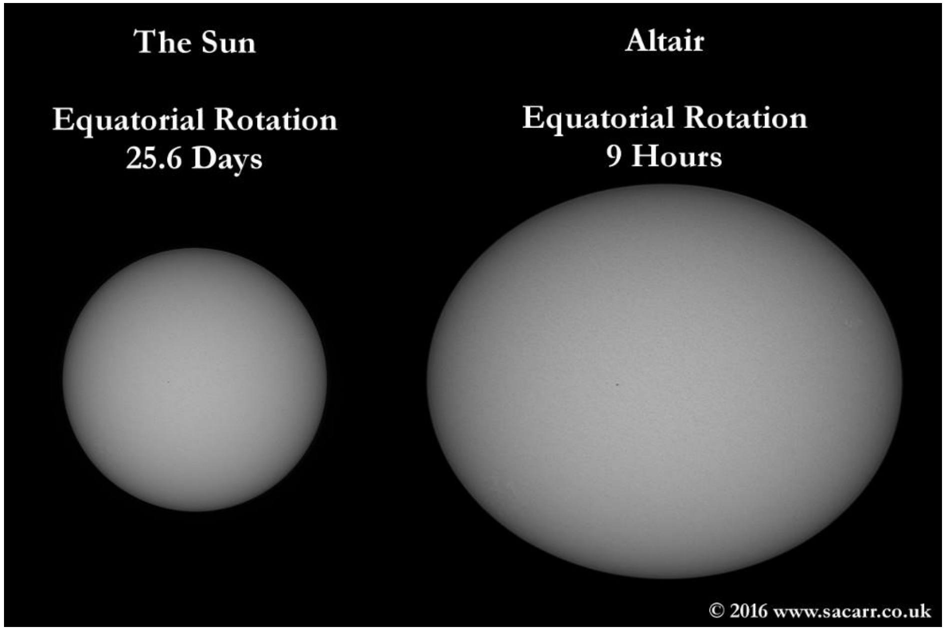
Accuracy of spherical symmetry assumption

$V = (4/3)\,\pi R^3$
$\omega = 2\pi/P$

In terms of mean density, we get

$$\omega^2 \ll \frac{GM}{r^3} \approx \pi G\bar{\rho} \quad \Rightarrow \quad \bar{\rho} \gg \frac{4\pi}{GP^2} \approx \frac{1.9\times10^8}{P^2}$$

For the majority of stars, departures from spherical symmetry can be **ignored**.



However, some stars do rotate rapidly and rotational effects must be included in the structure equations –
can change the output of models.

Accuracy of spherical symmetry assumption

Isolation?

In the Solar neighborhood, distances between stars are enormous: e.g. Sun's nearest stellar companion is Proxima Centauri at $d = 1.3$ pc. Ratio of Solar radius to this distance is:

$$\frac{R_{sun}}{d} \approx 2 \times 10^{-8}$$

Two important implications:

- Can ignore the gravitational field and radiation of other stars when considering stellar structure.
- Stars (almost) never collide with each other.

Once star has formed, initial conditions rather than interactions with other stars determine evolution.

However, stars in double systems are elongated due to gravitational attraction.

Recap

We obtained TWO of four basic equations to describe stellar structure:

1. Equation of hydrostatic equilibrium: at each radius, forces due to pressure differences balance gravity

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

2. Conservation of mass

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

We also found that basic assumptions on spherical symmetry and that the gravity and pressure forces are well balanced are very solid.

The Virial theorem (1)

Let's again take the hydrostatic equilibrium equation, in which enclosed mass m is used as the dependent variable (or combine the equation of hydrostatic equilibrium with the equation of mass conservation):

$$\frac{dP(r)}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dP(r)}{dr} \times \frac{dr}{dm} = -\frac{Gm}{r^2}\rho \times \frac{1}{4\pi r^2 \rho}$$

$$\frac{dmM}{dr} = 4\pi r^2 \rho(r)$$

Now multiply both sides by volume V=(4/3)πr^3:

$$3V(r)dP = -\frac{Gm}{r}dm$$

And integrate over the whole star:

$$3\int_{P_c}^{P_s} V(r)dP = -\int_0^M \frac{Gm}{r}dm$$

integrating by parts

$$3[PV]_c^s - 3\int_{V_c}^{V_s} PdV = -\int_0^M \frac{Gm}{r}dm$$

At centre, V_c=0 and at surface P_s=0

The Virial theorem (2)

Hence, we have

$$3\int_0^V P\,dV = \int_0^M \frac{Gm}{r}\,dm = -E_G$$

Now the right-hand term = – total gravitational binding energy of the star, or it is the energy needed to spread the star to infinity, or to assemble the star by bringing gas from infinity.

$$3\int_0^V P\,dV = -E_G$$

← version of the virial theorem

The left-hand side contains pressure integral. With some assumptions about the pressure, we can progress further.

The Virial theorem (3)

$$3\int_0^V P dV = -E_G$$

For **ideal** gas, $P = NkT$,

where N is concentration, T is the temperature, k is Boltzmann's constant,

while the thermal (kinetic) energy per particle is $e_{kin} = \frac{3}{2}NkT$

Thus, the LHS is

$$3\int_0^V P dV = 2\int_0^V e_{kin} dV = 2E_T$$

where E_T is the thermal energy of the star.

Thus, we can write the Virial Theorem:

or for the total energy $E = E_T + E_G$:

$$2E_T + E_G = 0$$

$$E = -E_T$$

This is of great importance in astrophysics and has many applications.

Timescales of stellar evolution (1)

1. Dynamical time scale

Measure of the time scale on which a star would expand or contract if the balance between pressure gradients and gravity was suddenly disrupted (same as free-fall time scale).

Previously we obtained a generalized form of the equation of hydrostatic support, and then assuming a non-zero inward acceleration to be $a = \beta g$, we obtained the spatial displacement d after a time t: $d = \frac{1}{2}\beta g t^2$

$$\frac{dP(r)}{dr} + g\rho(r) = a\rho(r)$$

$$t = \left(\frac{2d}{\beta g}\right)^{1/2}$$

Now, if we allow the star to collapse, i.e. set $\beta \approx 1$ and $d = R$ and substitute $g = GM/r^2$

$$t_{dyn} = \left(\frac{2R^3}{GM}\right)^{1/2}$$

$t_{\rm dyn}$ is known as the dynamical time.

The dynamical timescale (1)

We can get a better estimation if assume that the whole mass is concentrated in the centre. The equation of motion is

$$\ddot{r} + \frac{GM}{r^2} = 0$$

At home you will show that the time for collapse from radius R to 0 is

$$t_{dyn} = \frac{\pi}{2\sqrt{2}} \left(\frac{R^3}{GM} \right)^{1/2}$$

One can express that through mean density as:

$$t_{dyn} = \left(\frac{3\pi}{32} \right)^{1/2} \frac{1}{\sqrt{G\bar{\rho}}}$$

The dynamical timescale (2)

For different radii, we get

$$t_{dyn} = \frac{\pi}{2\sqrt{2}}\left(\frac{r^3}{Gm}\right)^{1/2} = \left(\frac{3\pi}{32}\right)^{1/2}\frac{1}{\sqrt{G\overline{\rho_r}}}$$

where $m = m(r)$ is the mass interior to r, $\overline{\rho_r}$ is the mean density in sphere of radius r.

We see that if density decreases with radius,

then $\overline{\rho_r}$ also decreases and time-scale grows.

Thus, shell at larger radii falls down longer.

This also confirmed our assumption of the whole mass concentrated inside.

For the Sun $R_\odot = 6.96 \times 10^{10}$ cm, $M_\odot = 1.99 \times 10^{33}$ g: $t_{dyn} = 1770 \text{ sec} \approx 0.5 \text{ hour}$

$\bar{\rho} = 1.4 \text{ g cm}^{-3}$

Timescales of stellar evolution (2)

2. Thermal time scale (Kelvin-Helmholtz time scale)

Suppose nuclear reaction were suddenly cut off in the Sun.

Thermal time scale is the time required for the Sun to radiate all its reservoir of thermal energy:

$$t_K \equiv \frac{E_T}{L} = \frac{GM^2}{2RL} \approx 1.5 \times 10^7 \text{ yr (for the Sun)}$$

Virial theorem: the thermal energy E_T is roughly equal to half the gravitational potential energy

Important timescale: determines how quickly a star contracts before nuclear fusion starts - i.e. sets roughly the pre-main sequence lifetime.

Timescales of stellar evolution (3)

3. Nuclear time scale

Time scale on which the star will exhaust its supply of nuclear fuel if it keeps burning it at the current rate.

Energy release from fusing one gram of hydrogen to helium is ~6 x 10^{18} erg, so:

$$t_n = \frac{qXM \times 6 \times 10^{18}}{L} \text{erg g}^{-1}$$

where

- X is the mass fraction of hydrogen initially present (X=0.7)
- q is the fraction of fuel available to burn in the core (q=0.1)

$$t_n \approx 7 \times 10^9 \text{ yr (for the Sun)}$$

Reasonable estimate of the main-sequence lifetime of the Sun.

Stellar timescales

Ordering time scales:

$$t_{dyn} \ll t_K \ll t_n$$

For the Sun: $t_{\rm dyn}$=30 min
$t_{\rm K}$ =15 million years
$t_{\rm n}$ = 7 billion years

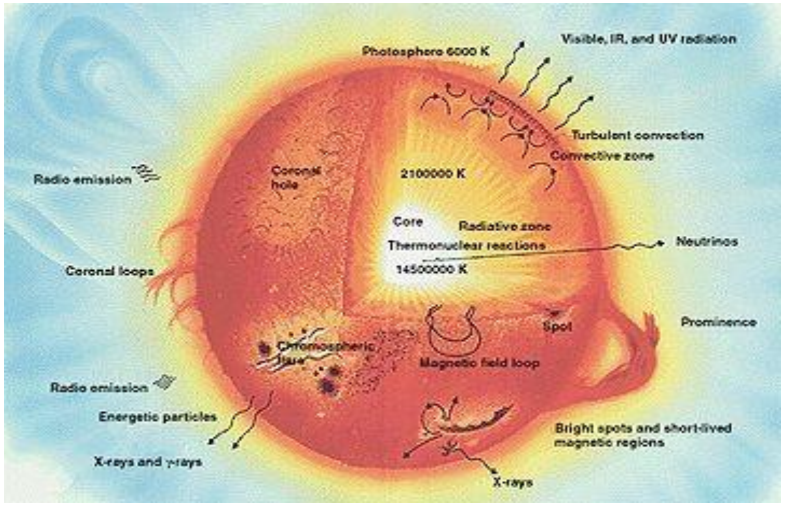
Most stars, most of the time, are in hydrostatic and thermal equilibrium, with slow changes in structure and composition occurring on the (long) time scale $t_{\rm n}$ as fusion occurs.

Do observe evolution on the shorter time scales also:

- Dynamical - stellar collapse / supernova
- Thermal / Kelvin-Helmholtz - pre-main-sequence

Conditions in stellar interiors

MINIMUM VALUE FOR CENTRAL PRESSURE OF A STAR
MINIMUM MEAN TEMPERATURE OF A STAR
STATE OF STELLAR MATERIAL



Content

Let us consider several applications of our current knowledge. We will derive mathematical formulae for the following

1. Minimum value for central pressure of a star
2. Minimum mean temperature of a star
3. State of stellar material

In doing this you will learn important assumptions and approximations that allow the values for minimum central pressure, mean temperature and the physical state of stellar material to be derived.

Minimum value for central pressure of star (1)

We have only 2 of the 4 equations, and no knowledge yet of material composition or physical state. But we can deduce a minimum central pressure.

Given what we know, **what is this likely to depend upon?**

Let's again take the hydrostatic equilibrium equation, in which enclosed mass m is used as the dependent variable (or combine the equation of hydrostatic equilibrium with the equation of mass conservation):

$$\frac{dP(r)}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dP(r)}{dr} \times \frac{dr}{dm} = -\frac{Gm}{r^2}\rho \times \frac{1}{4\pi r^2 \rho}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

Can integrate this over the whole star to give:

$$P_c - P_s = P_c = \int_0^M \frac{Gm(r)}{4\pi r^4} dm(r)$$

The integration requires functional forms of *m(r)*. Unfortunately, such explicit expression is not available.

However, replacing r by the stellar radius $R \geq r$, we obtain a lower limit for the central pressure:

$$P_c \geq \int_0^M \frac{Gm}{4\pi R^4} dm = \frac{GM^2}{8\pi R^4}$$

$$P_c \geq \frac{GM^2}{8\pi R^4}$$

Minimum value for central pressure of star (2)

We can improve the lower limit making a natural assumption that density does not increase towards the surface.

Define the average density within sphere of radius r as $\overline{\rho_r} \equiv m / \left(\frac{4\pi}{3} r^3\right)$

$$P_c = \int_0^M \frac{Gm(r)}{4\pi r^4} dm(r) = \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} G \int_0^M \overline{\rho_r}^{\,4/3} m^{-1/3}\, dm$$

For density **not increasing outwards** $\overline{\rho_r} \geq \bar{\rho} \equiv \overline{\rho_R}$, we get

$$P_c \geq \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} \bar{\rho}^{\,4/3} G \int_0^M m^{-1/3}\, dm = \left(\frac{\pi}{6}\right)^{1/3} G \bar{\rho}^{\,4/3} M^{2/3}$$

$$P_c \geq \frac{3}{8\pi} \frac{GM^2}{R^4}$$

Maximum value for central pressure of star

An upper limit on central pressure can be obtained just assuming $\rho_c \geq \overline{\rho_r}$ which is e.g., valid when density is largest in the centre:

$$P_c = \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} G \int_0^M \overline{\rho_r}^{\,4/3} m^{-1/3}\, dm \leq \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} \rho_c^{4/3} G \int_0^M m^{-1/3}\, dm$$

We can write it as

$$P_c \leq \frac{3}{8\pi}\frac{GM^2}{R_c^4}$$

where R_c is the radius of the star with mass M and density ρ_c defined by $\frac{4\pi}{3} R_c^3 \rho_c = M$

Thus, we get

$$\left(\frac{\pi}{6}\right)^{1/3} GM^{2/3}\bar{\rho}^{\,4/3} \leq P_c \leq \left(\frac{\pi}{6}\right)^{1/3} GM^{2/3}\rho_c^{4/3}$$

which is valid for $\rho_c \geq \overline{\rho_r} \geq \bar{\rho}$

Central pressure in stars

$$P_c \geq \frac{3}{8\pi}\frac{GM^2}{R^4}$$

$$P_c \geq \frac{3}{8\pi}\frac{GM^2}{R^4} = 1.35\times10^{15}\frac{(M/M_\odot)^2}{(R/R_\odot)^4}\frac{\text{dyn}}{\text{cm}^2} = 1.33\times10^{9}\frac{(M/M_\odot)^2}{(R/R_\odot)^4}\,\text{atm}$$

1 dyn=1 g·cm/s^2 1 dyn/cm^2 = 0.1 Pa = 9.8692×10^{-7} atm

For stars at main sequence (MS): $R \propto M^{\beta}$, where $\beta = (0.5 \div 1)$

- for low-mass stars: $\beta \approx 1$
- for masses above solar: $\beta \approx 2/3$
- for very high masses: $\beta \approx 0.5$

$$P_c \geq 1.3\times10^{9}(M/M_\odot)^{2-4\beta}\ \text{atm}$$

P_c strongly depends on the stellar structure and in reality, always much larger than simple estimates above. The central pressure in MS stars is about $10^{10} \div 10^{11}$ atm.

This seems rather large for gaseous material – we shall see that this is not an ordinary gas. This is huge pressure by Earth standards where experiments have reached only about 10^6 atm. But even those pressures are small compared to that inside white dwarfs (10^{19} atm), or neutron stars ($10^{29} \div 10^{30}$ atm).

Calculate P_c for the Sun

Minimum mean temperature of a star (1)

We have seen that pressure, P, is an important term in the equation of hydrostatic equilibrium and the virial theorem. We have derived a minimum value for the central pressure ($P_c > 10^9$ atmospheres).

What physical processes give rise to this pressure – which are the most important?

- Gas pressure P_g
- Radiation pressure P_{rad}
- We shall show that P_{rad} is negligible in majority of stellar interiors and pressure is dominated by P_g

To do this we first need to estimate the minimum mean temperature of a star.

Consider the gravitational binding energy:

$$-E_G = \int_0^M \frac{Gm}{r} dm = e_G \frac{GM^2}{R} \quad \text{where } e_G = \int_0^1 \frac{q}{x} dq$$

The dimensionless gravitational energy

and $q = m/M$, $x = r/R$

Minimum mean temperature of a star (2)

We can obtain a lower bound on e_G by noting: at all points inside the star x<1 and hence

The dimensionless gravitational energy $$e_G = \int_0^1 \frac{q}{x} dq \geq \int_0^1 q\, dq = 1/2$$

For a constant density sphere one can get: $q = m/M = (r/R)^3 = x^3 \Rightarrow e_G = 3/5$

And for density decreasing outwards $e_G \geq 3/5$ because one needs to move some mass towards centre which releases gravitational energy. For the Sun $e_G = 1.62$. Now, $dm = \rho dV$ and the *virial theorem* can be written

$$-E_G = 3\int_0^V P dV = 3\int_0^M \frac{P}{\rho} dm$$

Pressure is sum of radiation pressure and gas pressure: $P = P_g + P_r$

Assume, for now, that stars are composed of ideal gas with negligible P_r

Then, the equation of state of ideal gas:

$$P = NkT = \frac{\rho}{\mu m_p} kT$$

where N is concentration (number of particles per cm^3), T is the temperature, k is Boltzmann's constant, μ = mean molecular weight, i.e. the average mass of particles in unit of proton mass m_p.

Minimum mean temperature of a star (3)

Hence, we have

$$-E_G = 3\int_0^M \frac{P}{\rho}dm = 3\int_0^M \frac{kT}{\mu m_p}dm = e_G\frac{GM^2}{R}$$

Assuming chemically homogeneous star, μ=const,

and defining the average over mass temperature

$$\bar{T} \equiv \frac{1}{M}\int_0^M T dm$$

we get the mean temperature of the star

$$\bar{T} = \frac{e_G}{3}\frac{\mu m_p}{k}\frac{GM}{R}$$

This temperature is called virial temperature for obvious reason.

For density not increasing outwards, $e_G \geq 3/5$, and therefore

$$\bar{T} \geq \frac{1}{5}\frac{\mu m_p}{k}\frac{GM}{R}$$

Minimum mean temperature: Example

- As an an example, for a chemically homogeneous star, we have

$$\bar{T} = 7.7 \times 10^6 e_G \mu \frac{M/M_\odot}{R/R_\odot}\ \mathrm{K} \qquad \overline{kT} = 660 e_G \mu \frac{M/M_\odot}{R/R_\odot}\ \mathrm{eV}$$

- We know that H is the most abundant element in stars and for a fully ionized hydrogen star $\mu=1/2$ (as there are two particles, p + e^-, for each H atom). And for any other element μ is greater => $T_\odot > 2.3\times10^6$ K.
- Also, the average kinetic energy of particles at $\overline{T_\odot}$ is much higher than the ionization potential of H (13.6 eV) or for double ionization of He ($13.6\text{x}2^2=54$ eV). Thus, the gas must be highly ionized, i.e. is a plasma. As e_G is actually larger than 1, temperatures in the stellar interiors are $(1\div3)\times10^7$ K, which corresponds to the energies about 1-3 keV.

- Mean density of the Sun is higher than water and other ordinary liquids.
However, at such a temperature the gas is ionized. An ideal gas demands that the distances between the particles are much greater than their sizes, and nuclear dimension is 10^{-13} cm compared to atomic dimension of 10^{-8} cm (which would be of interest in neutral gas). It can thus withstand greater compression without deviating from an ideal gas.

Temperature in stellar interiors

- The dimensionless gravitational energy e_G is not very sensitive to the stellar structure and for MS stars it is about unity. The chemical composition is about the same for all MS stars and $\bar{T} \propto M/R$
- Because $R \propto M^{\beta}$, where $\beta = (0.5 \div 1)$
  the mean temperatures should decrease for smaller masses and change only by a factor of a few between the lower and upper end of stellar masses.
  However, variation in luminosities are much larger than variation in mass: $10^{10} \div 10^{11}$ and 10^3 times, respectively. As temperature varies only very slightly, we have to conclude that the energy production rate is a very strong function of temperature. This conclusion is supported by studies of nuclear reactions.

- The masses of red giants (RG) are about solar, but luminosities are large, therefore their temperatures should be at least as high as for the Sun. But their radii are 100 x solar R. We are forces to conclude that $e_G > 100$, which can be achieved by strong concentration of matter towards the centre. Thus, the RG structure should be very much different from MS stars. A large fraction of RG mass should be contained in its core (a white dwarf "under construction").

Physical state of stellar material

Let us revisit the issue of radiation vs gas pressure. We assumed that the radiation pressure was negligible. The pressure exerted by photons on the particles in a gas is:

$$P_{rad} = \frac{aT^4}{3}$$

where $a = \frac{4\sigma_{SB}}{c} = 7.56 \times 10^{-15}$ erg cm^{-3} K^{-4} = radiation density constant
Now compare gas and radiation pressure at a typical point in the Sun:

$$\frac{P_{rad}}{P_g} = \frac{aT^4}{3} \Big/ \frac{kT\rho}{\mu m_p} = \frac{\mu m_p a T^3}{3k\rho}$$

Taking $T \sim \bar{T} = 2 \times 10^6$ K, $\rho \sim \bar{\rho} = 1.4$ g cm^{-3}, and $\mu m_P = \frac{1}{2} 1.67 \times 10^{-24}$ g gives

$$\frac{P_{rad}}{P_g} = 10^{-4}$$

Hence radiation pressure appears to be negligible at a typical (average) point in the Sun. In summary, with no knowledge of how energy is generated in stars we have been able to derive a value for the Sun's internal temperature and deduce that it is composed of a near ideal gas plasma with negligible radiation pressure.

Mass dependency of radiation to gas pressure

However, we shall later see that P_{rad} becomes significant in higher mass stars. To give a basic idea of this dependency: replace ρ in the ratio equation above:

$$\frac{P_{rad}}{P_g} = \frac{\mu m_p a T^3}{3k\rho} = \frac{\mu m_p a T^3}{3k\left(\frac{3M}{4\pi R^3}\right)} = \frac{4\pi \mu m_p a}{9k} \frac{R^3 T^3}{M}$$

And from the virial theorem: $\bar{T} \propto \frac{M}{R}$

$$\Rightarrow \qquad \frac{P_{rad}}{P_g} \propto M^2$$

i.e. P_{rad} becomes more significant in higher mass stars.

This is one of the reasons why there are no stars of very high masses >100$M_\odot$

Summary

For our stars – which are isolated, static, and spherically symmetric –there are four basic equations to describe structure. All physical quantities depend on the distance from the centre of the star alone:

1. Equation of hydrostatic equilibrium: at each radius, forces due to pressure differences balance gravity
2. Conservation of mass
3. Conservation of energy : at each radius, the change in the energy flux = local rate of energy release
4. Equation of energy transport : relation between the energy flux and the local gradient of temperature

These basic equations supplemented with

- Equation of state (pressure of a gas as a function of its density and temperature)
- Opacity (how opaque the gas is to the radiation field)
- Core nuclear energy generation rate

With only two of the four equations of stellar structure, we have derived important relations for P_c and mean T.

We have used the *Virial theorem* – this is an important formula and concept in this course and astrophysics in general. You should be comfortable with the derivation and application of this theorem.

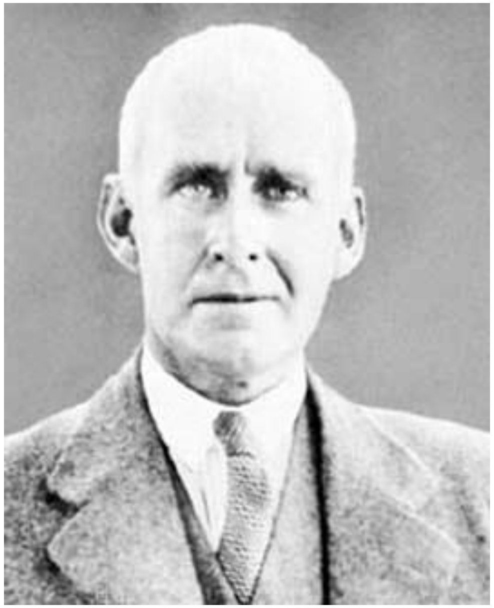
We were able to make interesting conclusions about the energy dissipation rate dependence on temperature, structure of red giants, and the role of radiation pressure.

The equations of stellar structure. Energy generation and transport.

THE LIKELY FORM OF ENERGY GENERATION

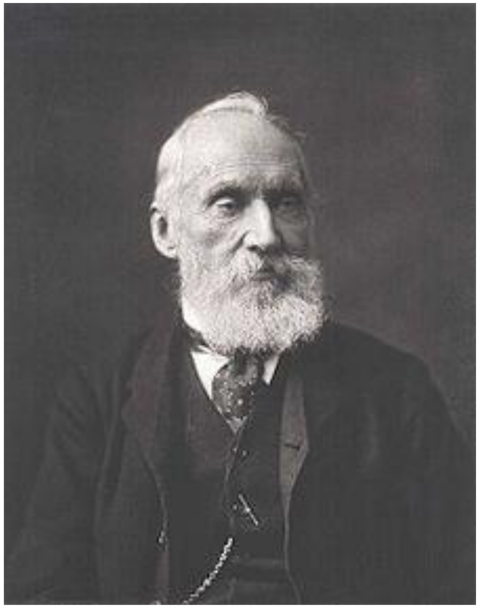
THE EQUATION OF CONSERVATION OF ENERGY

HOW ENERGY IS TRANSPORTED IN THE SUN: THE CRITERION FOR CONVECTION



Sir Arthur Stanley Eddington
(1882-1944)

Karl Schwarzschild
(1873-1916)



Lord Kelvin (William Thomson)
(1824-1907)

Energy generation in stars

So far, we have only considered the dynamical properties of the star, and the state of the stellar material. We need to consider **the source of the stellar energy**.

Let's consider the origin of the energy, i.e. the conversion of energy from some form in which it is not immediately available into some form that it can radiate.

How much energy does the Sun need to generate in order to shine with it's measured luminosity ?

$L_{\odot} = 4\times10^{26}$ W $= 4\times10^{33}$ erg s^{-1}

Sun has not changed flux in 10^9 yr (3×10^{16} s) $\Rightarrow$ Sun has radiated at least 1.2×10^{50} erg

The rest mass energy: $E = mc^2$ $\Rightarrow$ $m_{lost} = 10^{29}$ g $\approx 10^{-4} M_{\odot}$

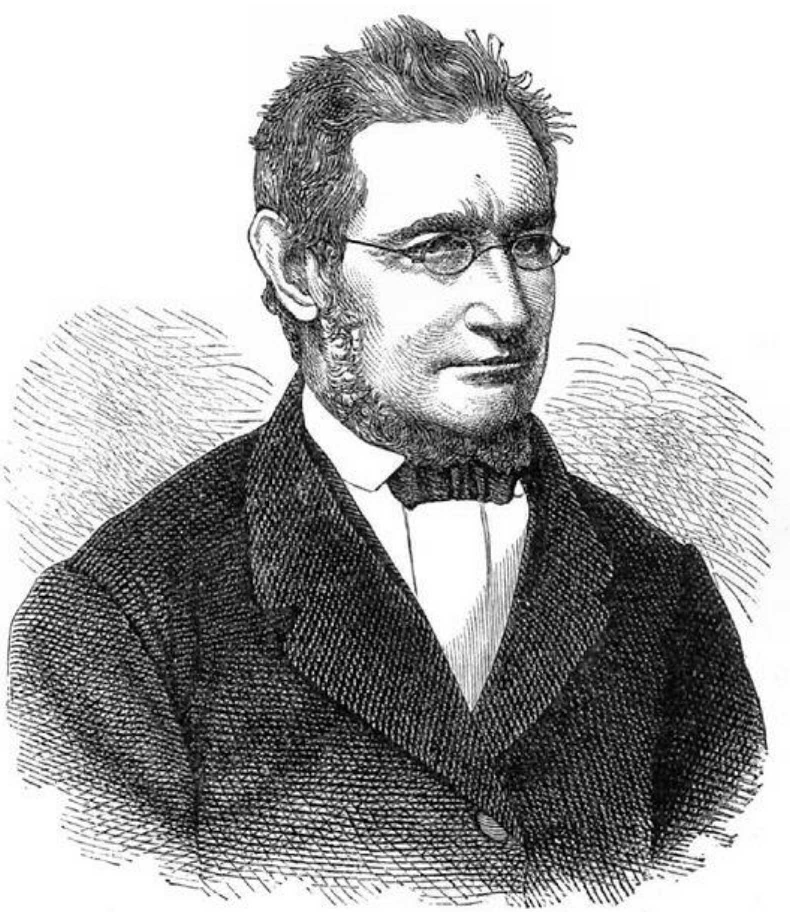
What is the source of this energy ?

- Accretion (Mayer)
- Cooling
- Contraction (gravitational; Helmholtz, Lord Kelvin)
- Chemical reactions
- Nuclear reactions

Source of energy generation

Accretion

- From the very beginning it was obvious that one of the energy source of the Sun could be gravitation. Mayer, one of the fathers of the energy conservation law, suggested that kinetic energy of meteorites would keep the Sun hot.
- This idea was not considered seriously. Now, however, we know that the most compact objects in the Galaxy, neutron stars in X-ray binaries, produce their large luminosity in a similar way, by accreting matter from a companion star.
- But for the Sun it cannot be important.



Julius Robert von Mayer
(1814-1878)

Source of energy generation

Cooling and contraction

These are closely related, so we consider them together. Cooling is simplest idea of all. Suppose the radiative energy of the Sun is due to the Sun being much hotter when it was formed and has since been cooling down. We can test how plausible this is.

Or is the Sun slowly contracting with consequent release of gravitational potential energy, which is converted to radiation?

Recently, assuming that stellar material is ideal monatomic gas (negligible P_{rad}), we obtained this form of the Virial theorem: $2E_T + E_G = 0$

The negative gravitational energy of a star is equal to twice its thermal energy. This means that the time for which the present thermal energy of the Sun can supply its radiation and the time for which the past release of gravitational potential energy could have supplied its present rate of radiation differ by only a factor two.

Total release of gravitational potential energy would have been sufficient to provide radiant energy at a rate given by the luminosity of the star L, for a time equal approximately to the thermal timescale (the time required for a star to radiate all its reservoir of thermal energy):

$$t_K \equiv \frac{E_T}{L} = \frac{GM^2}{2RL}$$

Cooling and contraction (1)

$t_K = \frac{GM^2}{2RL} \approx 1.5 \times 10^7$ yr (for the Sun).

This limit of the age of the Sun obtained in 1862 by Lord Kelvin (William Thomson) was used as an argument against Charles Darwin's (1859) theory of evolution. Darwin's theory required geological time to be much larger, so as to account for the slow evolution of species of plants and animals by natural selection.

In 1899 geologists Thomas C. Chamberlin noted:

"Is present knowledge relative to the behaviour of matter under extraordinary conditions as obtain in the interior of the Sun sufficiently exhaustive to warrant assertion that no unrecognized sources of heat reside there? What is the internal constituents of atoms may be is yet an open question. Is it not improbable that they are complex organizations and the seats of enormous energy?"

Later, Arthur Eddington in *Observatory* (1920) predicts that stellar energy is subatomic: "Only the inertia of tradition keeps the contraction hypothesis alive - or rather, not alive, but an unburied corpse… A star is drawing on some vast reservoir of energy by means unknown to us. This reservoir can scarcely be other than the subatomic energy which, it is known, exists abundantly in all matter… There is sufficient in the Sun to maintain its output for 15 billion years… If, indeed, the subatomic energy in the star is being freely used to maintain their great furnaces, it seems to bring a little nearer to fulfilment our dream of controlling this latent power for the well-being of the human race - or for its suicide."

Cooling and contraction (2)

Now the Earth age 4.6×10^9 years is determined using nuclear physics (from halftime decay of some radioactive elements). However, already in XVIII century Edmond Halley proposed a method of measuring the age from the speed the salt increases in the ocean. In XIX century, geological method were used that measure the time for sedimentation of various materials.

Let us now consider gravitational contraction in other stars.

$$\frac{R}{R_\odot} \propto \left(\frac{M}{M_\odot}\right)^\beta, \qquad \frac{L}{L_\odot} \propto \left(\frac{M}{M_\odot}\right)^l$$

$$t_{th} \sim \frac{GM^2}{RL}$$

$\beta = 1, \quad l = 4$ for $M < M_\odot$, $E_G \propto M \quad \Rightarrow t_{th} \propto M^{-3}$

$\beta = 3/4, l = 3\,1/4$ for $M > M_\odot$, $E_G \propto M^{5/4} \quad \Rightarrow t_{th} \propto M^{-2}$

Nuclear energy $E_N \propto M$

$E_N / E_G \approx 100$, and varies only by a factor 2 - 3

Cooling and contraction (3)

If the Sun was powered by either contraction or cooling, it would have changed substantially in the last 10 million years. A factor of ~100 too short to account for the constraints on age of the Sun imposed by fossil and geological records.

The Virial theorem: $2E_T + E_G = 0$

The total energy is $E = E_T + E_G = -E_T = E_G / 2 < 0$

If no energy source exist in a star, then its luminosity comes from the decreasing total energy:

$$L = -\dot{E} \quad \Rightarrow \quad L = \dot{E}_T \qquad \qquad as\ L > 0 \quad \Rightarrow \quad \dot{E}_T > 0 \quad \Rightarrow \quad \dot{E}_T \uparrow$$

Thus, a star is heating up when it loses its energy!

Another important consequence: $L = -\dot{E}_G/2$

a star can radiate only 1/2 of its gravitational energy released during contraction.

Source of energy generation

Chemical Reactions

Can quickly rule these out as possible energy sources for the Sun. We calculated above that we need to find a process that can produce at least 10^{-4} of the rest mass energy of the Sun. Chemical reactions such as the combustion of fossil fuels release $\sim 5\times10^{-10}$ of the rest mass energy of the fuel.

Nuclear Reactions

Hence the only known way of producing sufficiently large amounts of energy is through nuclear reactions. There are two types of nuclear reactions, fission and fusion.

Fission reactions, such as those that occur in nuclear reactors, or atomic weapons can release $\sim 5\times10^{-4}$ of the rest mass energy through fission of heavy nuclei (uranium or plutonium).

Class task: Can you show that the fusion reactions can release enough energy to feasibly power a star? Assume atomic weight of H=1.007825 and He^4=4.00260, atomic mass unit =$1.66054\text{x}10^{-24}$ g

$L_{\odot} = 4 \times 10^{33}$ erg s^{-1}

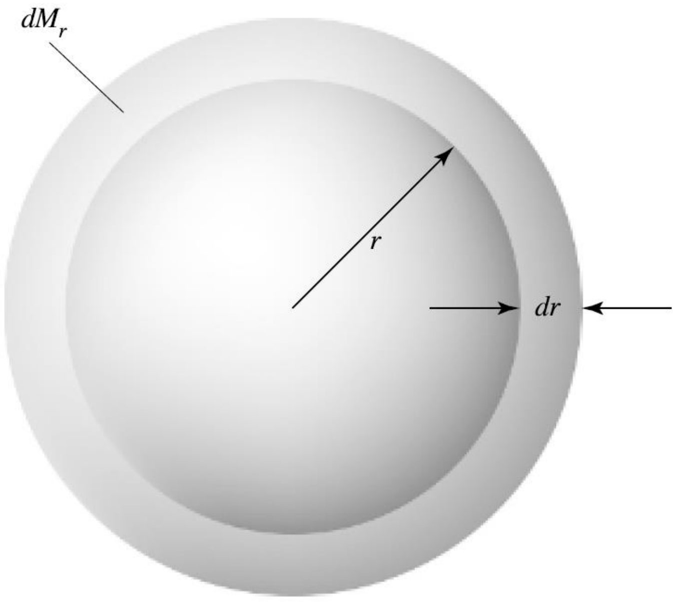
Yes, energy release from fusing one gram of hydrogen to helium is $\sim 6.4 \text{ x } 10^{18}$ erg

Hence, we can see that both fusion and fission could in principle power the Sun. Which is the more likely? As light elements are much more abundant in the solar system than heavy ones, we would expect nuclear fusion to be the dominant source. Given the limits on *P* and *T* that we have obtained - are the central conditions suitable for fusion ? We will return to this later.

Equation of energy production (1)

The third equation of stellar structure: relation between energy release and the rate of energy transport.

Consider a spherically symmetric star in which energy transport is radial and in which time variations are unimportant.



$L(r)$=rate of energy flow across sphere of radius r.

$L(r+dr)$=rate of energy flow across sphere of radius $r+dr$.

Because shell is thin:

$$dV(r) = 4\pi r^2 dr$$

and

$$dm = dV\rho(r) = 4\pi r^2 \rho(r) dr$$

Equation of energy production (2)

We define ε= energy release per unit mass per unit time (erg s^{-1} g^{-1})
Hence energy release rate in shell is written: $4\pi r^2 \rho(r)\varepsilon dr$
Conservation of energy leads us to

$$L(r+dr) = L(r) + 4\pi r^2 \rho(r)\varepsilon dr$$

$\Rightarrow$

$$\frac{L(r+dr) - L(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon$$

And for $dr \rightarrow 0$:

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon$$

This is the **equation of energy production**.

We now have three of the equations of stellar structure. However, we have five unknowns: $P(r), m(r), L(r), \rho(r)$, $\varepsilon(r)$.

In order to make further progress we need to consider energy transport in stars.

Method of energy transport

There are three ways energy can be transported in stars:

- **Conduction** – by exchange of energy during collisions of gas particles (usually e^-)
- **Radiation** – energy transport by the emission and absorption of photons
- **Convection** – energy transport by mass motions of the gas

Conduction and radiation are similar processes – they both involve transfer of energy by direct interaction, either between particles or between photons and particles.

Which is the more dominant in stars ?

Energy carried by a typical particle $\sim 3kT/2$ is comparable to energy carried by typical photon $\sim h\nu$

But number density of particles is much greater than that of photons. This would imply conduction is more important than radiation, but…

Mean free path of photon $l_{ph} \sim 1/(n\sigma_T) \sim 1$ cm

Mean free path of particle $l_p \sim 1/(n\sigma_o) \sim 10^{-8}$ cm

n is particle concentration in cm^{-3}, $\sigma_T \sim (8\pi/3) r_e^2 \sim 7 \times 10^{-25}$ cm^2, $\sigma_o \sim \pi a_0^2 \sim 10^{-16}$ cm^2.

$a_0 = 5 \times 10^{-9}$ cm is Bohr radius, $r_e = 3 \times 10^{-13}$ cm is the classical electron radius.

Photons can move across temperature gradients more easily, hence larger transport of energy.

Conduction is negligible, radiation transport in dominant.

The equation of radiative transport

We will derive an expression relating the change in temperature with radius in a star assuming all energy is transported by radiation. Hence, for now, we ignore the effects of convection and conduction which we will discuss later.

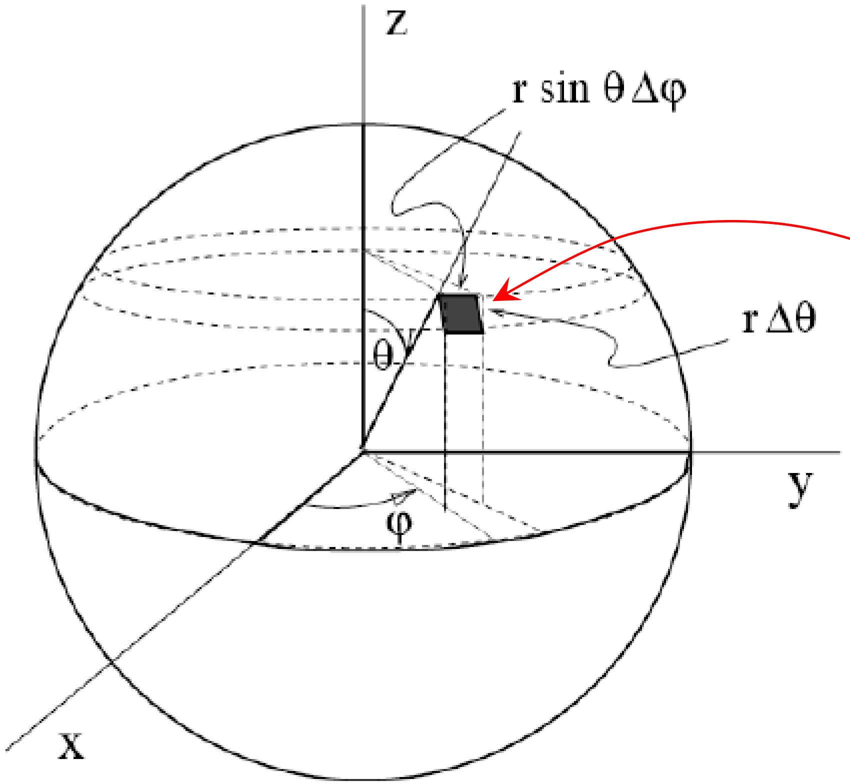
The radiative transfer equation describes how the physical properties of the material are coupled to the radiation spectrum.

First of all, we must have a carefully defined terminology to properly describe light and its interaction with the material in stellar interiors and atmospheres.

Basics about radiative transfer

RADIATION TERMS
SPECIFIC INTENSITY
CHANGE OF INTENSITY ALONG PATH ELEMENT
ABSORPTION AND EMISSION COEFFICIENTS
OPTICAL DEPTH, SOURCE FUNCTION
RADIATIVE TRANSFER EQUATION

Definition of solid angle and steradian



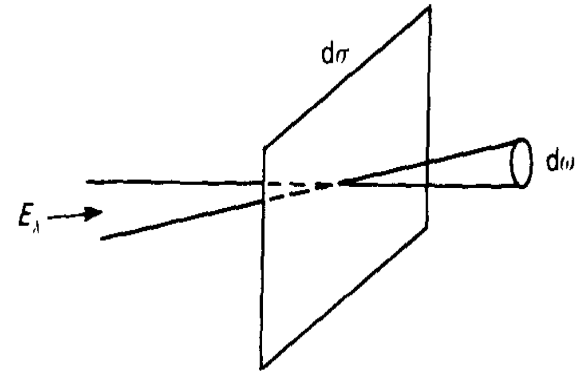
Put the polar axis along the star's radius.
Area of a patch dS on a sphere limited by $(\theta, \theta+\Delta\theta)$ and $(\varphi, \varphi+\Delta\varphi)$ is
$dS = r \sin\theta \;\; r\,\Delta\theta\,\Delta\varphi \equiv r^2\, d\omega$

dω is the solid angle subtended by the area dS at the center of the sphere.
For a unit radius sphere:
$d\omega = \sin\theta\, d\theta\, d\varphi$

Unit of solid angle is the steradian. 4π steradians cover whole sphere.

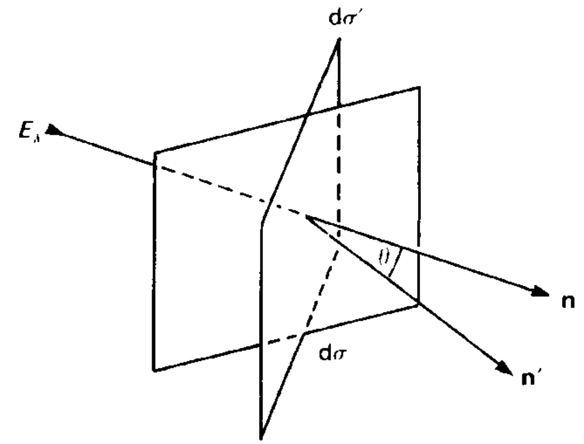
Specific Intensity (1)

Consider light passing through a *perpendicular* surface area dσ in a narrow cone of opening solid angle dω. The amount of energy E_λ passing through this area per second is given by



$$E_\lambda = I_\lambda \, d\lambda \, d\sigma \, d\omega \, dt$$

Now consider the energy passing through a surface area dσ at an angle θ with respect to the normal of this surface area, the effective beam width is reduced by cos(θ):



$$E_\lambda = I_\lambda \cos\theta \, d\lambda \, d\sigma \, d\omega \, dt$$

Specific intensity of the radiation

Specific Intensity (2)

Another, more intuitive name for the specific intensity is brightness.

$$I_\lambda = \frac{E_\lambda}{\cos\theta \, d\lambda \, d\sigma \, d\omega \, dt}$$

The (specific) intensity $\boldsymbol{I_\lambda}$ is then a measure of brightness with units of erg s^{-1} cm^{-2} steradian^{-1} Å^{-1}

(or erg s^{-1} cm^{-2} steradian^{-1} Hz^{-1}).

Bolometric Intensity

- One can alternatively define intensity in frequency units such that

$$I_\lambda d\lambda = I_\nu d\nu$$

- Note that $I_\lambda \neq I_\nu$! The two spectral distributions have different shapes for the same spectrum. The Solar spectrum has a maximum in the green in I_λ (5175Å), but for I_ν the maximum is in the far-red (8800Å).

Why?

$$c=\lambda\nu,\ d\nu/d\lambda=-c/\lambda^2$$

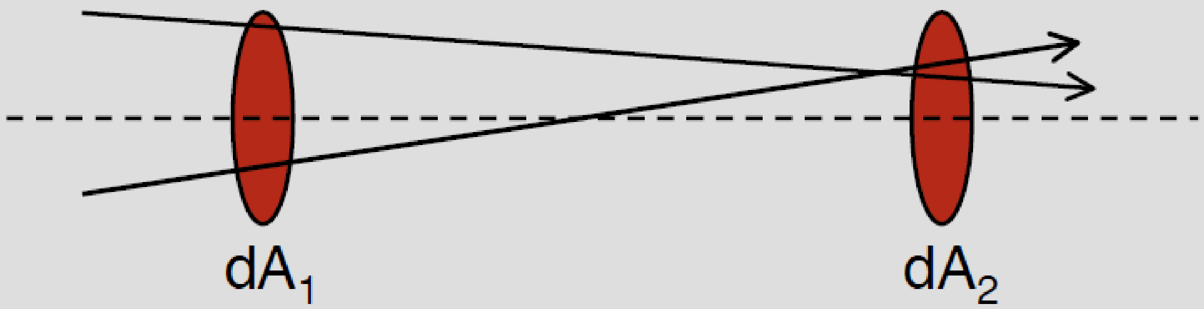
so equal intervals of λ correspond to different intervals of ν across the spectrum.

- Integrated (bolometric) intensity is

$$I = \int_0^\infty I_\lambda \, d\lambda = \int_0^\infty I_\nu \, d\nu$$

How does specific intensity change along a ray?

If there is no emission or absorption, specific intensity is just constant along the path of a light ray. Consider any two points along a ray, and construct areas dA_1 and dA_2 normal to the ray at those points. How much energy is carried by those rays that pass through both dA_1 and dA_2?



$$\left.\begin{aligned} E_1 &= I_{\lambda,1} d\lambda_1 dA_1\, d\omega_1\, dt \\ E_2 &= I_{\lambda,2} d\lambda_2 dA_2\, d\omega_2\, dt \end{aligned}\right\}$$

where $d\omega_1$ is the solid angle subtended by dA_2 at dA_1 etc

How does specific intensity change along a ray?

The same photons pass through both dA_1 and dA_2, without change in their frequency. Conservation of energy gives:

$E_1 = E_2$ - equal energy

$d\lambda_1 = d\lambda_2$ - same wavelength interval

Using definition of solid angle, if dA_1 is separated from dA_2 by distance r:

$$d\omega_1 = \frac{dA_2}{r^2}, \qquad d\omega_2 = \frac{dA_1}{r^2}$$

Substitute:

$$\left.\begin{array}{l} I_{\lambda,1} d\lambda_1 dA_1 dt\, d\omega_1 = I_{\lambda,2} d\lambda_2 dA_2 dt\, d\omega_2 \\ I_{\lambda,1} d\lambda_1 dA_1 dt \dfrac{dA_2}{r^2} = I_{\lambda,2} d\lambda_2 dA_2 dt \dfrac{dA_1}{r^2} \end{array}\right\} \begin{array}{c} E_1 = E_2 \\ d\lambda_1 = d\lambda_2 \end{array}$$

$$\boldsymbol{I_{\lambda,1} = I_{\lambda,2}}$$

Specific Intensity (3)

- Thus, specific intensity I_λ is ***independent of distance*** from the source, it remains the same as radiation propagates through **free** space.
- Justifies use of alternative term '**brightness**' - e.g. brightness of the disk of a star remains same no matter the distance - flux goes down but this is compensated by the light coming from a smaller area.
- If we measure the distance along a ray by variable s, then we can express result equivalently in differential form:

$$\frac{dI_\lambda}{ds} = 0$$

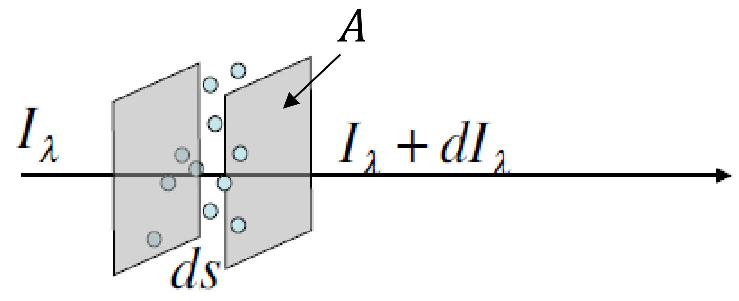
- Specific intensity can only be measured **directly** if we **resolve** the radiating surface (e.g. Sun, nebulae, planets).

Interaction radiation – matter

- As noticed above, specific intensity I_λ is ***independent of distance*** from the source and remains the same as radiation propagates through **free** space.
- However, space is **not** always **free** (consider, for instance, stellar interiors).
- Thus, energy can be removed from, or delivered to, the radiation field.
- In this case, the intensity of light will change.

Absorption coefficient (1)

- Consider radiation shining through a medium



- The intensity of light is found experimentally to decrease by an amount dI_λ where

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds$$

Here κ_λ is the so-called mass absorption coefficient (alias opacity) [cm^2 g^{-1}], ρ is the density (in mass per unit volume), and ds is a length.

It can also be represented as $\alpha_\lambda = \kappa_\lambda \rho$, where α_λ is the absorption coefficient [cm^{-1}].

Absorption coefficient (2)

Number density of absorbers (particles per unit volume) = n
Each absorber has cross-sectional area = σ_λ (units cm^2)

If a light beam travels through ds, total area of absorbers is:

number of absorbers × cross-section = $n\ A$ ds × σ_λ

Fraction of radiation absorbed = fraction of area blocked:

$$\frac{dI_\lambda}{I_\lambda} = -\frac{n\ A\ ds\ \sigma_\lambda}{A} = -n\sigma_\lambda ds$$

$$dI_\lambda = -n\sigma_\lambda I_\lambda ds = -\alpha_\lambda I_\lambda ds$$

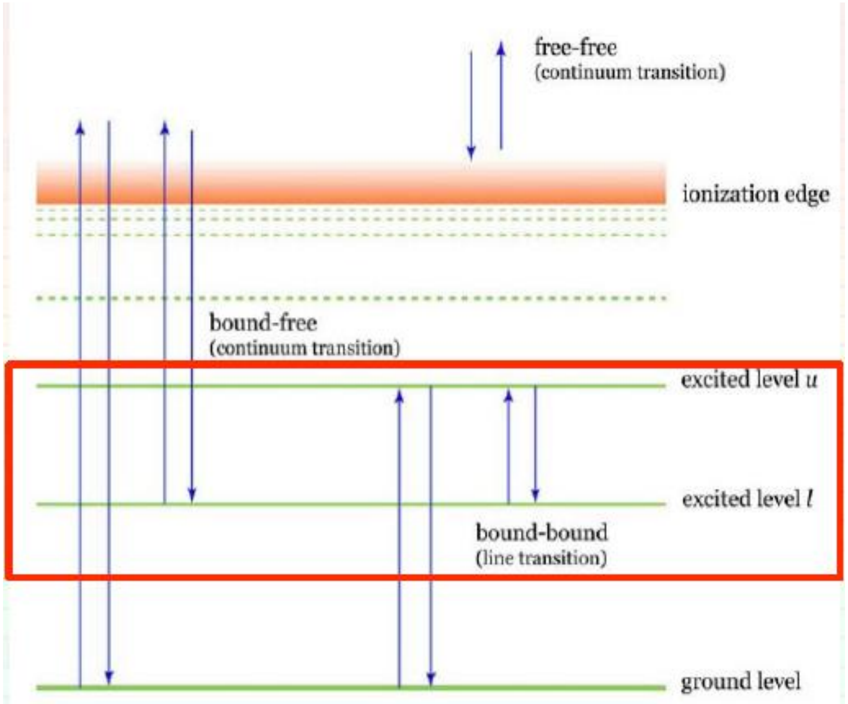
Thus, $\alpha_\lambda = \kappa_\lambda \rho = n\sigma_\lambda$, where α_λ is the absorption coefficient [cm^{-1}].

The photon mean free path is inversely proportional to α_λ.

Opacity

Opacity is the resistance of material to the flow of radiation through it. In most stellar interiors it is determined by all the processes which scatter and absorb photons:

- bound-bound absorption
- bound-free absorption
- free-free absorption
- scattering



The first three are known as true absorption processes because they involve the disappearance of a photon, whereas the fourth process only alters the direction of a photon.

Bound-bound absorption

- Bound-bound absorptions occur when an electron is moved from one orbit in an atom or ion into another orbit of higher energy due to the absorption of a photon. If the energy of the two orbits is E_1 and E_2, a photon of frequency ν_{bb} will produce a transition if

$$E_2 - E_1 = h\nu_{bb}$$

- Bound-bound processes are responsible for the spectral lines visible in stellar spectra, which are formed in the atmospheres of stars.

- In stellar interiors, however, bound-bound processes are **not** of great importance as most of the atoms are highly ionised and only a small fraction contain electrons in bound orbits.
- In addition, most of the photons in stellar interiors are so energetic that they are more likely to cause bound-free absorptions, as described below.

Bound-free absorption

- Bound-free absorptions involve the ejection of an electron from a bound orbit around an atom or ion into a free hyperbolic orbit due to the absorption of a photon. A photon of frequency ν_{bf} will convert a bound electron of energy E_1 into a free electron of energy E_3 if

$$E_3 - E_1 = h\nu_{bf}$$

- Provided the photon has sufficient energy to remove the electron from the atom or ion, any value of energy can lead to a bound-free process.
- Bound-free processes hence lead to continuous absorption in stellar atmospheres.

- In stellar interiors, however, the importance of bound-free processes is **reduced** due to the rarity of bound electrons.

Free-free absorption & emission

- Free-free absorption occurs when a free electron of energy E_3 absorbs a photon of frequency ν_{ff} and moves to a state with energy E_4, where

$$E_4 - E_3 = h\nu_{ff}$$

- There is no restriction on the energy of a photon which can induce a free-free transition and hence free-free absorption is a **continuous** absorption process which operates in **both** stellar atmospheres and stellar interiors.

- Note that, in both free-free and bound-free absorption, low energy photons are more likely to be absorbed than high energy photons.

Scattering

- In addition to the above absorption processes, it is also possible for a photon to be scattered by an electron or an atom. One can think of scattering as a collision between two particles which bounce of one another.

- For example, **electron scattering** – deflection of a photon from its original path by a free electron, without changing its wavelength.

- There are a lot of free electrons in stellar interiors, so this is an important process which **operates** in stellar interiors.
- Although this process does not lead to the true absorption of radiation, it does slow the rate at which energy escapes from a star because it continually changes the direction of the photons.

Example: Thomson scattering

A free electron has a cross section to radiation given by the Thomson value:

$$\sigma_T = 6.7\times10^{-25}\ \text{cm}^2$$

…independent of frequency. The opacity is therefore:

Avogadro constant

$$\kappa_\lambda = \frac{n}{\rho}\sigma_T = N_A\sigma_T = 0.4\ \text{cm}^2\text{g}^{-1}$$

If the gas is pure hydrogen (protons and electrons only)

(note: we should distinguish between absorption and scattering, but don't need to worry about that here…)

Optical Depth

- Two physical processes contribute to the opacity κ_λ (note that subscript λ just means that absorption is photon-wavelength dependent);
  - (i) true absorption where the photon is destroyed and the energy thermalized;
  - (ii) scattering where the photon is shifted in direction and removed from the solid angle under consideration.

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds \Rightarrow \frac{dI_\lambda}{I_\lambda} = -\kappa_\lambda \rho ds \Rightarrow \ln I_\lambda = -\int_0^s \kappa_\lambda \rho ds + \ln C$$

$$I_\lambda = Ce^{-\int_0^s \kappa_\lambda \rho ds} = Ce^{-\tau_\lambda}$$

If s=0, then $C=I_\lambda{}^0$ $\quad = I_\lambda{}^0 e^{-\tau_\lambda}$ the usual simple extinction law

$$\boldsymbol{\tau_\lambda = \int_0^s \kappa_\lambda \rho ds}$$

the "optical depth"

Importance of optical depth

- We can write the change in specific intensity over a path length as

$$dI_\lambda = -I_\lambda d\tau_\lambda$$

This is a "passive" situation where no emission occurs and is the simplest example of the radiative transfer equation.

- An optical depth of $\tau=0$ corresponds to no reduction in intensity (i.e. the top of photosphere for a star).
- An optical depth of $\tau=1$ corresponds to a reduction in intensity by a factor of $e=2.7$.
- If the optical depth is large ($\tau \gg 1$) negligible intensity reaches the observer.
- In stellar atmospheres, typical photons originate from $\tau=2/3$ (the proof will follow later on).

Emission coefficient and Source function

- We can also treat emission processes in the same way as absorption via a (volume) emission coefficient ε_λ [erg/s/cm³/str/Å] , or a (mass) emission coefficient j_λ [erg/s/g/str/Å]

$$dI_\lambda = \varepsilon_\lambda ds = j_\lambda \rho ds$$

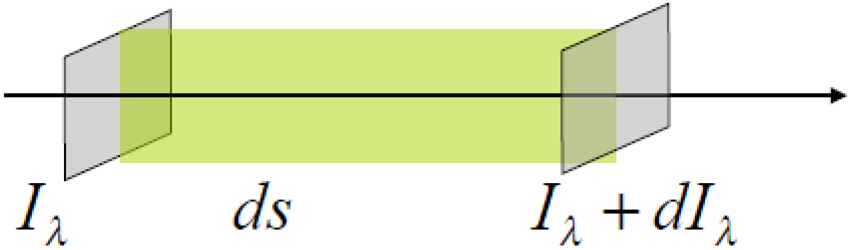
- Physical processes contributing to ε_λ , are
  (i) True (real) emission – the creation of photons;
  (ii) scattering of photons into a given direction from other directions.

- The ratio of emission to absorption coefficients is called the Source function

$$S_\lambda = j_\lambda / \kappa_\lambda$$

Radiative transfer equation (1)

- The primary mode of energy transport through the surface layers of a star is by radiation.
- The radiative transfer equation describes how the physical properties of the material are coupled to the spectrum we ultimately measure.
- Recall, energy can be removed from (true absorption or scattered), or delivered to (true emission or scattered) a ray of radiation:



- The rate of change of (specific) intensity is:

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds + j_\lambda \rho ds$$

Radiative transfer equation (2)

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds + j_\lambda \rho ds$$

$$d\tau_\lambda = \kappa_\lambda \rho ds$$
$$S_\lambda = j_\lambda / \kappa_\lambda$$

- We can re-write this equation in terms of the optical depth τ_λ and the source function S_λ

$$dI_\lambda / d\tau_\lambda = -I_\lambda + S_\lambda$$

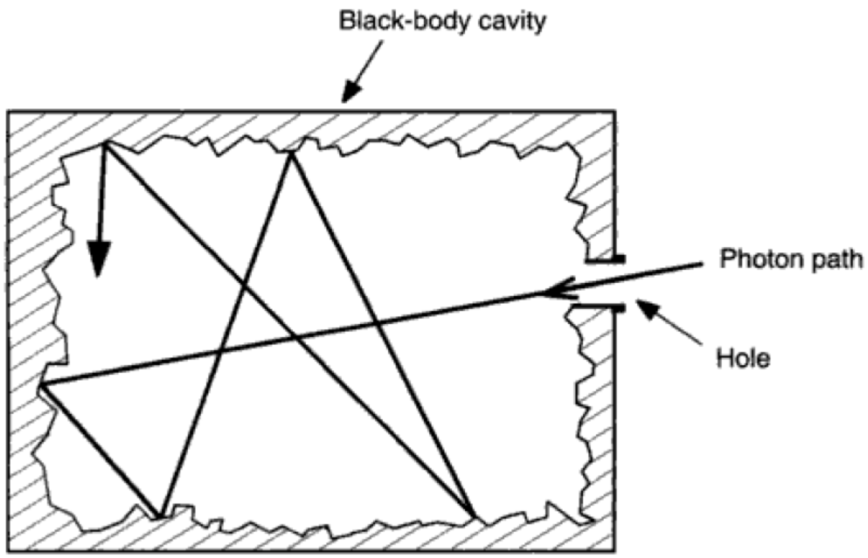
- **This is the (parallel-ray) equation of radiative transfer (RTE).** It will need a small modification before it is applicable to stars, but we can already gain some insight from its solution.
- If $S_\lambda < I_\lambda$, the intensity will decrease with increasing τ_λ, it will stay constant if $S_\lambda = I_\lambda$ and increase if $S_\lambda > I_\lambda$. *When* $\tau_\lambda \rightarrow \infty$, $I_\lambda \rightarrow S_\lambda$

Thermodynamic Equilibrium (TE)

The Black Body

Imagine a box which is completely closed except for a small hole. Any light entering the box will have a very small likely hood of escaping & will eventually be absorbed by the gas or walls. For constant temperature walls, this is in **thermodynamic equilibrium**.

If this box is heated the walls will emit photons, filling the inside with radiation. A small fraction of the radiation will leak out of the hole, but so little that the gas within it remains in equilibrium. **The emitted radiation is that of a black-body.**



The black body intensity is defined (following discovery by Max Planck in 1900) as either

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{\frac{hc}{\lambda kT}} - 1} \quad \text{or} \quad B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1}$$

where c=2.99x10^{10} cm, h=6.57x20^{-27} erg s, k=1.38x10^{-16} erg/s.

Physical interpretation of S_λ

$$dI_\lambda/d\tau_\lambda = -I_\lambda + S_\lambda$$

- If $S_\lambda < I_\lambda$, the intensity will decrease with increasing τ_λ
- The intensity will increase if $S_\lambda > I_\lambda$
- It will stay constant if $S_\lambda = I_\lambda$

- Thermodynamic Equilibrium (TE) → nothing changes with time
- A beam of light passing through such a gas volume will not change either

$$dI_\lambda/d\tau_\lambda = 0 \quad \longrightarrow \quad S_\lambda = I_\lambda = B_\lambda$$

in TE, the source function equals the Planck function

$$\kappa_\lambda B_\lambda = j_\lambda \quad \text{or} \quad \alpha_\lambda B_\lambda = \varepsilon_\lambda$$

The law of Kirchhoff

Radiative transfer equation (3)

$$dI_\lambda / d\tau_\lambda = -I_\lambda + S_\lambda$$

One can formally solve this form of the RTE, assuming that S_λ is constant along the path. **Class task: solve it** using an integrating factor e^{τ_λ}

i.e.

$$e^{\tau_\lambda}\frac{dI_\lambda}{d\tau_\lambda} = -e^{\tau_\lambda} I_\lambda + e^{\tau_\lambda} S \quad \text{so} \quad e^{\tau_\lambda}\frac{dI_\lambda}{d\tau_\lambda} + e^{\tau_\lambda} I_\lambda = e^{\tau_\lambda}\frac{dI_\lambda}{d\tau_\lambda} + I_\lambda \frac{de^{\tau_\lambda}}{d\tau_\lambda} = e^{\tau_\lambda} S_\lambda$$

$$e^{\tau_\lambda}\frac{dI_\lambda}{d\tau_\lambda} + I_\lambda \frac{de^{\tau_\lambda}}{d\tau_\lambda} = \frac{d}{d\tau_\lambda}(e^{\tau_\lambda} I_\lambda) \quad \text{so} \quad \frac{d}{d\tau_\lambda}(e^{\tau_\lambda} I_\lambda) = e^{\tau_\lambda} S_\lambda$$

Now integrate:

constant

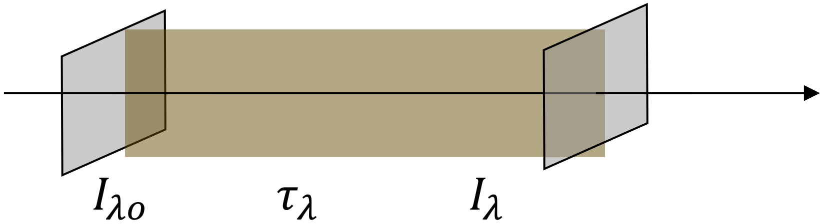
$$\int_0^{\tau_\lambda} \frac{d}{d\tau_\lambda}(e^{\tau_\lambda} I_\lambda)\, d\tau_\lambda = [e^{\tau_\lambda} I_\lambda]_0^{\tau_\lambda} = \int_0^{\tau_\lambda} e^{\tau_\lambda} S_\lambda \, d\tau_\lambda = [e^{\tau_\lambda} S_\lambda]_0^{\tau_\lambda}$$

Radiative transfer equation (4)

Inserting boundary conditions:

$$I_\lambda e^{\tau_\lambda} - I_{\lambda 0} = S_\lambda (e^{\tau_\lambda} - 1)$$

$I_\lambda(\tau_\lambda = 0) = I_{\lambda 0}$



Rearrange:

$$I_\lambda = S_\lambda (1 - e^{-\tau_\lambda}) + I_{\lambda 0} e^{-\tau_\lambda}$$

The second term of the RHS describes the amount of radiation left over from the intensity entering the box, after it has passed through an optical depth τ, the first term gives the contribution of the intensity from the radiation emitted along the path.

Constant S_λ along the path is **a very rude assumption!**
See D. Gray (pp. 127-129) for more accurate integration.

$$I_\lambda(\tau_\lambda) = \int_0^{\tau_\lambda} S_\lambda(t_\lambda) e^{-(\tau_\lambda - t_\lambda)}\, dt_\lambda + I_{\lambda 0} e^{-\tau_\lambda}$$

This is the formal solution of the RTE which assumes that the source function is known (D. Gray)

Solution to transfer equation

$$I_\lambda = S_\lambda(1 - e^{-\tau_\lambda}) + I_{\lambda 0} e^{-\tau_\lambda}$$

Imagine first the case in which $I_{\lambda o}=0$,
i.e. solely emission from the volume of gas:

$$I_\lambda = S_\lambda(1 - e^{-\tau_\lambda})$$

We have two limiting cases:

- Optically thin case ($\tau_\lambda \ll 1$)

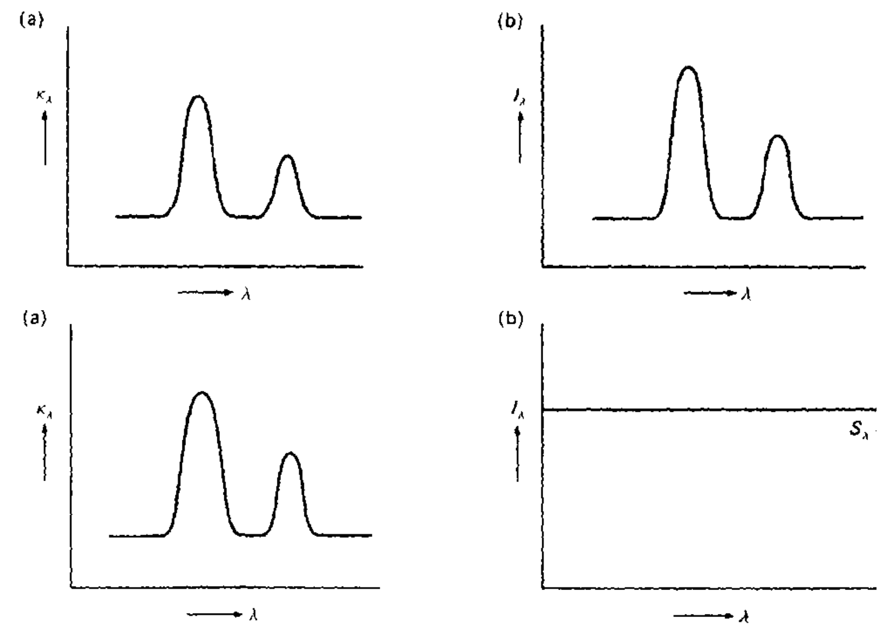
  $e^{-\tau_\lambda} \approx 1 - \tau_\lambda \Rightarrow I_\lambda = \tau_\lambda S_\lambda$

EXAMPLE: Hot, low density nebula

- Optically thick case ($\tau_\lambda \gg 1$)

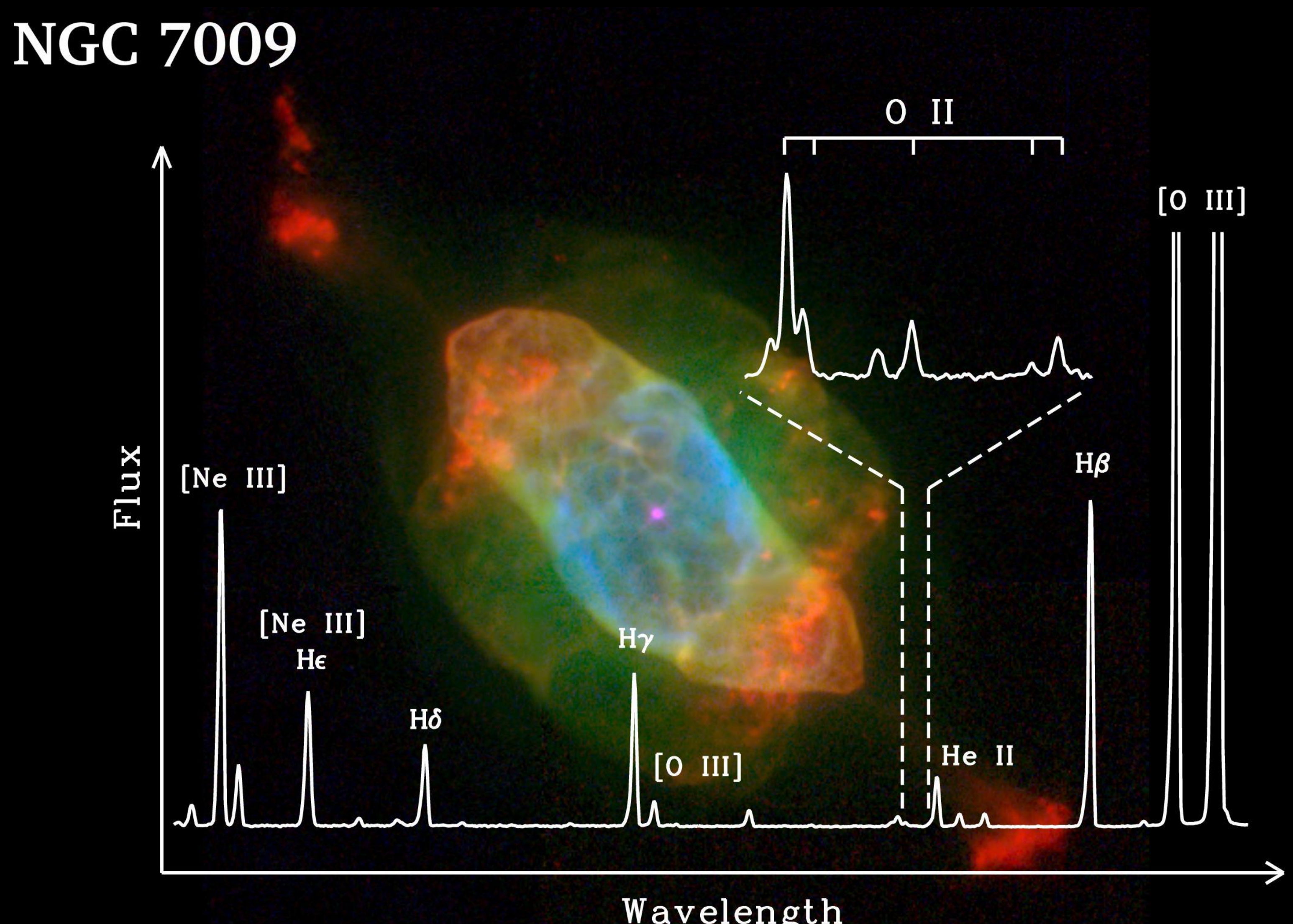
  $e^{-\tau_\lambda} \approx 0 \Rightarrow I_\lambda = S_\lambda$

EXAMPLE: Black body, $S_\lambda = B_\lambda(T)$



Opacity κ versus λ ➔ Intensity versus λ

Hot nebular gas: emission lines –optically thin



Absorption versus emission

$$I_\lambda = S_\lambda(1 - e^{-\tau_\lambda}) + I_{\lambda 0} e^{-\tau_\lambda}$$

Imagine now $I_{\lambda o} \neq 0$,
again with two extreme cases:

- Optically thin case ($\tau_\lambda \ll 1$) $\quad I_\lambda = I_{\lambda 0}(1 - \tau_\lambda) + \tau_\lambda S_\lambda = I_{\lambda 0} + \tau_\lambda (S_\lambda - I_{\lambda 0})$

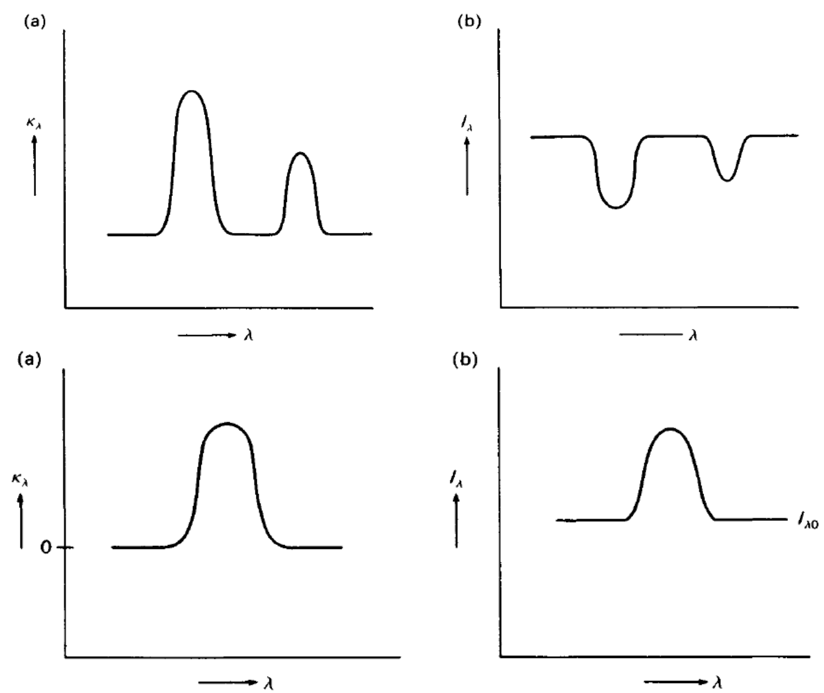
(a) If $I_{\lambda o} > S_\lambda$, so there is something subtracted from the original intensity which is proportional to the optical depth – we see absorption lines on the continuum intensity I_λ.
EXAMPLE: stellar photospheres

(b) If $I_{\lambda o} < S_\lambda$, we will see emission lines on top of the background intensity.

Example: Solar UV spectrum

- Optically thick case ($\tau_\lambda \gg 1$): $\quad I_\lambda = S_\lambda$

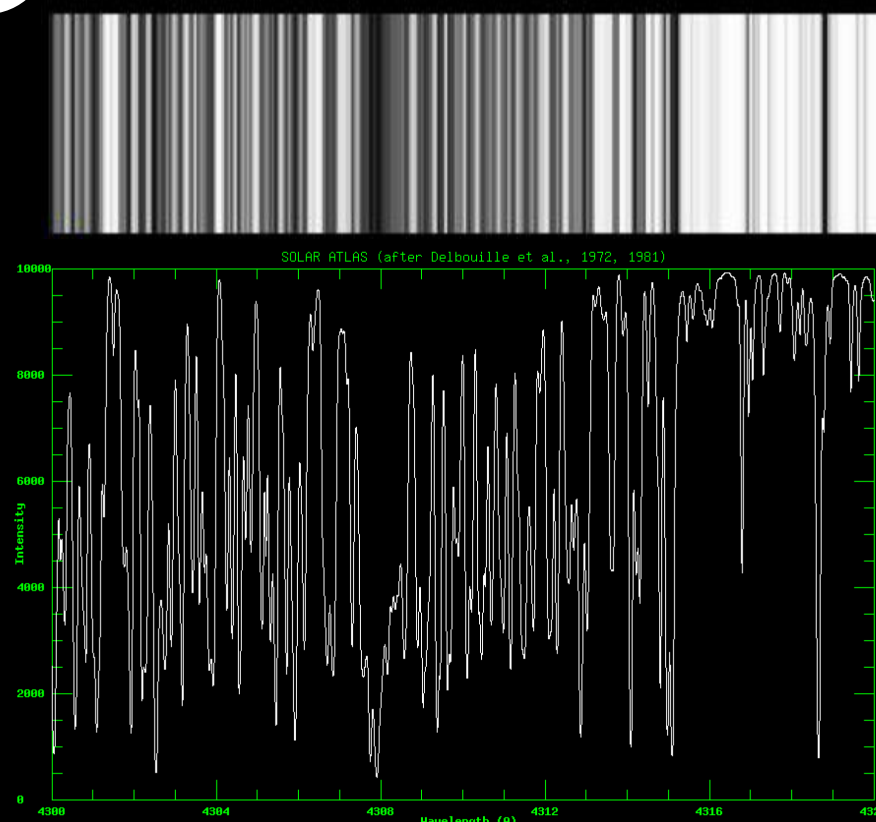
Planck function as before.



Opacity κ versus λ ➔ Intensity versus λ

Outward decreasing temperature

- In a star absorption lines are produced if $I_{\lambda 0} > S_{\lambda}$ i.e. the intensity from deep layers is larger than the source function from top layers.
- In local TE (LTE), the source function is $B_{\lambda}(T)$, so the Planck function for the deeper layers is larger than the shallower layers. Consequently the deeper layers have a higher temperature than the top layers (since the Planck function increases at all wavelengths with T).
- (Instances occur where LTE is not valid, and the source function declines outward in parallel with an increasing temperature).



Solar Spectrum (4300-4320Ang)

Absorption versus emission lines

Emission line spectra:

- Optically thin volume of gas with no background illumination (emission nebula)
- Optically thick gas in which the source function increases outwards (UV solar spectrum)

Absorption line spectra:

- Optically thin gas in which source function declines outward, generally T decreases outwards (Stellar photospheres)
- Optically thin gas penetrated by background radiation (ISM between us and the star)

Things we already learned about RT

- We defined the specific intensity I_λ, emission (j_λ and ε_λ) and absorption coefficients (κ_λ and α_λ), optical depth $d\tau_\lambda$, the source function S_λ.
- We have derived and solved (assuming constant S_λ) the (parallel-ray) equation of radiative transfer (RTE):

$$\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda + S_\lambda$$

$$I_\lambda = S_\lambda(1 - e^{-\tau_\lambda}) + I_{\lambda 0}e^{-\tau_\lambda}$$

- In TE (thermodynamic equilibrium), the source function equals the Planck function, $S_\lambda = B_\lambda$.
- The law of Kirchhoff: $B_\lambda = j_\lambda/\kappa_\lambda = \varepsilon_\lambda/\alpha_\lambda$

- Today, we will define other important terms which we will use later.

Specific and mean Intensity

From the previous lecture:

$$I_\lambda = \frac{E_\lambda}{\cos\theta \ d\lambda \, d\sigma \, d\omega \, dt}$$

Let's try in another way:

- The (specific) intensity $\boldsymbol{I_\lambda}$ is a measure of brightness:

$$I_\lambda = \frac{dE_\lambda}{\cos\theta \ d\lambda \, d\sigma \, d\omega \, dt}$$

$d\lambda, d\sigma, d\omega, dt \rightarrow 0$ dE diminishes to zero as well

- In this way, we define the specific intensity at a "point" on the surface, at a given time, in a direction θ, at a wavelength λ - *brightness*.

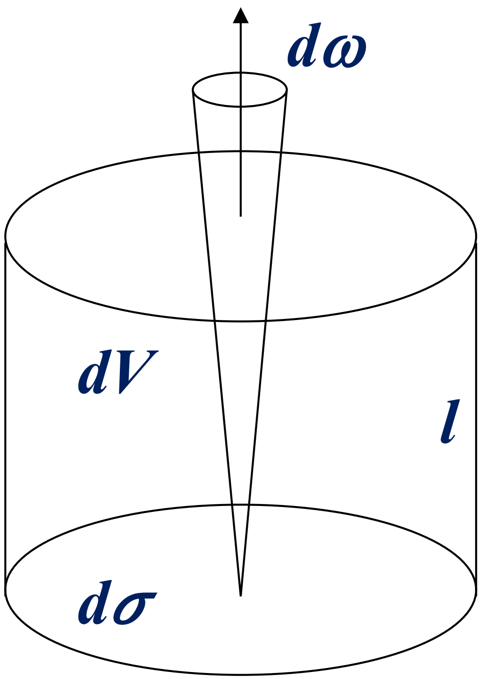
The *mean* intensity $\boldsymbol{J_\lambda}$ is the directional average of the specific intensity (over 4π steradians):

$$J_\lambda = \frac{1}{4\pi} \oint I_\lambda d\omega$$

Integrated over the whole unit sphere centered on the point of interest.

Mean intensity and Energy density

$$J_\lambda = \frac{1}{4\pi}\oint I_\lambda d\omega$$



- The *mean* intensity $\boldsymbol{J_\lambda}$ is related to the energy density $\boldsymbol{u_\lambda}$:
- Energy radiated through area element $d\sigma$ during time dt:

$$dE_\lambda = I_\lambda \, d\lambda \, d\sigma \, d\omega \, dt$$

$$l = c\,dt \Rightarrow dV = l\,d\sigma = c\,dt\,d\sigma$$

- Hence, the energy contained in volume element dV per wavelength interval is:

$$u_\lambda dV d\lambda = \oint I_\lambda \, d\omega \, d\lambda \, d\sigma \, dt = 4\pi J_\lambda \frac{dV}{c} d\lambda$$

$$u_\lambda = \frac{4\pi}{c} J_\lambda \quad \left[\frac{erg}{cm^3 Å}\right]$$

$$u = \int_0^\infty u_\lambda \, d\lambda = \frac{4\pi}{c}\int_0^\infty J_\lambda \, d\lambda \quad \left[\frac{erg}{cm^3}\right]$$

Total radiation emerge in volume element

Flux (1)

- From an observational point of view, we are generally more interested in the energy flux or flux (L_λ, L) and the flux density (F_λ, F).
  Flux density gives the power of the radiation per unit area and hence has dimensions of erg s^{-1} cm^{-2} $Å^{-1}$ (or erg s^{-1} cm^{-2} Hz^{-1}).
  Observed flux densities are usually extremely small and therefore (especially in radio astronomy) flux densities are often expressed in units of the Jansky (Jy), where 1 Jy=10^{-23} erg s^{-1} cm^{-2} Hz^{-1}.

- You should be aware - and beware - that different authors define the terms flux density, flux and intensity differently, and they are sometimes used interchangeably!

- We will often call flux density as just flux.

- Standard definition:
  Flux describes any effect that appears to pass or travel through a surface or substance. In transport phenomena (radiative transfer, heat transfer, mass transfer, fluid dynamics), flux is defined as the rate of flow of a property per unit area, which has the dimensions $[\text{quantity}]\times[\text{time}]^{-1}\times[\text{area}]^{-1}$.
  - For example, the magnitude of a river's current, i.e. the amount of water that flows through a cross-section of the river each second is a kind of flux.

Flux (2)

In radiative transfer, flux is related to the intensity ("specific" is often omitted):

- Flux F_λ, is a measure of the net energy flow across an area $d\sigma$, over a time dt, in a $d\lambda$. The only directional significance is whether the energy crosses $d\sigma$ from the top or from the bottom. Then we can write:

The solid angle $d\omega$ appears for I_λ but not for F_λ

$$F_\lambda = \frac{\oint dE_\lambda}{d\lambda\, d\sigma\, dt}$$

Integrated over all directions.

substitute

$$I_\lambda = \frac{dE_\lambda}{\cos\theta\, d\lambda\, d\sigma\, d\omega\, dt}$$

$$F_\lambda = \oint I_\lambda \cos\theta\, d\omega \left[\frac{erg}{Å\, cm^2\, s}\right]$$

Thus, flux F_λ is the projection of the specific intensity I_λ in the radial direction (integrated over all solid angles)

The amount of energy going through 1 cm² per second per 1Å into the solid angle dω in the direction inclined by the angle θ to the normal of the area.

Flux (3)

$d\omega$ = $\sin\theta\, d\theta\, d\varphi$

Expressing $d\omega$ by means of θ and φ,

$$F_\lambda = \oint I_\lambda \cos\theta\, d\omega = \int_0^{2\pi} d\varphi \int_0^{\pi} I_\lambda \cos\theta \sin\theta\, d\theta$$

If there is no azimuthal dependence for I_λ then

$$F_\lambda = \oint I_\lambda \cos\theta\, d\omega = 2\pi \int_0^{\pi} I_\lambda \cos\theta \sin\theta\, d\theta$$

In the plane-parallel or spherical case, we do not find any dependence of I_λ on the longitude φ

$$F_\lambda = -2\pi \int_0^{\pi} I_\lambda \cos\theta\, d(\cos\theta)$$

Meaning of flux:

Radiation flux = **netto** energy going through area
Decomposition into two half-spaces:

$$F = -2\pi \int_0^\pi I_\lambda \cos\theta \, d(\cos\theta) = 2\pi \int_{-1}^{1} I(\mu)\, \mu \, d\mu$$

$\mu = \cos\theta$

$$= 2\pi \int_0^1 I(\mu)\, \mu\, d\mu + 2\pi \int_{-1}^{0} I(\mu)\, \mu\, d\mu$$

$$= 2\pi \int_0^1 I(\mu)\, \mu\, d\mu - 2\pi \int_0^1 I(-\mu)\mu\, d\mu = \boldsymbol{F^+ - F^-}$$

Netto = Outwards - Inwards

Special cases: at the surface of a star $F^- = 0$, so that $F = F^+$
at the centre of a star, isotropic radiation field: $F=0$

Intensity, Flux, and Luminosity

- $\boldsymbol{I}$ is independent of distance from the source and can only be measured directly if we resolve the radiating surface.
  In contrast, $\boldsymbol{F}$ obeys the inverse square law and is all that may be measured for most stars.

$\mathrm{d}S \equiv r^2\, \mathrm{d}\omega$

- Indeed, if we consider a star as the source of radiation, then the flux emitted by the star into a solid angle $\mathrm{d}\omega$ is $\mathrm{d}L = \mathrm{d}\omega r^2 F$, where F is the flux density observed at a distance r from the star. If the star radiates isotropically then radiation at a distance r will be distributed evenly on a spherical surface of area $4\pi r^2$ and hence we get the relationship:

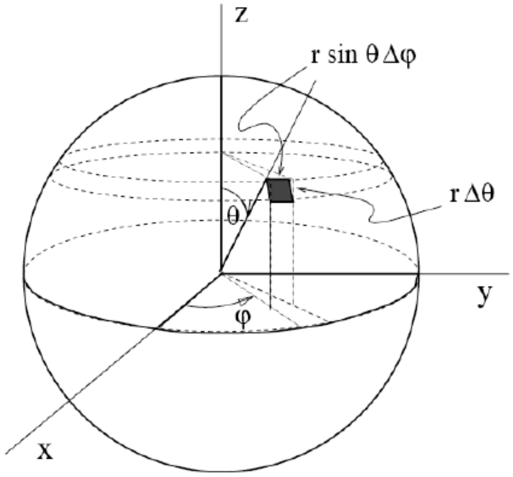
$$L = 4\pi r^2 F$$

- It is also usual to refer to the total flux from a star as the Luminosity, L.

Surface brightness

- Flux density arriving from a point source is inversely proportional to the distance. But what about an extended luminous object such as a nebula or galaxy? The situation is slightly more complicated.
- The surface brightness is defined as the flux density per unit solid angle. The geometry of the situation results in the interesting fact that the observed surface brightness is independent of the distance of the observer from the extended source.
- This slightly counter-intuitive phenomenon can be understood by realizing that although the flux density arriving from a unit area is inversely proportional to the square of the distance to the observer, the area on the surface of the source enclosed by a unit solid angle at the observer is directly proportional to the square of the distance.
- Thus, the two effects cancel each other out.

$$dS \equiv r^2 \, d\omega$$



Mean Intensity, Flux and K-integral

- The *mean* intensity $\boldsymbol{J_\lambda}$ is the directional average of the specific intensity (over 4π steradians):

$$J_\lambda = \frac{1}{4\pi} \oint I_\lambda d\omega$$

- Flux $\boldsymbol{F_\lambda}$, is the projection of the specific intensity in the radial direction (integrated over all solid angles):

$$F_\lambda = \oint I_\lambda \cos\theta \, d\omega$$

- There is also a K-integral which we will use later:

$$K_\lambda = \frac{1}{4\pi} \oint I_\lambda \cos^2\theta \, d\omega$$

K-integral and radiation pressure

- K-integral is related to the radiation pressure: $K_\lambda = \frac{1}{4\pi}\oint I_\lambda \cos^2\theta \, d\omega$
- A photon has momentum $p_\lambda = E_\lambda / c$

- Consider photons transferring momentum to a solid wall.
  Force:

$$F = \frac{dp_{\lambda\perp}}{dt} = \frac{1}{c}\frac{dE_\lambda}{dt}\cos\vartheta$$

- Pressure: $$dP_\lambda = \frac{F}{d\sigma} = \frac{1}{c}\frac{dE_\lambda}{dt}\frac{\cos\vartheta}{d\sigma} = \frac{1}{c} I_\lambda \cos^2\vartheta \; d\omega \, d\lambda$$

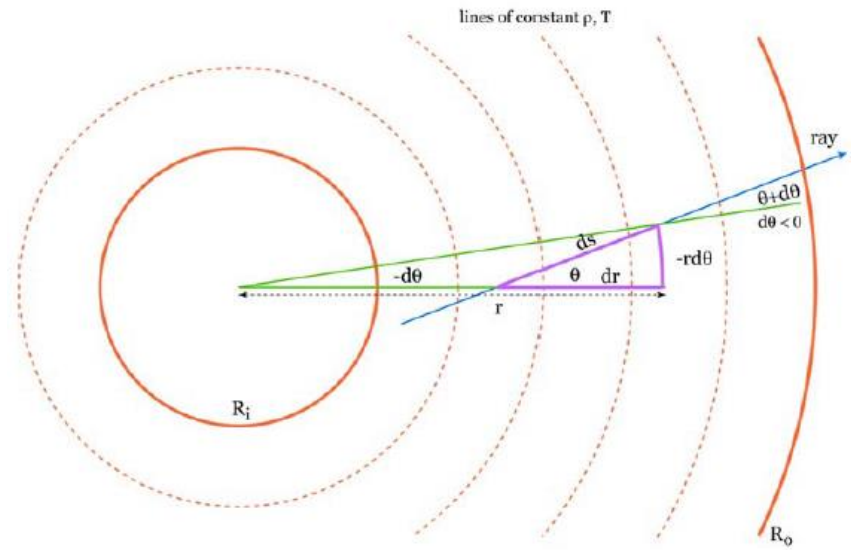
$$I_\lambda = \frac{dE_\lambda}{\cos\theta \, d\lambda \, d\sigma \, d\omega \, dt}$$

$$P_{rad}(\lambda) = \frac{1}{c}\oint_{4\pi} I_\lambda \cos^2\vartheta \; d\omega = \frac{4\pi}{c} K_\lambda$$

Plane-parallel vs spherical geometry

- Parallel-ray RTE is a very simple approach.
- In principal, we need to consider spherical geometry when calculating the transfer equation in stars.
- Fortunately, the geometrical thickness of most stellar photospheres is small compared to their radii, permitting the plane-parallel approximation, $r \rightarrow \infty$

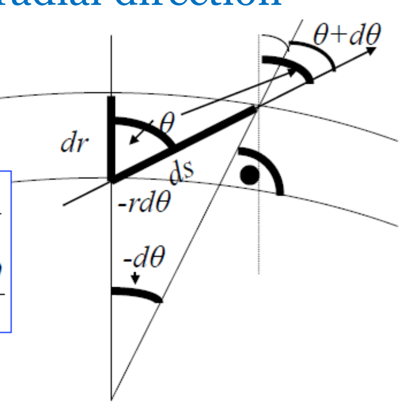
$$\frac{dI_\lambda}{ds} = -\cos\vartheta \frac{\partial I_\lambda}{\partial r}$$



angle ϑ between ray and radial direction is not constant

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \theta}\frac{d\theta}{ds}$$

$$\frac{dr}{ds} = \cos\theta, \quad \frac{d\theta}{ds} = -\frac{\sin\theta}{r}$$



Transfer Equation for Stars

The plane-parallel transfer equation
(for stars with thin photospheres)

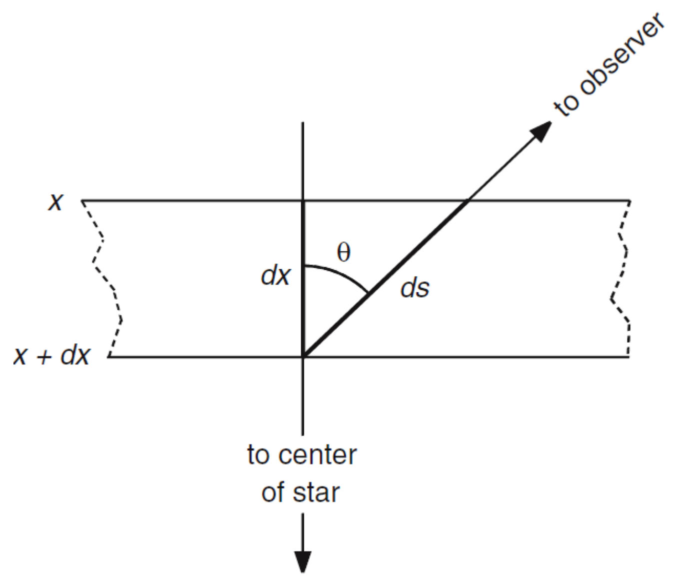
$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

is identical to the parallel-ray transfer equation (for ISM studies),

$$\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda + S_\lambda$$

except for

1. the cos(θ) term, because the optical depth is measured along the radial direction x and not along the line of sight, i.e $d\tau_\lambda = -\kappa_\lambda \rho\, dx$
2. sign change, since we are now looking from the outside in, along direction x.



The full spherical geometry transfer equation is necessary for supergiants.

The plane-parallel RTE

- We will try to solve **the plane-parallel RTE** later when we start discussing stellar photospheres.
- But now let's concentrate on stellar interiors.

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

- The plane-parallel RTE leads to two particularly useful relations between the various quantities describing the radiation field.

- First, recall that S depends only on the local conditions of the gas, independent of direction. Then, integrating over all solid angles, we get

$$\frac{d}{d\tau_\lambda}\oint I_\lambda \cos\theta \, d\omega = \oint I_\lambda d\omega - S\oint d\omega$$

$$\frac{dF_\lambda}{d\tau_\lambda} = 4\pi(J_\lambda - S)$$

Radiative diffusion (1)

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

- The second relation: multiply the plane-parallel RTE by cos(θ) and again integrate over all solid angles:

$$\frac{d}{d\tau_\lambda}\oint I_\lambda \cos^2\theta\, d\omega = \oint I_\lambda \cos\theta\, d\omega - S_\lambda \oint \cos\theta\, d\omega$$

$d\omega = \sin\theta\, d\theta\, d\varphi$

$$P_{rad,\lambda} = \frac{1}{c}\oint_{4\pi} I_\lambda \cos^2\vartheta\ d\omega$$

$$\frac{dP_{rad,\lambda}}{d\tau_\lambda} = \frac{1}{c}F_\lambda$$

$$\oint \cos\theta\, d\omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \cos\theta \sin\theta\, d\theta = 0$$

$d\tau_\lambda = -\kappa_\lambda \rho\, dx$

$$\frac{dP_{rad,\lambda}}{dr} = -\frac{\kappa_\lambda \rho}{c}F_\lambda$$

- Integrating the radiation pressure and flux over wavelengths, and replacing κ_λ by a weighted mean of opacity κ_R – the Rosseland mean opacity [we will introduce it later]:

$$\frac{dP_{rad}}{dr} = -\frac{\rho\kappa_R}{c}F$$

Radiative diffusion (2)

$$\frac{dP_{rad}}{dr} = -\frac{\rho\kappa_R}{c}F$$

- This relation can be interpreted as that the net radiative flux is driven by differences in the radiation pressure, with a "photon wind" blowing from high to low $P_{\rm rad}$.
- Thus, the transfer of energy by radiation is a process involving the slow upward diffusion of randomly walking photons, drifting toward the surface in response to tiniest differences in the radiation pressure.
- As we see, the description of a "ray" of light is in fact only a convenient fiction, used to define the direction of motion instantly shared by the photons that are continually absorbed and scattered into and out of the beam.
  - It can be shown that a photon generated near the centre of the Sun will be absorbed and re-emitted $\sim 10^{22}$ times before it escapes at the surface and the time it takes to do this is approximately equal to the thermal timescale of the Sun (a few $\times 10^7$ years). This means that when we observe energy radiated at the solar surface, we are usually seeing the results of nuclear reactions which occurred tens of millions of years ago.

The Radiative Temperature Gradient

Recall: the pressure exerted by photons on the particles in a gas is:

$$P_{rad} = \frac{aT^4}{3}$$

where radiation density constant

$$a = \frac{4\sigma_{SB}}{c}$$

- The radiation pressure gradient:

$$\frac{dP_{rad}}{dr} = -\frac{\kappa_R \rho}{c} F = \frac{4}{3} a T^3 \frac{dT}{dr}$$

- Then

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa_R \rho}{T^3} F$$

- Let's write Flux in terms of the local radiative luminosity of the star at radius r:

$$F(r) = \frac{L(r)}{4\pi r^2}$$

- The temperature gradient for radiative transport becomes:

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa_R \rho}{T^3} \frac{L(r)}{4\pi r^2} = -\frac{3}{64\pi \sigma_{SB} r^2} \frac{\kappa_R \rho}{T^3} L(r)$$

The fourth equation of stellar structure.

Summary of the lectures on RT

- In addition to the specific intensity I_λ, emission (j_λ and ε_λ) and absorption coefficients (κ_λ and α_λ), optical depth $d\tau_\lambda$, the source function S_λ, we defined the mean intensity J_λ and the energy density, radiative flux F_λ and luminosity L, K-integral and the radiation pressure F_{rad}.

- We derived the plane-parallel equation of radiative transfer (RTE):

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

- We have also derived the fourth differential equation of stellar structure (the temperature gradient for radiative transport):

$$\frac{dT}{dr} = -\frac{3}{64\pi\sigma_{SB}r^2}\frac{\kappa_R\rho}{T^3}L(r)$$

- Now we have all four equations, which govern the structure of stars. Let's now start searching for possible ways to solve them.

The equations of stellar structure

THE EQUATIONS OF STELLAR STRUCTURE AND POSSIBLE WAYS TO SOLVE THEM.

BOUNDARY CONDITIONS.

CONVECTION AND CONDITIONS FOR ITS OCCURRENCE

Solving the equations of stellar structure

Now we have all four differential equations, which govern the structure of stars
(Note! in the absence of convection)

- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$
- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$
- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$
- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma r^2}\frac{\rho(r)\kappa_R(r)}{T^3(r)}L(r)$

Where

- r = radius
- P = pressure at r
- m = mass of material within r
- ρ = density at r
- L = luminosity at r (rate of energy flow across sphere of radius r)
- T = temperature at r
- κ_R = Rosseland mean opacity at r
- ε = energy release per unit mass per unit time

We will consider the quantities:

- $P = P(\rho, T,$ chemical composition)
- $\kappa_R = \kappa_R(\rho, T,$ chemical composition)
- $\varepsilon = \varepsilon(\rho, T,$ chemical composition)

→ **The equation of state**

Boundary conditions

- Two of the boundary conditions are fairly obvious, at r=0, the centre of the star, m=0, L=0.

- At the surface of the star its not so clear, but we use approximations to allow solution.
  - There is no sharp edge to the star, but for the Sun ρ(surface)~10^{-7} g cm^{-3}. It is much smaller than mean density $\bar{\rho}$~1.4 g cm^{-3}.
  - We also know the surface temperature (T_{eff}=5780K) is much smaller than its minimum mean temperature (2×10^6 K).
- Thus, we make two approximations for the surface boundary conditions: ρ=0, T=0 at r=R, i.e. that the star does have a sharp boundary with the surrounding vacuum.

Use of mass as the independent variable

The above formulae would (in principle) allow theoretical models of stars with a given radius. However, from a theoretical point of view it is the mass of the star which is chosen, the stellar structure equations solved, then the radius (and other parameters) are determined. We observe stellar radii to change by orders of magnitude during stellar evolution, whereas mass appears to remain constant. Hence it is much more useful to rewrite the equations in terms of m rather than r.

We did it before: divide the equations by the equation of mass conservation:

- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$
- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$
- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$
- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma_{SB} r^2}\frac{\rho(r)\kappa_R(r)}{T^3(r)}L(r)$

- $\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$
- $\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$
- $\frac{dL}{dm} = \varepsilon$
- $\frac{dT}{dm} = -\frac{3\kappa_R L}{256\times\pi^2\sigma_{SB} r^4 T^3}$

We specify m and the chemical composition and now have a well-defined set of relations to solve. It is possible to do this analytically if simplifying assumptions are made, but in general these need to be solved numerically on a computer.

Stellar evolution (1)

- We have a set of equations that will allow the complete structure of a star to be determined, given a specified mass and chemical composition.
  However, what do these equations not provide us with?

- In deriving the equation for hydrostatic support, we have seen that provided the evolution of star is occurring slowly compared to the dynamical time t_d, we can ignore temporal changes (e.g. pulsations). Indeed, for the Sun $t_d \sim 30$ min, hence this is certainly true.

- And we have also assumed that time dependence can be omitted from the equation of energy generation, if the nuclear timescale (the time for which nuclear reactions can supply the stars energy) is greatly in excess of t_{th}.

Stellar evolution (2)

- If there are no bulk motions in the interior of the star, then any changes of chemical composition are localized in the element of material in which the nuclear reactions occurred. So, the star would have a chemical composition which is a function of mass $m(r)$.
- In the case of no bulk motions – the set of equations we derived must be supplemented by equations describing the rate of change of abundances of the different chemical elements. Let $C_{\mathrm{X,Y,Z}}$ be the chemical composition of stellar material in terms of mass fractions of hydrogen (X), helium (Y), and metals (Z)
[e.g., for the Solar system X=0.7, Y=0.28, Z=0.02].

$$\frac{\partial (C_{X,Y,Z})_m}{\partial t} = f(\rho, T, C_{X,Y,Z})$$

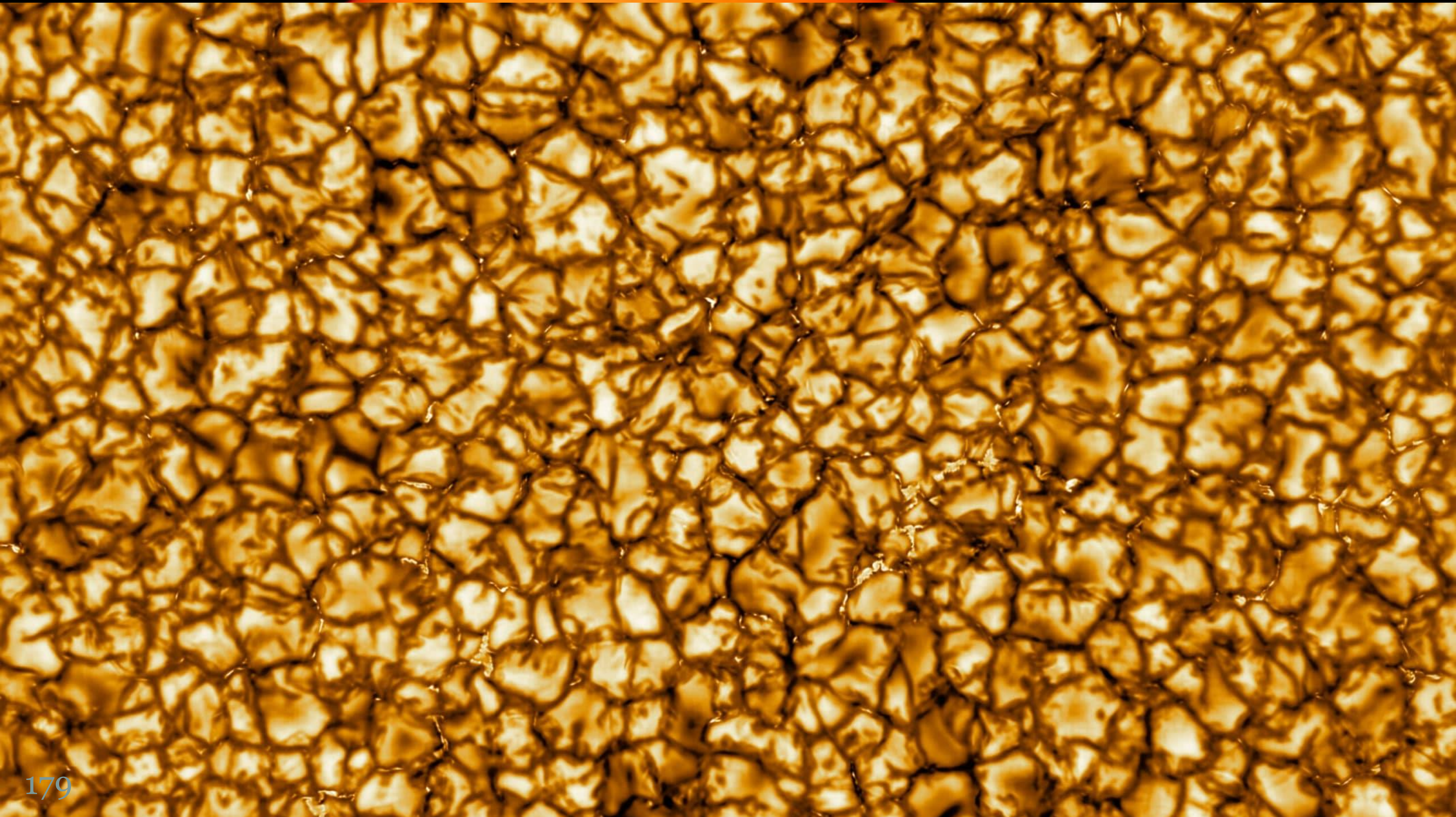
- Now let's consider how we could evolve a model

$$(C_{X,Y,Z})_{m,t_0+\delta t} = (C_{X,Y,Z})_{m,t_0} + \frac{\partial (C_{X,Y,Z})_m}{\partial t}$$

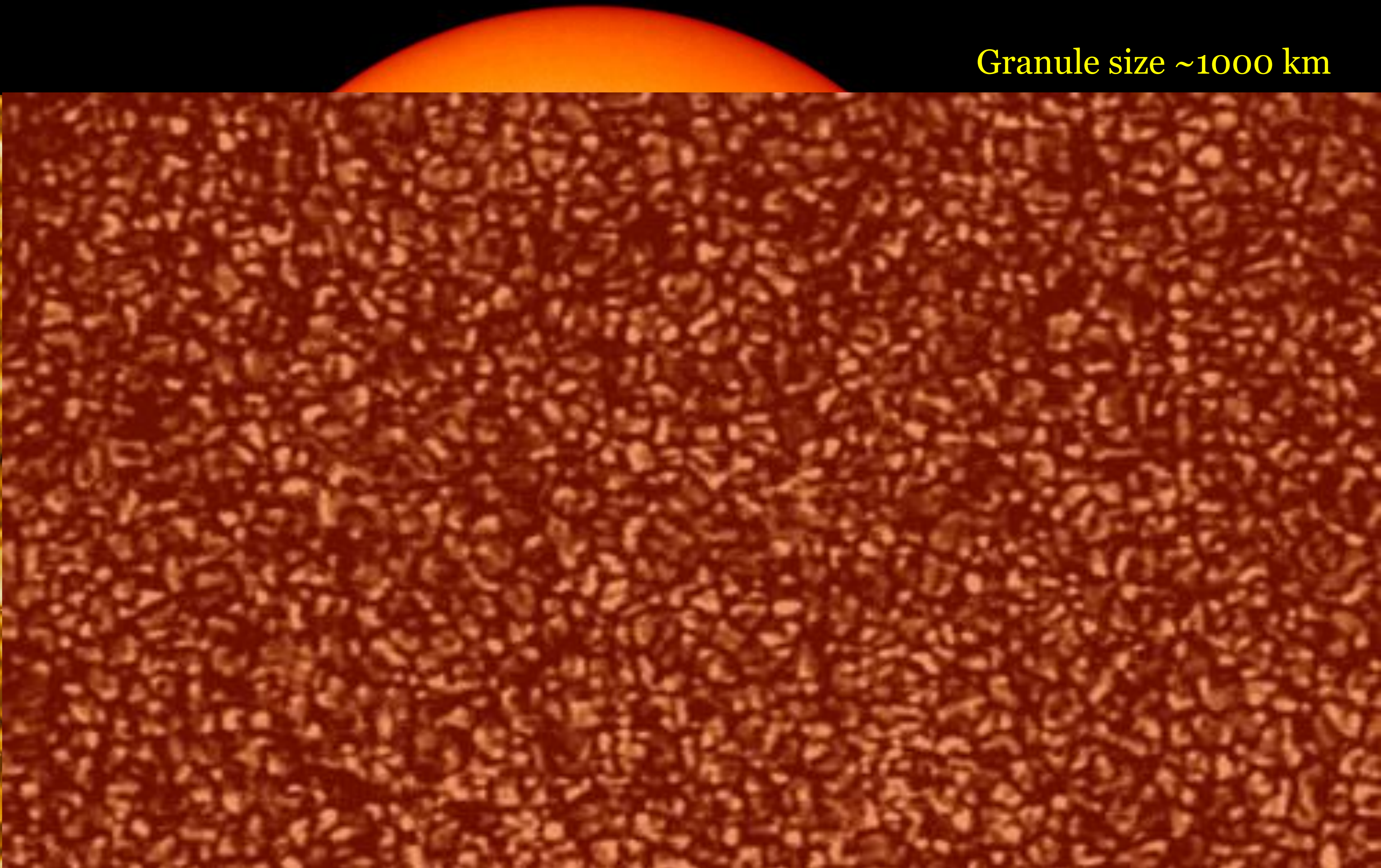
However …

Solar surface

Granule size ~1000 km



Solar surface



Convection (1)

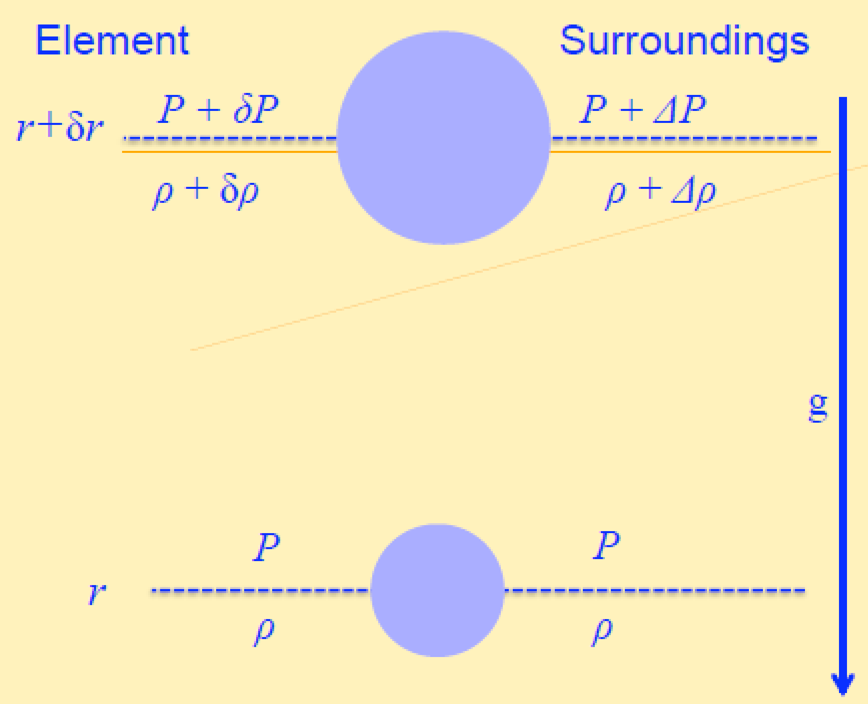
Convection is the mass motion of gas elements – only occurs when temperature gradient exceeds some critical value.
We can derive an expression for this.

Consider a convective element at distance r from the centre of star. Element is in equilibrium with the surrounding.

Now let's suppose it rises to $r+\delta r$. Element expands to stay in pressure balance with the new environment, $P(r)$ and $\rho(r)$ are reduced to $P + \delta P$ and $\rho + \delta\rho$.

But these may not generally equal the new surrounding gas conditions.
Define those as $P + \Delta P$ and $\rho + \Delta\rho$.

Convective element of stellar material



If gas element is denser than surroundings at $r + \delta r$ then will sink (i.e. stable).
If it is less dense then it will keep on rising – ***convectively unstable***.

Convection (2)

The condition for instability is therefore

$$\rho + \delta\rho < \rho + \Delta\rho$$

Whether or not this condition is satisfied depends on two things:

- The rate at which the element expands due to decreasing pressure
- The rate at which the density of the surroundings decreases with height

Let's make two assumptions

1. The element rises adiabatically, i.e. no heat is exchanged with the surrounding;
2. The element rises at a speed much less than the sound speed.
   During motion, sound waves have time to smooth out the pressure differences between the element and the surroundings. Hence $\delta P = \Delta P$ at all times.

The first assumption means that the element must obey the adiabatic relation between pressure and volume

$$PV^{\gamma} = \text{constant}$$

where $\gamma = c_p / c_V$ is the adiabatic index or heat capacity ratio defined as specific heat (i.e. the energy to raise temperature of 1 g of material by 1K) at constant pressure, divided by specific heat at constant volume.

Convection (3)

Given that V is inversely proportional to ρ, we can write

$$\frac{P}{\rho^{\gamma}} = \text{constant}$$

Hence equating the term at r and $r+\delta r$:

$$\frac{P+\delta P}{(\rho+\delta\rho)^{\gamma}} = \frac{P}{\rho^{\gamma}}$$

If $\delta\rho$ is small, we can expand $(\rho+\delta\rho)^{\gamma}$ using the binomial theorem as follows $(\rho+\delta\rho)^{\gamma} \sim \rho^{\gamma} + \gamma\,\delta\rho\,\rho^{\gamma-1}$. Combining last two expressions we obtain

$$\delta\rho = \frac{\rho}{\gamma P}\delta P$$

Now we need to evaluate the change in density of the surroundings, $\Delta\rho$.

Let's consider a very small rise of δr

$$\Delta\rho = \frac{d\rho}{dr}\delta r$$

Convection (4)

$\rho + \delta\rho < \rho + \Delta\rho$

And substituting these expressions for $\delta\rho$ and $\Delta\rho$ into the condition for convective instability derived above:

$$\delta\rho = \frac{\rho}{\gamma P}\delta P \qquad \frac{\rho}{\gamma P}\delta P < \frac{d\rho}{dr}\delta r \qquad \Delta\rho = \frac{d\rho}{dr}\delta r$$

And this can be rewritten by recalling our 2nd assumption that element will remain at the same pressure as its surroundings, so that in the limit

$$\delta r \to 0, \qquad \frac{\delta P}{\delta r} = \frac{dP}{dr}$$

$$\frac{\rho}{\gamma P}\frac{dP}{dr} < \frac{d\rho}{dr}$$

The LHS is the density gradient that would exist in the surroundings if they had an adiabatic relation between density and pressure. RHS is the actual density in the surroundings.

We can convert this to a more useful expression, by first dividing both sides by $\mathrm{d}P/\mathrm{d}r$.
Note that $\mathrm{d}P/\mathrm{d}r$ is negative, hence the inequality sign must change.

Convection (5)

$$\frac{\rho}{\gamma P} > \frac{d\rho}{dr} \Big/ \frac{dP}{dr} \quad \Rightarrow \quad \frac{\rho}{\gamma P} > \frac{d\rho}{dP}$$

$$\left(\frac{P}{\rho}\right)\frac{d\rho}{dP} < \frac{1}{\gamma} \quad \text{or} \quad \frac{d\ln\rho}{d\ln P} < \frac{1}{\gamma}$$

For an ideal gas in which radiation pressure is negligible (where μ is the mean molecular weight of particles in the stellar material in unit of proton mass m_{p})

$$P = \frac{\rho k T}{\mu m_p} \quad \Rightarrow \quad \ln P = \ln\rho + \ln T + \text{const}$$

And can differentiate to give

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad \text{or} \quad 1 = \frac{d\ln\rho}{d\ln P} + \frac{d\ln T}{d\ln P}$$

And combining this with the equation above gives ….

Schwarzschild condition for occurrence of convection

$$\frac{P}{T}\frac{dT}{dP} = \frac{d\ln T}{d\ln P} > \frac{\gamma - 1}{\gamma}$$

which is the **Schwarzschild** condition for the occurrence of convection (in terms of the temperature gradient).

A gas is convectively unstable if the actual temperature gradient is steeper than the adiabatic gradient. If the condition is satisfied, then large scale rising and falling motions transport energy upwards.

Condition for occurrence of convection

A gas is convectively unstable if the actual temperature gradient is steeper than the adiabatic gradient. The criterion can be satisfied in two ways:

1. The temperature gradient is very steep

   For example, if a large amount of energy is released at the centre of a star, it may require a large temperature gradient to carry the energy away. Hence where nuclear energy is being released, convection may occur.

2. The ratio of specific heats γ is close to unity

   Alternatively, in the cool outer layers of a star, gas may only be **partially ionized**, hence much of the heat used to raise the temperature of the gas goes into ionization and hence the specific heat of the gas at constant V is nearly the same as the specific heat at constant P (because T~const), and γ~1.

   In such a case, a star can have a **cool outer convective layer**. We will come back to the issues of convective cores and convective outer envelopes later.

Condition for occurrence of convection

Convection is an extremely complicated subject, and it is true to say that the lack of a good theory of convection is one of the worst defects in our present studies of stellar structure and evolution.

We know the conditions under which convection is likely to occur but don't know how much energy is carried by convection.

Fortunately, we will see that we can often find occasions where we can manage without this knowledge.

Influence of convection

Let's back to the equations of stellar structure.

Ideally, we would like to know exactly **how much energy is transported by convection** – but lack of a good theory makes it difficult to predict exactly. Fully self-consistent models of stellar convection are an active area of research and require considerable computational resources to accurately capture the three-dimensional fluid dynamics.

However, it can be shown that **even a very small difference between the actual temperature gradient and adiabatic gradient is sufficient to carry all energy**. This suggests that the actual gradient is not greatly in excess of the adiabatic gradient. We can assume that the temperature gradient has exactly the adiabatic value in a convective region in the interior of a star and hence can rewrite the condition of occurrence of convection in the form

$$\frac{d\ln T}{d\ln P} = \frac{\gamma - 1}{\gamma}$$

Thus, the simplest model of convection is to assume that the process is highly efficient – so much so that it drives the system to saturate the Schwarzschild criterion.

Equations of stellar structure in a convective region

Thus, **in a convective region**, we must solve the four differential equations, together with equations for ε and P:

- $\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$
- $\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$
- $\frac{dL}{dm} = \varepsilon$
- $\frac{P}{T}\frac{dT}{dP} = \frac{\gamma - 1}{\gamma}$

The equation for luminosity due to radiative transport is still true:

$$L_{rad} = -\frac{256 \times \pi^2 \sigma_{SB} r^4 T^3}{3\kappa_R} \frac{dT}{dm}$$

And once the other equations have been solved, $L_{\rm rad}$ can be calculated. This can be compared with L (from $dL/dm = \varepsilon$) and the difference gives the value of luminosity due to convective transport $L_{\rm conv} = L - L_{\rm rad}$

In solving the equations of stellar structure, the equations appropriate to a convective region must be switched on whenever the temperature gradient reaches the adiabatic value, and switched off when all energy can be transported by radiation.

Summary

- We have derived the 4th equation to describe the stellar structure and explored the ways to solve these equations.
- As they are not time dependent, we must iterate with the calculation of changing chemical composition to determine short steps in the lifetime of stars. The crucial changing parameter is the H/He content of the stellar core (and afterwards, He burning will become important – to be explored in next lectures).
- We have discussed the boundary conditions applicable to the solution of the equations and made approximations, that do work with real models.
- We have also derived the condition for convection and explored the influence of convection on energy transport within stars. We have shown that it must be considered, but only in areas where the temperature gradient approaches the adiabatic value. In other areas, the energy can be transported by radiation alone and convection is not required. We saw that convection may be important in hot stellar cores and cool outer envelopes, but that a good quantitative theory is lacking.
- The next lectures will explore stellar interiors and the nuclear reactions.

The equations of stellar structure - II

EQUATION OF STATE (EOS)
STELLAR OPACITY

Introduction

- We have 4 differential equations of stellar structure.
- Accurate expressions for pressure, opacity and energy generation are extremely complicated, but we can find simple approximate forms.
- Equations of stellar structure too complicated to find exact analytical solution, hence must be solved with computer.
- Sometimes simplifications can be made to find analytical solutions that still have most of the physics.

The equations of stellar structure

- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$
- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$
- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$
- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma r^2}\frac{\rho(r)\kappa_R(r)}{T^3(r)}L(r)$
- $\frac{P}{T}\frac{dT}{dP} = \frac{\gamma - 1}{\gamma}$

- r = radius
- P = pressure at r
- m = mass of material within r
- ρ = density at r
- L = luminosity at r (rate of energy flow across sphere of radius r)
- T = temperature at r
- κ_R = Rosseland mean opacity at r
- ε = energy release per unit mass per unit time

To these four differential equations we need to add three equations connecting the pressure, the opacity, and the energy production rate of the gas with its density, temperature, and composition:

$P = P(\rho, T,$ chemical composition$)$ ⟶ **usually called the equation of state (EOS)**

$\kappa_R = \kappa_R(\rho, T,$ chemical composition$)$

$\varepsilon = \varepsilon(\rho, T,$ chemical composition$)$

The equation of state (EOS)

- The equation of state (EOS) describes the microscopic properties of stellar matter for given density ρ, temperature T and composition X_i.
- It is usually expressed as the function that relates the pressure P to ρ, T, and mean molecular weight μ at any place in the star.
- Since it is a solely an internal property of the gas, it can, in principle, be computed once externally, and used via a lookup table, i.e., $P_{\text{gas}} = P(\rho, \mu, T)$.

EOS in stars

- We have seen that stellar gas is ionized plasma, and although density is so high that typical inter-particle spacing is of the order of an atomic radius, the effective particle size is more like a nuclear radius (10^5) times smaller.
- Thus, interior of a star contains a mixture of ions, electrons, and radiation (photons). For most stars (except for very low mass stars and stellar remnants), the ions and electrons can be treated as **an ideal gas** and quantum effects can be neglected.

- The net pressure can be divided into three components, pressure from ions, pressure from electrons, and pressure from radiation.

Total pressure: $P = P_{\mathrm{i}} + P_{\mathrm{e}} + P_{\mathrm{rad}} = P_{\mathrm{gas}} + P_{\mathrm{rad}}$

P_{i} is the pressure of the ions
P_{e} is the electron pressure
P_{rad} is the radiation pressure

However, P_{gas} may not obey the ideal gas law due to the effects of degeneracy.

EOS of an ideal gas

The equation of state for an ideal gas is:

$$P_{\text{gas}} = nkT$$

where n is concentration (number of particles per cm^3 = $n_I + n_e$, where n_I and n_e are the number densities of ions and electrons respectively), T is the temperature, k is Boltzmann's constant.

But we want this equation in the form: $P = P\,(\rho, T,$ chemical composition$)$

This can be written as:

$$P_{\text{gas}} = \frac{\rho k T}{\mu m_p} = \frac{\Re \rho T}{\mu} \qquad \text{where} \qquad \Re = \frac{k}{m_p}$$ is the gas constant, and

μ = mean molecular weight,
i.e. the average mass of particles in unit of proton mass m_p.

Mean molecular weight (1)

The mean molecular weight μ (the average mass of particles in unit of proton mass m_p) depends upon the composition of the gas and the state of ionization. For example:

- Neutral hydrogen: $\mu = 1$
- Fully ionized hydrogen: $\mu = 0.5$

An exact solution is complex, depending on fractional ionization of all the elements in all parts of the star.

For simplicity, let's now assume that all of the material in the star is fully ionized. This is justified as hydrogen and helium are most abundant and they are certainly fully ionized in stellar interiors (however, this assumption will break down near stellar surface).

Mean molecular weight (2)

Denote abundances of different elements per unit mass by:

X = fraction of material by mass of H

Y = fraction of material by mass of He

Z = fraction of material by mass of all heavier elements ("metals")

$$X + Y + Z = 1$$

Hence in 1 cm^3 of stellar gas of density ρ, there is mass $X\times(\rho$ of H), $Y\times(\rho$ of He), $Z\times(\rho$ of metals). In a fully ionized gas,

H gives 2 particles per m_H

He gives 3/4 particles per m_H (α particle, plus two e^-)

Metals, average mass Am_H, give ~1/2 particles per m_H

(^{12}C has nucleus plus $6e^-$ = 7/12)

(^{16}O has nucleus plus $8e^-$ = 9/16)

where A is the atomic weight of the species.

Mean molecular weight (3)

If the density of the plasma is ρ, then add up number densities of hydrogen, helium, and metal nuclei, plus electrons from each species:

	H	He	metals
Number density of nuclei	$\frac{X\rho}{m_H}$	$\frac{Y\rho}{4m_H}$	$\frac{Z\rho}{Am_H}$
Number density of electrons	$\frac{X\rho}{m_H}$	$\frac{2Y\rho}{4\,m_H}$	$\frac{A}{2}\times\frac{Z\rho}{Am_H}$

The total number of particles per cm^3 is then given by the sum:

$$n = 2\frac{X\rho}{m_H} + \frac{3}{4}\frac{Y\rho}{m_H} + \frac{2Z\rho + \mathbf{A}Z\rho}{2\mathbf{A}m_H} \approx \frac{\rho}{m_H}\left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right] = \frac{\rho}{\mu\, m_H}$$

...assuming that A≫1

Thus,

$$\mu = \left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]^{-1}$$

Mean molecular weight (4)

$$\mu = \left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]^{-1}$$

This is a good approximation to μ except in cool, outer stellar regions.

For solar abundances, X = 0.73, Y = 0.25, Z = 0.02, and therefore $\mu = 0.60$, i.e. the mean mass of particles in a star of solar composition is a little over half the mass of the proton.

In the central regions of the Sun, about half of the hydrogen has already been converted into helium by nuclear reactions, and as a result X = 0.34, Y = 0.64, and Z = 0.02, giving $\mu = 0.85$.

When Z is negligible: Y = 1 − X; $\mu = 4/(3 + 5X)$

The electron number density n_e plays a considerable role for the properties of the gas. It is convenient to introduce the mean molecular weight per electron, μ_e, such that

$$n_e = \frac{\rho}{\mu_e m_H} \quad \Longrightarrow \quad \mu_e \approx \frac{2}{1+X}$$

Prove it!

The Ionization Fraction

- The accurate calculation of mean molecular weight μ requires knowledge of the chemical composition of the material and the ionization fraction. To calculate ionization fraction, one needs **the Saha equation**, which we will derive later, in the Stellar atmospheres part of this course :

$$\frac{N_1^+}{N_1} = \frac{2g_1^+}{N_e g_1} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{ion}/kT}$$

  where m_e is the mass of the electron, χ_{ion} is the ionization energy, N_1^+ and N_1 are the number density of ions and neutral atoms in their ground state, N_e is the electron number density , g_1^+ and g_1 are the statistical weight of the ground state of the ion and neutral atom.

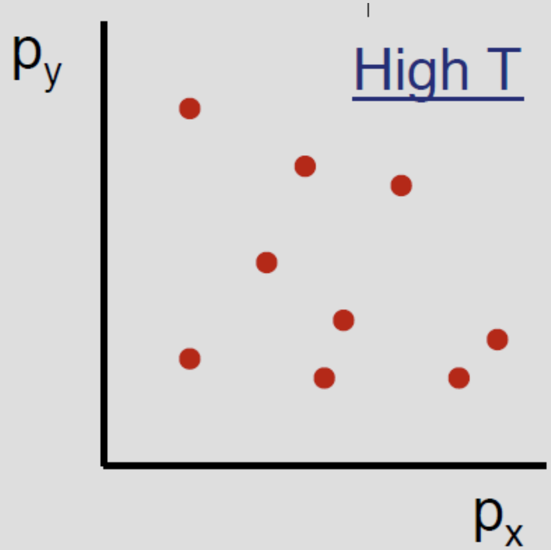
- In general, the Saha equation can be used to compute ionization fractions over most of the star. It does, however, require that the gas be in the thermodynamic equilibrium. This is true throughout almost the whole star, as at high densities, collisions will control the level populations. This approximation only breaks down in the solar corona, where the densities become very low.
- However, the Saha equation also breaks down in the centers of stars, where high densities cause the ionization energies of atoms to be reduced. Indeed, if the mean distance between atoms is d, then there can be no bound states with radii greater than $\sim d/2$). In practice, the Saha equation begins to break down at nuclear distances of $\sim 10a_0$ ($\sim$10 Bohr radii).
- To correct for this effect, the Saha equation is normally used until it begins to show decreasing ionization fractions toward the center of the star. When this happens, complete ionization is assumed.

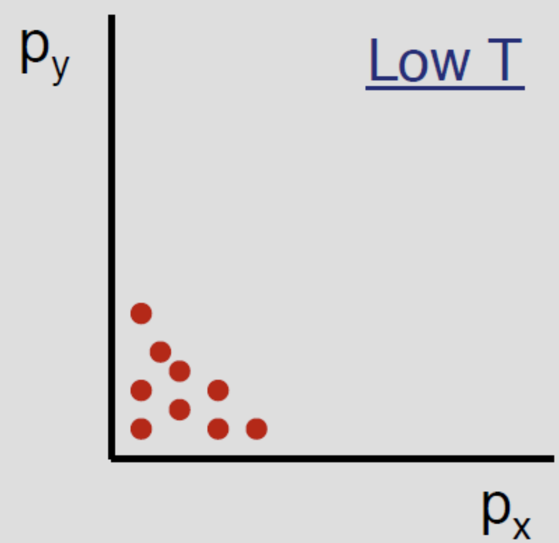
Degeneracy Pressure (1)

- An ordinary classical gas: $P_{gas} \propto T \rightarrow 0$ as $T \rightarrow 0$
- Simultaneously, the mean speed of particles in the gas also goes to zero:

$$v = \sqrt{2kT/m}$$

- The momenta are given by: $p_x = mv_x$; $p_y = mv_y$; $p_z = mv_z$
- ... if we plot the momenta of particles in a 3D space of p_x, p_y, and p_z then as T decreases the particles become concentrated near the origin:





Degeneracy Pressure (2)

At low enough temperatures / high enough densities, the concentration of particles with similar (low) momenta would violate the Pauli exclusion principle:

No two electrons can occupy the same quantum state

i.e. have the same momentum, spin, and location.

To avoid violating the exclusion principle, electrons in a dense, cold gas **must have larger** momenta than we would predict classically.

Since the pressure P is mean rate of transport of momentum across unit area

$$P = \frac{1}{3}\int_0^\infty vpn(p)\,dp$$

...where $n(p)dp$ is the number of particles with momentum between p and $p+dp$

... larger momentum means higher pressure.

This quantum mechanical source of pressure is **degeneracy pressure**.

Degeneracy Pressure (3)

This **quantum mechanical** source of pressure is **degeneracy pressure**.
We will discuss it later.

$$P_{ideal} = \frac{k}{\mu m_p} \rho T$$

- **Non-relativistic degeneracy pressure** (speeds v ≪ c) **:**

$$P_{\text{deg}} = K_1 \rho^{5/3} = K_1 \rho^{1+\frac{1}{n}}$$

← A polytrope of index n=1.5

  - K_1 is constant
  - Does not depend upon temperature for low enough T
  - Depends upon composition via the relation between N_e and ρ

- **Relativistic degeneracy pressure :**

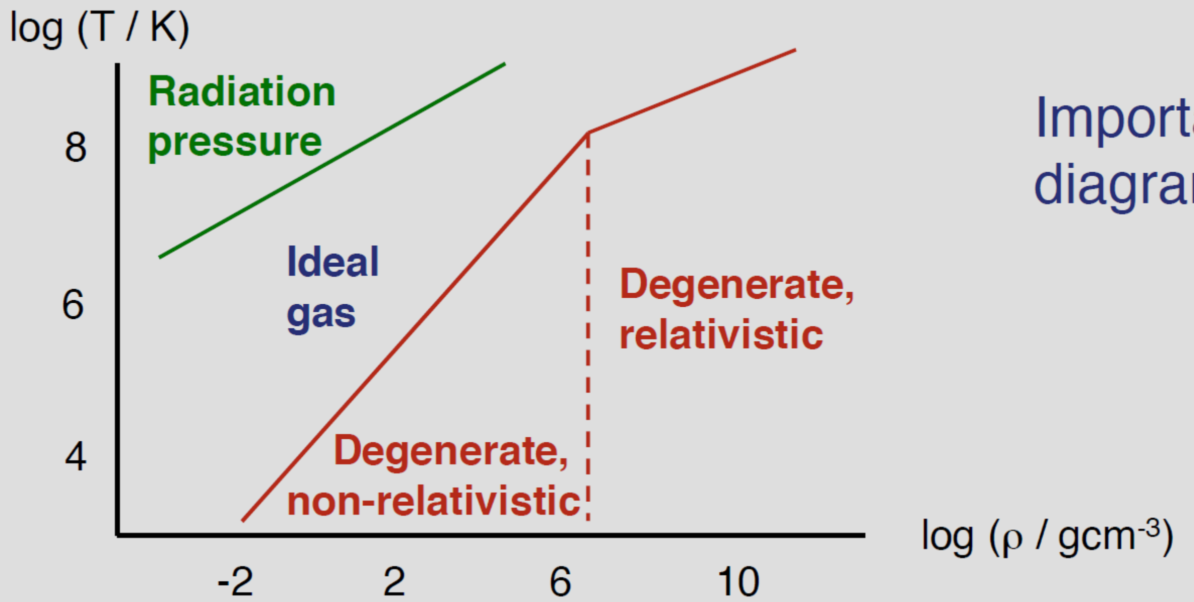
$$P_{\text{deg}} = K_2 \rho^{4/3}$$

← A polytrope of index n=3

  - K_2 is another constant
  - Equation of state for relativistic degenerate matter, which applies at high density. This is a "softer" equation of state, since P rises more slowly with increasing density than for the non-relativistic case.

A relation of the form $P = K\rho^{1+\frac{1}{n}}$ where K and n are constants is called a polytropic relation, and n is the polytropic index.

When do the different pressures matter?



Important diagram!

Different types of star occupy different portions of the plane:

- Solar-type stars - ideal gas throughout
- Massive stars - radiation pressure
- White (and brown) dwarfs - non-relativistic degeneracy pressure

Relativistic degeneracy implies an **unstable** equation of state, so **no stable stars** in that part of the plane.

Radiation Pressure

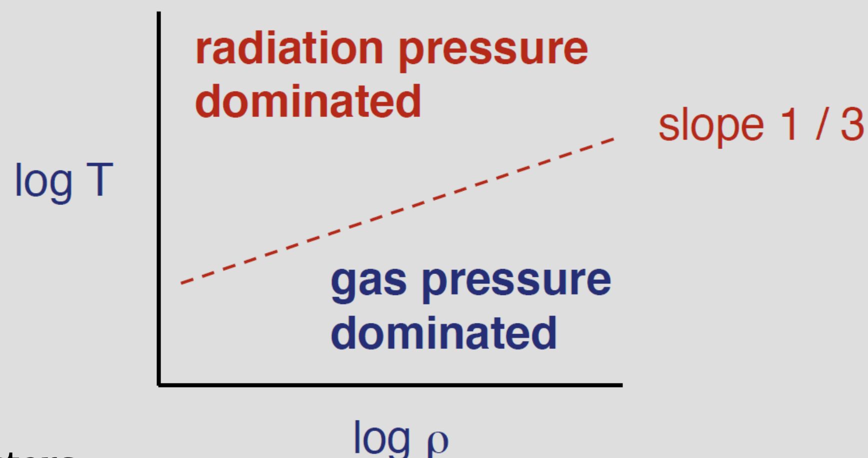
We have already showed before that radiation pressure can be neglected for Solar-type stars:

$$\frac{P_{rad}}{P_g} = \frac{aT^4}{3} \Big/ \frac{kT\rho}{\mu m_p} = \frac{\mu a}{3\Re}\frac{T^3}{\rho} \approx 10^{-4} \text{ (for the Sun)}$$

But becomes **very important** for early-type stars due to the T^4 sensitivity.

In which stars are gas and radiation pressure important?

$$\left.\begin{aligned} P_{rad} &= \frac{aT^4}{3} \\ P_g &= \frac{\Re T\rho}{\mu} \end{aligned}\right\} \text{ equal when } T^3 = \frac{3\Re}{a\mu}\rho$$



From the virial theorem (see Lecture 3):

$$\bar{T} \propto \frac{M}{R} \Rightarrow \frac{P_{rad}}{P_g} \propto M^2$$

i.e. $P_{\rm rad}$ becomes more significant in higher mass stars.

Effect of radiation pressure

For stars in which radiation pressure plays a non-negligible role we can write the generalized form of the equation of hydrostatic support (Lecture 2):

$$\frac{dP(r)}{dr} + g\rho(r) = a\rho(r)$$

Then

$$\frac{dP(r)}{dr} = -g\rho(r) - \frac{dP_{rad}}{dr} = -g_{\text{eff}}(r)\rho(r)$$

From Lecture 6 (slide 169):

$$\frac{dP_{rad}}{dr} = -\frac{\rho\kappa_R}{c}F \quad \Longrightarrow \quad g_{\text{eff}}(r) = g - \frac{\kappa_R}{c}F$$

Consider relative contributions of radiation and (ideal) gas pressures:

$$P_g = \beta P = \frac{\Re T\rho}{\mu}, \quad P_{rad} = (1-\beta)P = \frac{aT^4}{3}$$

Exclude temperature: $P = \left[\left(\frac{\Re}{\mu}\right)^4 \frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3} \rho^{4/3} \quad \Longrightarrow \quad P = K\rho^{1+\frac{1}{n}}$ ⟵ A polytrope of index n=3

Opacity (1)

Concept of opacity was introduced when deriving the equation of radiation transport.
It will be discussed extensively in the Stellar atmospheres part of this course.
Opacity is the resistance of material to the flow of radiation through it. In most stellar interiors it is determined by all the processes which scatter and absorb photons.

Four main processes:

- Bound-bound absorption:
  - is related to photon-induced transitions of a (bound) electron in atoms or ions to a higher energy state by the absorption of a photon. The atom is then de-excited either spontaneously or by collision with another particle, whereby a photon is emitted. Although this is limited to certain transition frequencies, the process can be efficient because the absorption lines are strongly broadened by collisions.
- Bound-free absorption:
  - which is another name for photoionization - the removal of an electron from an atom (ion) caused by the absorption of a photon. The inverse process is radiative recombination.
- Free-free absorption:
  - the absorption of a photon by a free electron, which makes a transition to a higher energy state by briefly interacting with a nucleus or an ion. The inverse process, leading to the emission of a photon, is known as bremsstrahlung.
- [Electron] Scattering:
  - the scattering of a photon by a free electron, the photon's energy remaining unchanged (known as Thomson scattering).

Which process is most important in the deep stellar interiors?

Opacity (2)

Which process is most important in the deep stellar interiors?

- Temperatures are very high there, thus the last two processes are dominant, simply because the material is almost completely ionized, there are very few bound electrons.
- Furthermore, the energy of most photons in the Planck distribution is of the order of keV, whereas the separation energy of atomic levels is only a few tens eV. Hence most photons interacting with bound electrons would set them free. Thus bound-bound (and even bound-free) transitions have extremely low probabilities, interactions occurring predominantly between photons and free electrons.
- Thus, the most dominant process in the deep stellar interiors is electron scattering.

The opacity per unit mass of material in this case is $\kappa_e = \frac{n_e \sigma_T}{\rho}$,

where $\sigma_T = 6.65\times10^{-25}$ cm^2 is the Thomson cross section. The opacity is therefore:

$$\kappa_e \cong 0.2(1 + X)\ \mathrm{cm}^2\mathrm{g}^{-1}$$

$$n_e = \frac{\rho}{\mu_e m_H}$$

The opacity resulting from electron scattering is temperature and density **independent**!

Opacity (3)

We need an expression for opacity to solve the equations of stellar structure. For stars in thermodynamic equilibrium with only a slow outward flow of energy, the opacity should have the form $\kappa = \kappa(\rho, T, \text{chemical composition})$

Opacity coefficients may be calculated, taking into account all possible interactions between the elements and photons of different frequencies. This requires an enormous amount of calculation and is beyond the scope of this course. Such calculation are done e.g. by the OPAL opacity project at Lawrence Livermore National Laboratory.

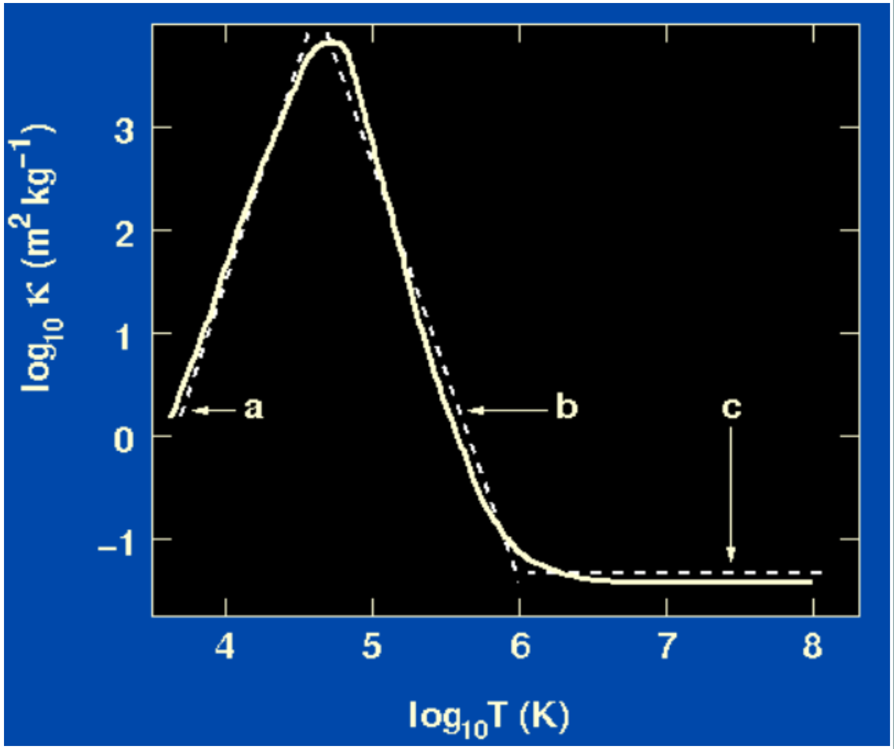
When it is done, the results are usually approximated by the relatively simple formula:

$$\kappa = \kappa_0 \rho^{\alpha} T^{\beta}$$

where α, β are slowly varying functions of density and temperature and κ_0 is a constant for a given chemical composition.

Opacity (4)

Figure shows opacity as a function of temperature for a star of given ρ (10^{-4} g cm^{-3}). Solid curve is from detailed opacity calculations. Dotted lines are approximate power-law forms.



- At high T: κ is low and remains constant. Most atoms are fully ionized, high photon energy, hence free-free absorption unlikely. Dominant mechanism is electron scattering, independent of T, $\alpha=\beta=0$:

$$\kappa = \kappa_1 = \sigma_T/m_H\mu_e \quad \text{(curve c)}$$

- Opacity is low a low T, and increases towards higher T. Most atoms are not ionized, few electrons available to scatter photons or for free-free absorption. Approximate analytical form is $\alpha=1/2$, $\beta=4$:

$$\kappa = \kappa_2\rho^{0.5}T^4 \quad \text{(curve a)}$$

- At intermediate T, κ peaks, when bound-free and free-free absorption are very important, then decreases with T (Kramers opacity law):

$$\kappa = \kappa_3 T^{-7/2} \quad \text{(curve b)}$$

- κ_1, κ_2, κ_3 are constants for stars of given chemical composition but all depend on composition.

Summary and conclusions

- We have learned the approximate forms of the equation of state and the opacity.
- Next lecture: A method of simplifying the solution of the stellar structure equations.
- After that we will discuss nuclear reactions and move on to discussing the output of full numerical solutions of the equations and realistic predictions of modern theory

Nuclear Energy Production

Introduction

- We have seen that the 4 equations of stellar structure must be supplemented with expressions for P, κ, ε.
- We have discussed that
  - P is given by the equation of state of the stellar matter.
  - κ is determined by the atomic physics of the stellar material.
- The last function we still need to describe is the power density $\varepsilon = \varepsilon(\rho, T,$ chemical composition$)$.
- In Lecture 4 we have concluded that the energy source behind ε must be nuclear burning.
- Thus, ε is defined by the nuclear energy source in the interiors. We need to develop a theory and understanding of nuclear physics and reactions.

We now move from the studies of a stellar body to studies of its "soul" (V.V. Ivanov).

How to compute ε?

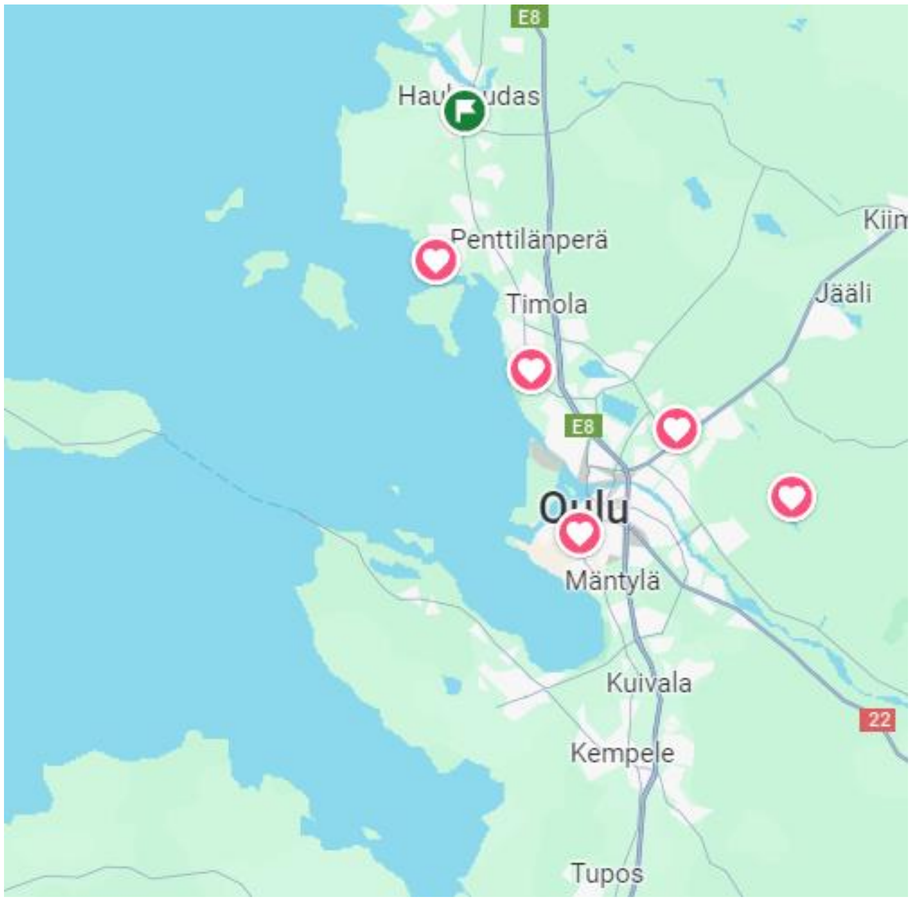
- Our goal is to compute the rate of energy generation ε per unit mass per unit time.
- The computation can be separated into three parts:
  - the cross section for a reaction between a pair of nuclei, which is determined predominantly by the properties of the nuclei;
  - the amount of energy generated per reaction, which again is a property of the nuclei;
  - the total reaction rate which, beside the cross section, also depends on the statistics of the motion of the nuclei.
- An additional consequence of the nuclear processes is a gradual change of the chemical composition, which controls the evolution of the star. Hence, we must determine the rate of change of the abundances.

Nuclear dimensions

- A convenient unit for measuring the radius of an atom is the Bohr radius, $\sim 0.5\times10^{-10}$ m = 0.5 Å.
- Atomic radii are typically a few to a few tens of Bohr radii, or of order 10^{-10} m, or a few Å.
- Proton radius = 0.83×10^{-15} m ≈ 10^{-13} cm = 10^{-5} Å.
- The radius R of a nucleus containing A nucleons, i.e. having an atomic mass number of A, is $R \approx r_0 A^{1/3}$, where $r_0 = 1.2\times10^{-15}$ m.
- Nuclei are therefore of order a few $\times 10^4$ times smaller in radius than an atom.

- How compact the nucleus compared to the atom within which it resides?

Nuclear dimensions



~40 km

~1 m

Thus, atoms are very empty, but nuclei are very compact and incompressible dense.

Nuclear processes in stars

- The most important series of fusion reactions are those converting hydrogen into helium (H-burning). As we will see later, this dominates ~90% of lifetime of nearly all stars.
- Mass of nuclei with several protons and/or neutrons does not exactly equal mass of the constituents - slightly smaller because of the binding energy of the nucleus.
- Hence, there is a decrease of mass, and from the Einstein mass-energy relation **$E=mc^2$** the mass deficit is released as energy.

Example: $4\times{}^1H \rightarrow {}^4He$

4 protons × each of mass 1.0078 atomic mass units: 4.0313 amu

mass of helium nucleus: 4.0026 amu

Mass difference: 0.0287 amu = 4.76×10^{-26} g

$$\Delta E = \Delta M c^2 = 4.278\times10^{-5}\ \text{erg} = 26.7\ \text{MeV}$$

- Since binding energy differs for different nuclei, it can release or absorb energy when nuclei either fuse or fission.

The binding energy

- General calculation: define the binding energy of a nucleus as the energy required to break it up into constituent protons and neutrons.
- Suppose nucleus has:
  - Proton number Z
  - N neutrons
  - Atomic mass number A (number of protons + neutrons, $Z+N$), or the baryon number, or nucleon number, or nuclear mass
  - Nucleus mass $m_{nuc} = m(Z,N)$
- Binding energy of the nucleus (also known as the difference between the masses of nucleons and nucleus) is:

$$Q(Z,N) \equiv [Zm_p + Nm_n - m(Z,N)]c^2$$

- Some useful units:
  - 1 eV $=1.602\times10^{-12}$ erg $=11.65\times10^{3}$ K
  - 1amu $=1/12\ m(^{12}C) = 931.49$ MeV$/c^2 = 1.660\times10^{-24}$ g
  - $m_e c^2 = 0.511$ MeV
  - $m_p = 1.007825$ amu $m_p c^2 = 938.72$ MeV
  - $m_n = 1.008665$ amu $m_n c^2 = 939.56$ MeV
  - $m(^4He) = 4.002603$ amu

The binding energy per nucleon (1)

- For our purposes, a more useful quantity for considering which nuclear reactions yield energy is the total binding energy per nucleon:

$$q = \frac{Q(Z,N)}{A} \qquad {}^{A}_{Z}X$$

- We can then consider this number relative to the hydrogen nucleus.
- For reaction $4\times{}^{1}\mathrm{H} \rightarrow {}^{4}\mathrm{He}$

$$q = \frac{28.30}{4} = 7.07\ \mathrm{MeV}$$

- This corresponds to 7.07 MeV / 931 MeV = 0.7% of the rest mass converted to energy when hydrogen burns. The time for the Sun to radiate away just 10% of the energy available from this source is

$$\tau_{nuc} = \frac{0.1 \times 0.007 \times M_{\odot} \times c^2}{L_{\odot}} \approx 10^{10}\ \mathrm{yr}$$

  In terms of energy budget, hydrogen fusion can easily produce the solar luminosity over the age of the Solar System.

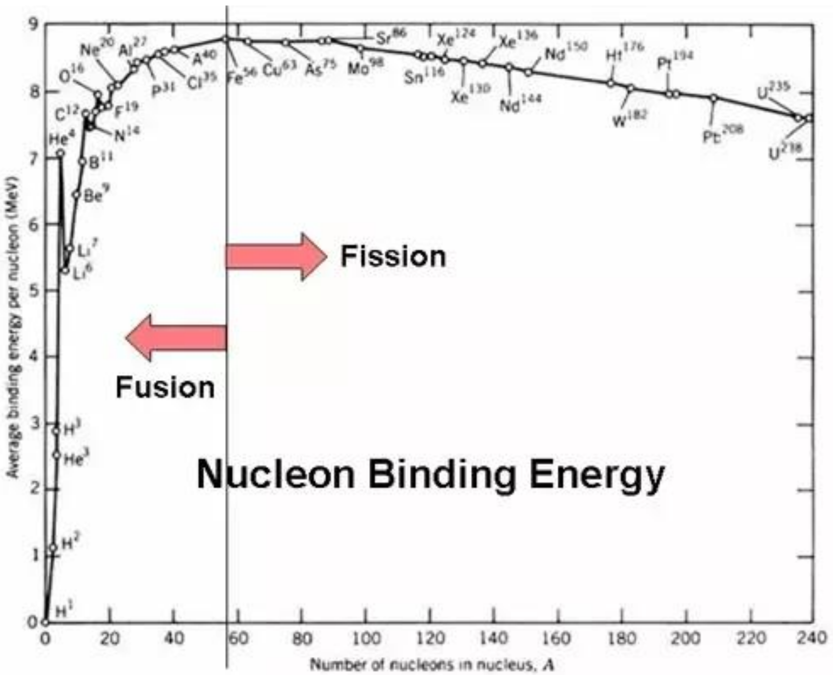
- Compare with fusion of hydrogen to iron ${}^{56}\mathrm{Fe}$: q=8.8 MeV per nucleon. Most of this is already obtained in forming helium (7.07 MeV).

The binding energy per nucleon (2)

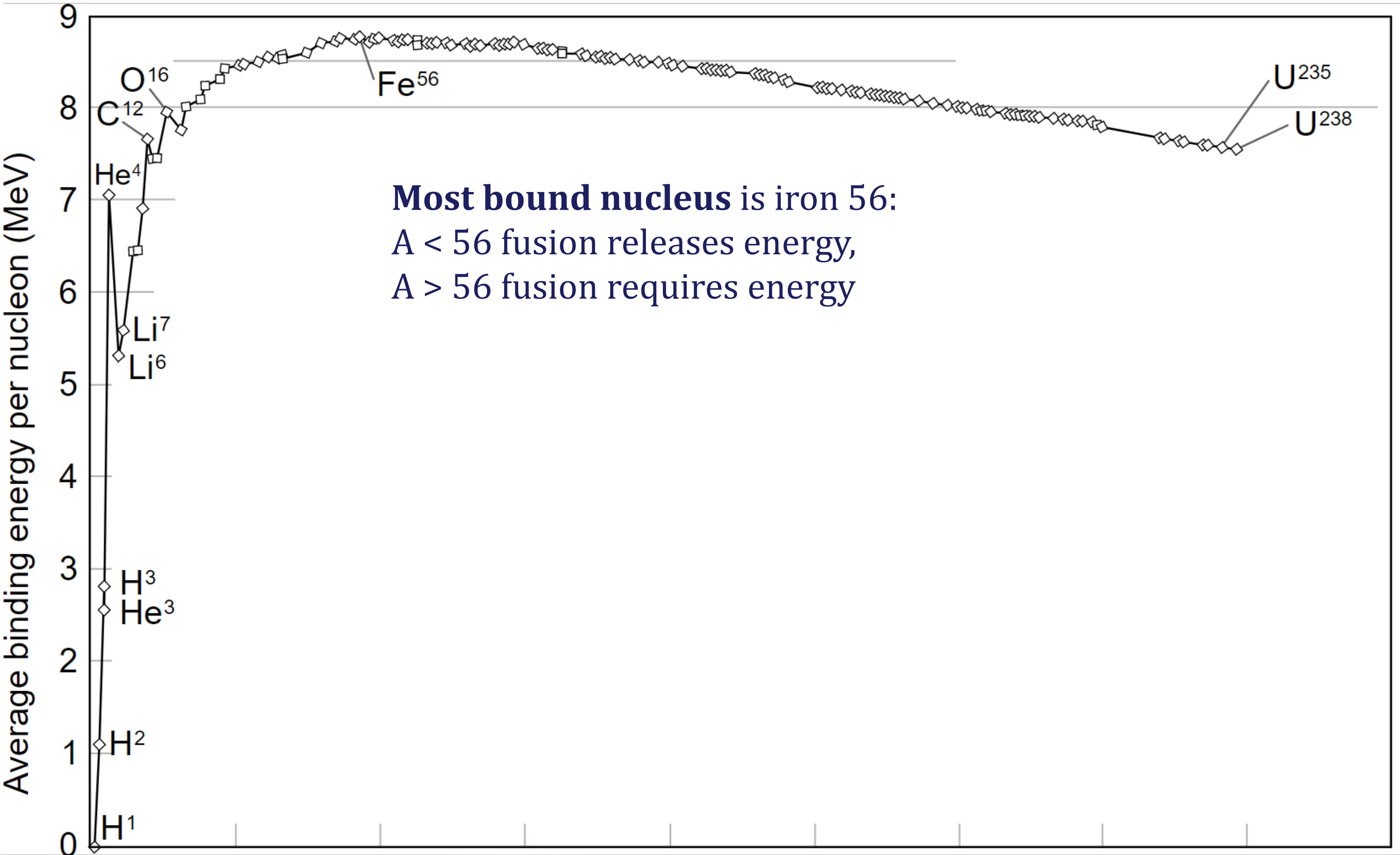
On average, the binding energy per nucleon is about $q \approx 8$ MeV/nucleon

More accurately it depends on the baryon number A.

- General trend is an increase of q with atomic mass up to $A = 56$ (Fe). Then slow monotonic decline.
- There is steep rise from H through ^{2}H, ^{3}He, to ^{4}He $\rightarrow$ fusion of H to He should release larger amount of energy *per unit mass* than say fusion of He to C.
- Energy may be gained by *fusion* of light elements to heavier, up to iron
- Or from *fission* of heavy nuclei into lighter ones down to iron.



The binding energy per nucleon (3)



Most bound nucleus is iron 56:
A < 56 fusion releases energy,
A > 56 fusion requires energy

Occurrence of fusion reactions (1)

Let's now discuss the conditions under which these fusion reactions can occur and whether such conditions exist in stellar interiors.

Consider two nuclei with proton numbers Z_1 and Z_2. Nuclei interact through four forces of physics, but only electromagnetic and strong nuclear important here:

- Two positively charged nuclei repel each other and must overcome a repulsive Coulomb potential

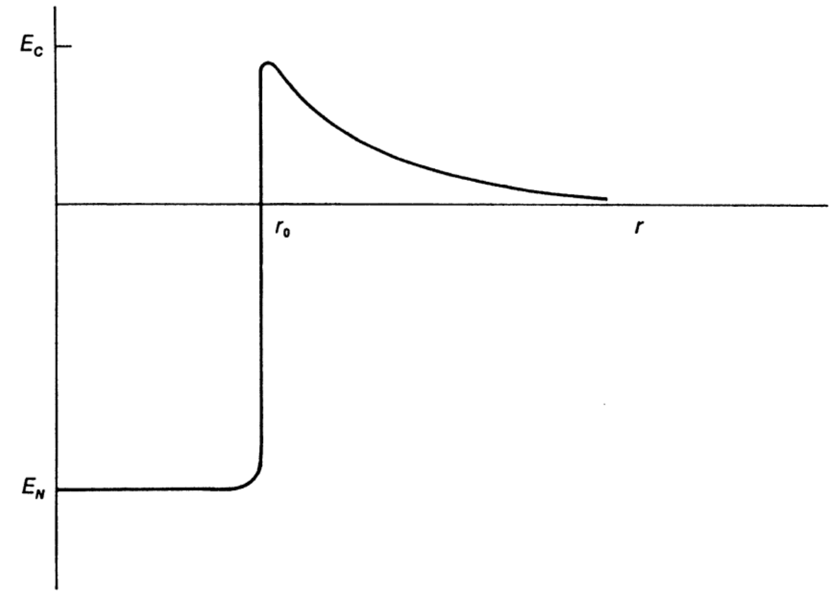
$$V_C = \frac{Z_1 Z_2 e^2}{r} = 1.44 \frac{Z_1 Z_2}{r[\mathrm{fm}]} \text{ MeV}$$

$r[\mathrm{fm}]$ – r in fermis ($\times 10^{-13}$ cm)

- to reach separation distances where strong force dominates (typical size of nucleus)

$$r_0 \approx 1.44 \times 10^{-13} A^{1/3} \text{ cm}$$

A – Atomic mass number



Schematic potential energy between two nuclei. For $r < r_0$ the attractive nuclear forces dominate; for $r > r_0$ Coulomb repulsion dominates.

At $r = r_0$, height of the Coulomb barrier is:

$$V_0 = \frac{Z_1 Z_2 e^2}{r_0[fm]} = Z_1 Z_2 \text{ MeV}$$

i.e. of the order of 1 MeV for two protons…

Femtometer [**fm**] (derived from the Danish and Norwegian word *femten* 'fifteen' equal to 10^{-15} m. This distance is also called **fermi**.

Classical approach

At $r = r_0$, height of the Coulomb barrier is $V_0 = Z_1 Z_2$ MeV, of the order of 1 MeV for two protons.

Class task:

Compare this to the average kinetic energy of a particle ($3kT/2$). What T do we require for fusion? How does this compare with the minimum mean T of the Sun? Any comments on these two temperatures?

- At a typical internal stellar temperature of 10^7 K, the kinetic energy of a nucleus is $3kT/2 \sim 1$ keV. The characteristic kinetic energy is thus of order 10^{-3} of the energy required to overcome the Coulomb barrier.
- Typical nuclei will approach each other only to a separation $r \sim 10^{-10}$ cm, 1000 times larger than the distance at which the strong nuclear binding force operates.
- Can nuclei that are in the high-energy tail of the Maxwell–Boltzmann distribution overcome the barrier? The fraction of nuclei with such energies is

$$e^{E/kT} \approx e^{-1000} \approx 10^{-434}$$

- Even considering the high-energy tail of the Maxwell-Boltzmann distribution, the fraction of particles with $E > E_C$ is **vanishingly small**.
- With purely classical considerations nuclear reactions have **no chance of happening** at such temperatures.

Boltzmann's constant $k = 1.38 \times 10^{-16}$ cm^2 g s^{-2} K^{-1} = 8.62×10^{-5} eV K^{-1}

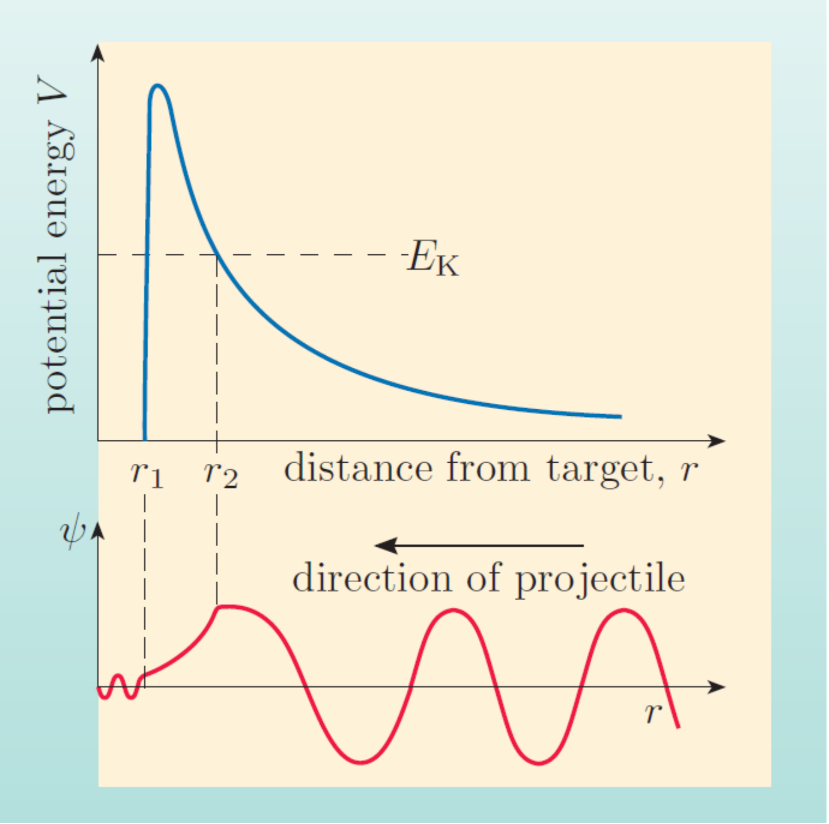
Occurrence of fusion reactions (2)

At $r = r_0$, height of the Coulomb barrier is $V_0 = Z_1 Z_2$ MeV, i.e. of the order of 1 MeV for two protons…

If we imagine what will happen as the nuclei approach one another with a certain kinetic energy E_K then, ignoring quantum mechanics, they will simply come to rest temporarily when their kinetic energy has been converted into electrical potential energy, before 'bouncing' apart again.

The distance of least separation is given by

$$r_2 = \frac{Z_1 Z_2 e^2}{E_K}$$



Thus, **classically**, a particle with kinetic energy E_K cannot get closer to the origin than $r = r_2$ because of the potential barrier of height $V > E_K$. **Quantum-mechanically**, however, the wave function shows that there is a **non-zero probability** of finding the particle beyond the barrier, i.e. at $r < r_1$.

Quantum tunnelling

- We need to turn to quantum mechanics to see how nuclear reactions are possible at stellar temperatures.
- As was discovered by George Gamow (1928), there is a finite non-zero probability for a particle to penetrate the repulsive Coulomb barrier even if $E_K \ll V_C$ as if "tunnel" existed.
- For the Coulomb barrier, the penetration probability can be expressed in terms of the particle energy $E=E_K$, and the Gamow energy E_G which depends on the atomic number (and therefore charge) of the interacting nuclei, and hence the size of the Coulomb barrier:

$P_{pen} = g(E) \approx e^{-\sqrt{E_G/E}}$ with the Gamow energy $E_G = 2m_r c^2(\pi\alpha Z_1 Z_2)^2$, where $m_r = m_1 m_2/(m_1+m_2)$ is the reduced mass of particles, α is the fine structure constant ≈ 1/137.

  - P_{pen} which is also called the Gamow factor $g(E)$ increases rapidly with E
  - P_{pen} decreases with $Z_1 Z_2$ - lightest nuclei can fuse more easily than heavy ones
  - Higher energies / temperatures needed to fuse heavier nuclei, so different nuclei burn in well-separated phases during stellar evolution.

Quantum tunnelling for two protons

$$E_G = 2m_r c^2 (\pi \alpha Z_1 Z_2)^2$$

$$P_{pen} \approx e^{-\sqrt{E_G/E}}$$

$m_r = m_1 m_2/(m_1+m_2)$ is the reduced mass of particles, α is the fine structure constant ≈ 1/137.

- **Class task:** calculate E_G for two protons and find P_{pen} for the typical kinetic energy of particles in the Sun's core, E ~ 1 keV.
  It is convenient to remember that the rest energy of a proton, $m_p c^2$, is 938 MeV.

  - For two protons, $E_G \approx 500$ keV
  - $P_{pen} \approx e^{-22} \approx 2 \times 10^{-10}$. It is very small, but considerably larger than the classical probability.

Reaction cross-section

- Even if tunneling occurs, and two nuclei are within the strong force's interaction range, the probability of a nuclear reaction will still depend on a nuclear cross section, which will generally depend inversely on the kinetic energy. Thus, the total cross-section for a nuclear reaction can be written as

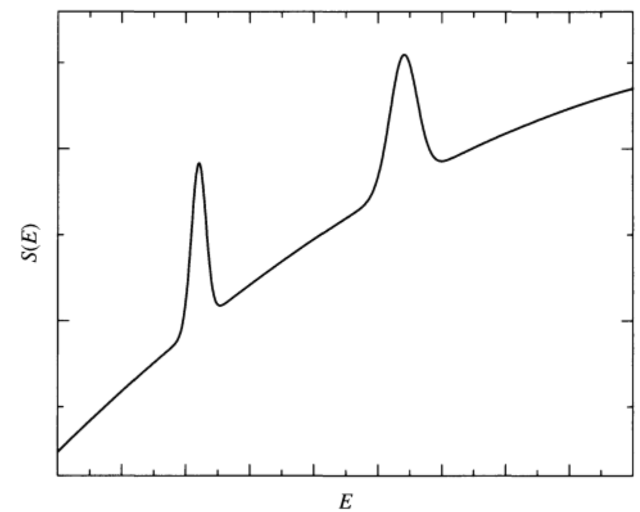
The cross-section is defined as

$$\sigma = \frac{\text{number of reactions per sec}}{\text{number of incident particles per sec per cm}^2}$$

$$\sigma = \frac{S(E)}{E} g(E) = \frac{S(E)}{E} e^{-\sqrt{E_G/E}}$$

where $S(E)$ is a **slowly varying** S-factor:

- $S(E)$ contains all remaining effects, i.e. the intrinsic nuclear properties of the reaction including possible resonances.
- Its evaluation requires laboratory data.
- Generally, it is insensitive to particle energy or velocity.
- In some cases, however, $S(E)$ can vary quite rapidly, peaking at specific energies. These energies correspond to energy levels within the nucleus, analogous to the orbital energy levels of electrons. It is a resonance between the energy of the incoming particle and differences in energy levels within the nucleus that accounts for these strong peaks.



The speed-averaged cross-section

- The number of reactions per second in a unit volume is

$$r_{ij} = \frac{n_i n_j}{1 + \delta_{ij}} \sigma v$$

where n_i and n_j are number densities of particles, and v is a relative velocity.
If a reaction between identical particles is considered (i.e., protons on protons) then r needs to be divided by 2, to avoid double counting , i.e. $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$

- In reality, the nuclei in a gas will have a distribution of velocities (a range of kinetic energies), so every velocity has some probability of occurring. Hence (after some algebra, which we will not describe)

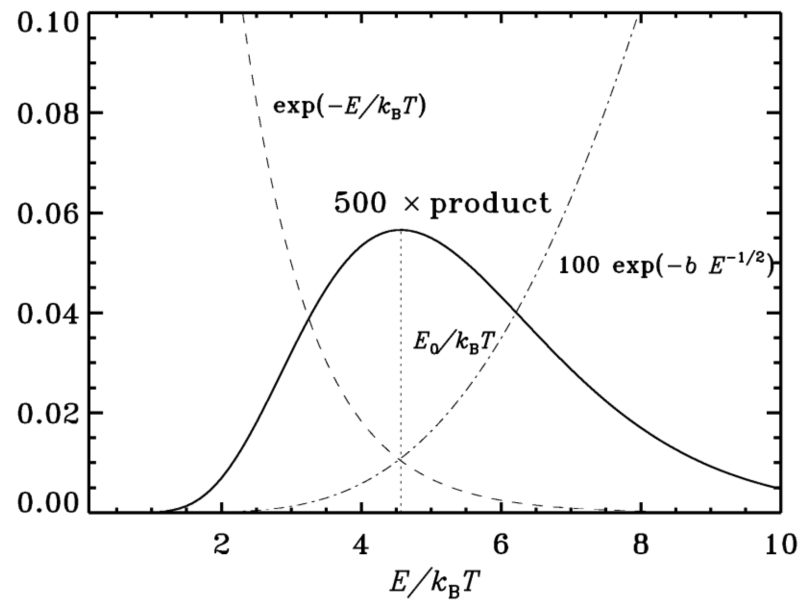
$$\langle \sigma v \rangle = \left(\frac{8}{\pi m_r} \right)^{1/2} \frac{S(E_0)}{(kT)^{3/2}} \int_0^{\infty} e^{-E/kT} e^{-\sqrt{E_G/E}} dE$$

- The integrand in this expression, $f(E) = e^{-E/kT} e^{-\sqrt{E_G/E}}$, is composed of the product of two exponential functions, one (from the Boltzmann distribution) falling with energy, and the other (due to the Gamow factor embodying the Coulomb repulsion) rising with energy. Obviously, $f(E)$ will have a narrow maximum at some energy E_0, at which most of the reactions take place:

$$E_0 = \left(\frac{kT}{2} \right)^{2/3} E_G^{1/3}$$

The Gamow peak

- The Gamow peak is the product of the Maxwellian distribution and tunnelling probability. The area under the Gamow peak determines the reaction rate.
- The higher the electric charges of the interacting nuclei, the greater the repulsive force, hence the higher the E_k and T are needed for reaction to occur.
- Highly charged nuclei are obviously the more massive, so reactions between light elements occur at lower T than reactions between heavy elements.



The Boltzmann probability distribution for $kT = 1$ keV, the Gamow factor for the case of two protons, with $E_G = 500$ keV, and their product. Scaled up by large factors for display purposes.

Nuclear Reaction Rates

$Q \equiv [Zm_p + Nm_n - m(Z,N)]c^2$

$n_i = \frac{X_i \rho}{A m_H}$

$r_{ij} = \frac{n_i n_j}{1 + \delta_{ij}} \langle \sigma v \rangle$

- Each reaction releases an amount of energy Q_{ij}, so then $Q_{ij} r_{ij}$ is the energy generated per unit volume and per second. The energy generation rate per unit mass from the reaction between nuclei of type *i* and *j* is then

$$\varepsilon_{ij} = \frac{Q_{ij} r_{ij}}{\rho} = \frac{Q_{ij}}{(1 + \delta_{ij}) A_i A_j m_H^2} \rho X_i X_j \langle \sigma v \rangle_{ij}$$

- The final expression for the power density due to a given nuclear reaction:

$$\varepsilon_{ij} = \frac{2^{\frac{5}{3}} \sqrt{2}\, Q_{ij}}{(1 + \delta_{ij}) \sqrt{3}} \frac{\rho X_i X_j}{m_H^2 A_i A_j \sqrt{m_r}} S_0 \frac{E_G^{1/6}}{(kT)^{2/3}} \exp\left[-3 \left(\frac{E_G}{4kT}\right)^{\frac{1}{3}}\right]$$

where $S_0 = S(E_0)$.

- The total power density at a point in a star with a given temperature, density, and abundance will be the sum of the power densities due to all the possible nuclear reactions, each described by this equation.
- Because of the exponential term in the Eqn, there will be a strong preference for reactions between species with low atomic number, and hence small E_G.
- Furthermore, the higher the Gamow energy, the more strongly will the reaction rate depend on temperature.

Timescale of a nuclear reaction

- The typical timescale of a nuclear reaction is inversely proportional to the reaction rate r_{ij}. The mean time it takes for a particular nucleus of type i to undergo fusion with a nucleus of type j is

$$\tau_i = \frac{n_i}{(1 + \delta_{ij})\, r_{ij}}$$

- The extremely high sensitivity of nuclear reaction rates to temperature leads to the concept of "ignition" of a nuclear fuel: each reaction (or nuclear process) has a typical narrow temperature range over which its rate increases by orders of magnitudes, from negligible values to very significant ones.
- Around this range, the temperature dependence of the reaction rate may be well approximated by a power law (with a high power) and an ignition or threshold temperature may be defined. Hence, the equation of the power density ε should characteristically have

$$\varepsilon \approx \varepsilon_0 (T/T_0)^n$$

H-burning: n = 5÷15; He-burning n = 40.

- The process of creation of new nuclear species by fusion reactions is called nucleosynthesis. Since the kinetic energy of particles is that of their thermal motion, the reactions between them are called thermonuclear.

Electron shielding (1)

- We found that the repulsive Coulomb force between nuclei plays a crucial role in determining the rate of a thermonuclear reaction. In our derivation of the cross-section, we have ignored the influence of the surrounding free electrons, which provide overall charge neutrality in the gas.
- In a dense medium, the attractive Coulomb interactions between atomic nuclei and free electrons cause each nucleus to be effectively surrounded by a cloud of electrons. This electron cloud reduces the Coulomb repulsion between the nuclei at large distances, and may thus increase the probability of tunneling through the Coulomb barrier. This effect is known as **electron screening** or **electron shielding**.

Electron shielding (2)

- According to the weak screening approximation, which applies to relatively low densities and high temperatures such as found in the centre of the Sun and other main-sequence stars, clouds of negatively charged electrons can increase r_{jk} by about 10%.
- The description of electron screening becomes complicated at high densities and relatively low temperatures, where the weak screening approximation is no longer valid. A general result is that with increasing strength of electron screening, the temperature sensitivity of the reaction rate diminishes, and the density dependence becomes stronger. At very high densities, $\rho > 10^6$ g/cm^3, the screening effect is so large that it becomes the dominating factor in the reaction rate.

Nuclear reactions in stellar interiors

ENERGY GENERATION
PP-CHAINS
CNO-CYCLE
HELIUM BURNING
CARBON BURNING AND BEYOND
IRON AND HEAVIER ELEMENTS
COMPOSITION CHANGES

Notations for nuclear reactions

The general description of a nuclear reaction is

- $I(A_i, Z_i) + J(A_j, Z_j) \rightarrow K(A_k, Z_k) + L(A_l, Z_l)$
- $I(A_i, Z_i) + J(A_j, Z_j) \rightarrow K(A_k, Z_k) + L(A_l, Z_l) + e^+ + \nu$
- $I(A_i, Z_i) + J(A_j, Z_j) \rightarrow K(A_k, Z_k) + L(A_l, Z_l) + \gamma$

e^+ – positron, γ – photon, ν – neutrino

Recall that in any nuclear reaction the following must be conserved:

- The baryon number – heavy particles (protons, neutrons and their anti-particles)
- The lepton number – light particles (electrons, positrons, neutrinos, and antineutrinos)
- Charge

Note also that the anti-particles have the opposite baryon/lepton number to their particles.

Examples of nuclear reactions

Deuterium is a stable isotope of hydrogen, which, unlike "normal" hydrogen atoms, also contains a neutron. The nucleus of a deuterium atom, called a deuteron, contains one proton and one neutron.

1. The first, and most obvious reaction is

$$^{1}H + {}^{1}H \rightarrow {}^{2}D + e^{+} + \nu$$

S_0 factor for this reaction is 22 (!) orders of magnitude smaller than that for other reactions. It proceeds via weak, not strong, interaction. Therefore, reaction proceeds extremely slowly.
Note that all three conservation laws are obeyed – baryon number, lepton number, and charge.

2. $$^{1}H + {}^{2}D \rightarrow {}^{3}He + \gamma$$

Fast reaction (strong interaction), the first reaction when gas is getting hotter.
But the abundance of deuterium is extremely small, and it burns away rapidly.

3. Reaction $^{2}D + {}^{2}D \rightarrow$ is not important, as D burns on H faster (τ smaller and abundance of H is larger)

4. $$^{1}H + {}^{4}He \rightarrow ?$$

There are no nuclei with A=5 in nature. H does not burn on He.

The main nuclear burning cycles

In principle, many different nuclear reactions can occur simultaneously in a stellar interior.

For a very precise analysis (i.e., for deriving the detailed isotopic abundances), a large network of reactions must be calculated.

However, for the calculation of the structure and evolution of a star usually a much simpler procedure is sufficient, for the following reasons:

- The very strong dependence of nuclear reaction rates on the temperature, combined with the sensitivity to the Coulomb barrier $Z_1 Z_2$, implies that nuclear fusions of different possible fuels – hydrogen, helium, carbon, etc. – are well separated by substantial temperature differences.
- The evolution of a star therefore proceeds through several distinct nuclear burning cycles. For each nuclear burning cycle, only a handful of reactions contribute significantly to energy production and/or cause major changes to the overall composition.
- In a chain of subsequent reactions, often one reaction is by far the slowest and determines the rate of the whole chain. Then only the rate of this bottleneck reaction needs to be taken into account.

Hydrogen burning

The most important series of fusion reactions are those converting H to He (H-burning). This dominates ~90% of lifetime of nearly all stars. Since a simultaneous reaction between four protons is extremely unlikely, a chain of reactions is always necessary for hydrogen burning. Hydrogen burning in stars takes place at temperatures ranging between 8×10^6 K and 50×10^6 K, depending on stellar mass and evolution stage.

We will consider here the main ones: the **PP-chain** and the **CNO cycle**.

Let's first discuss the most obvious, so-called p-p reaction: $^1H + {}^1H \rightarrow {}^2H$

- The resulting particle would be, however, unstable and it would immediately disintegrate back into two separate protons.
- Hans Bethe in 1939: The Pauli principle prevents two protons with parallel spins from occupying the same position, and even a very close approach between them (10^{-13} cm) becomes improbable.
- For this reaction to proceed, one proton should be transformed to neutron (weak force). However, $m_n > m_p$, energy is taken from binding energy of deuterium (2.24 MeV).
- On distance of 10^{-13} cm, the particles exist for just 10^{-21} s, and in that time β-decay should occur.
- Probability is very small, and cross-section is extremely small, 10^{-47} cm^2, impossible to measure in the laboratories. Computations give S-factor:

$$S_{pp}(E_0) = S_0 = 3.88 \times 10^{-22} \text{ keV barn} \quad (1 \text{ barn} = 10^{-24} \text{ cm}^2).$$

$$\sigma = \frac{S(E)}{E} g(E)$$

- Therefore, the following reaction proceeds extremely slowly:

$$^1H + {}^1H \rightarrow {}^2D + e^+ + \nu \quad (\text{or} \quad p + p \rightarrow d + e^+ + \nu)$$

You will show at home that the typical time a proton has to wait until it reacts with another proton $\tau_{pp} \sim 2 \times 10^{10}$ yr, the lifetime of the Sun. But this is sufficient to power the Sun.

The p-p chains

The first reaction is the p-p reaction. After some deuterium is produced, it rapidly reacts with another proton to form ^3He. Subsequently three different branches are possible to complete the chain towards ^4He:

$$p + p \rightarrow d + e^+ + \nu$$
$$d + p \rightarrow {}^3He + \gamma$$

$$^3He + {}^3He \rightarrow {}^4He + 2\,p$$

pp1

$$^3He + {}^4He \rightarrow {}^7Be + \gamma$$

$$^7Be + e^- \rightarrow {}^7Li + \nu$$
$$^7Li + p \rightarrow {}^4He + {}^4He$$

pp2

$$^7Be + p \rightarrow {}^8B + \gamma$$
$$^8B \rightarrow {}^8Be + e^+ + \nu \quad \leftarrow \beta \text{ decay}$$
$$^8Be \rightarrow {}^4He + {}^4He$$

pp3

The p-p chains

$$p + p \rightarrow d + e^+ + \nu$$
$$d + p \rightarrow {}^3He + \gamma$$

$${}^3He + {}^3He \rightarrow {}^4He + 2\,p$$

pp1

$${}^3He + {}^4He \rightarrow {}^7Be + \gamma$$

$${}^7Be + e^- \rightarrow {}^7Li + \nu$$
$${}^7Li + p \rightarrow {}^4He + {}^4He$$

pp2

$${}^7Be + p \rightarrow {}^8B + \gamma$$
$${}^8B \rightarrow {}^8Be + e^+ + \nu \quad \leftarrow \beta \text{ decay}$$
$${}^8Be \rightarrow {}^4He + {}^4He$$

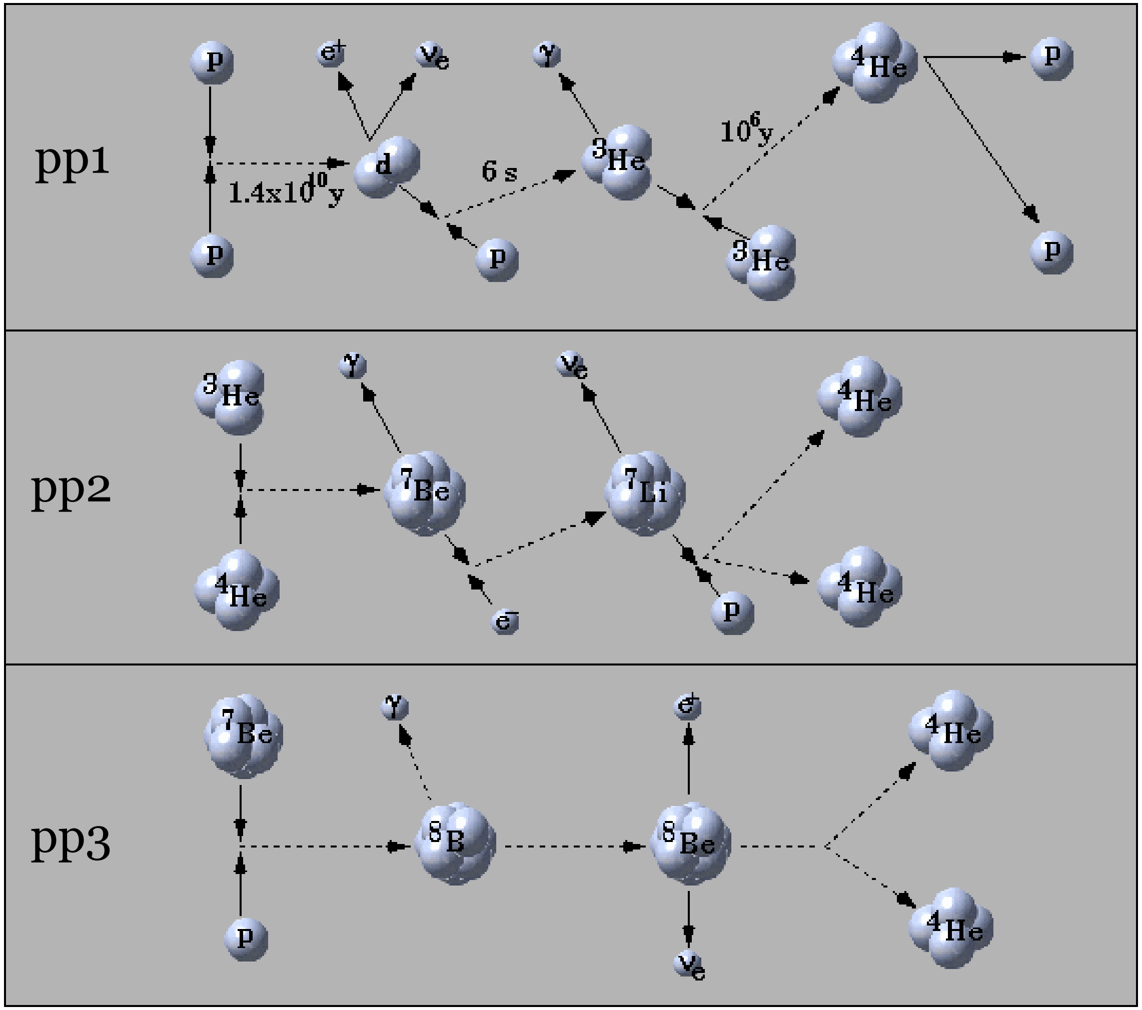
pp3

The **pp1** branch requires two ${}^3He$ nuclei, so the first two reactions in the chain have to take place twice.
The alternative **pp2** and **pp3** branches require only one ${}^3He$ nucleus and an already existing ${}^4He$ nucleus (either present primordially or produced previously by hydrogen burning).
The resulting ${}^7Be$ nucleus is unstable and can either capture an electron or fuse with another proton, giving rise to the second branching into **pp2** and **pp3**.

Three of the reactions in the chains are accompanied by neutrino emission, and the (average) neutrino energy is **different** in each case.

Therefore, the total energy release Q_H to produce one ${}^4He$ nucleus is different for each chain:
26.20 MeV (**pp1**), 25.66 MeV (**pp2**) and only 19.76 MeV for **pp3**.

Importance of different branches of the p-p chains



The relative importance of the **pp1** and **pp2** chains (branching ratios) depends on conditions of H-burning (T, ρ, abundances). The transition from **pp1** to **pp2** occurs at temperatures in excess of 1.3×10^7 K. Above 3×10^7 K the **pp3** chain dominates over the other two, but another process takes over in this case.

In **pp2** and **pp3** ^{4}He plays a role of a catalyst, it accelerates synthesis of itself. Abundance of ^{4}He changes, and therefore, the role of these branches grows even for T=const. **pp1** needs two pp-reactions, and therefore, is slow.

The overall rate of the whole reaction chain is set by the rate of the bottleneck p-p reaction, r_{pp}. In this steady-state or “equilibrium” situation the rate of each subsequent reaction adapts itself to the pp rate.

The **pp3** chain is never very important for energy generation, but it does generate abundant high energy neutrinos.

Neutrino emission

- A neutrino is released by weak interactions ($p \rightarrow n + e^+ + \nu$), which escape without interacting with the stellar matter. It is customary **not** to include the neutrino energies into the overall energy release Q, but to take into account only the energy that is used to heat the stellar gas. This includes energy released in the form of γ–rays (including the γ-rays resulting from pair annihilation after e^+ emission) and in the form of kinetic energies of the resulting nuclei.
- Thus, the effective Q-value of hydrogen burning is therefore somewhat smaller than 26.734 MeV and depends on the reaction in which the neutrinos are emitted.
- Assuming that neutrinos take away a small fraction of energy and knowing the solar luminosity, we can get the total formation rate of helium.

- In fact, it is these neutrinos that directly confirm the occurrence of nuclear reactions in the interior of the Sun. **No other direct observational test of nuclear reactions is possible.** The mean neutrino energy is ~0.26 MeV for a deuterium creation (**pp1 /2**) and ~7.2 MeV for β decay (**pp3**). But as **pp3** is negligible, the energy released for each He nucleus assembled is ~26 MeV (or 6×10^{18} erg g^{-1})

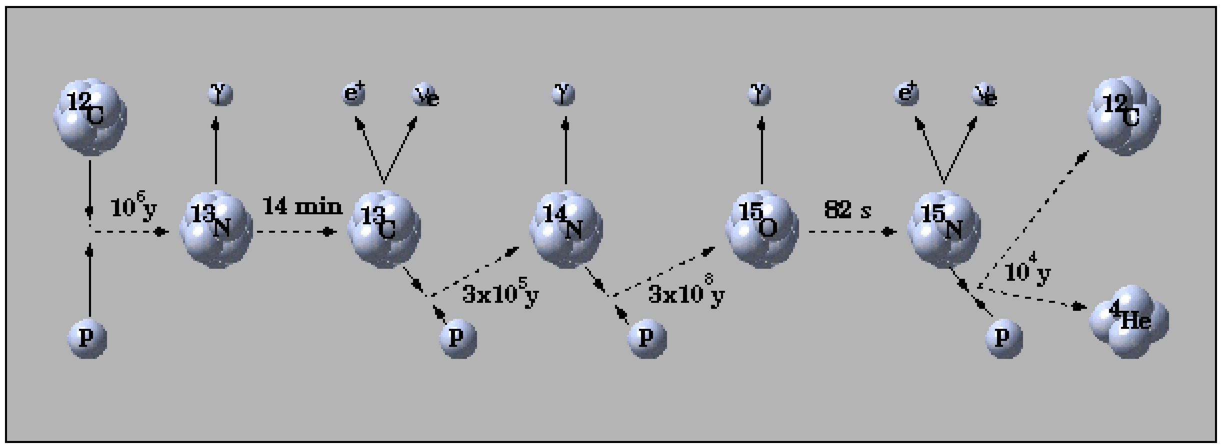
The CNO Cycle

At birth (most) stars contain a small (2%) mix of heavy elements, some of the most abundant of which are carbon, nitrogen and oxygen (CNO). If the temperature is sufficiently high, these nuclei may induce a chain of H-burning reactions in which they act as catalysts. The process is known as the CNO Cycle. This is a cyclical sequence of reactions that typically starts with a proton capture by a ^{12}C nucleus:

$$^{12}C + p \rightarrow {}^{13}N + \gamma$$
$$^{13}N \rightarrow {}^{13}C + e^+ + \nu$$
$$^{13}C + p \rightarrow {}^{14}N + \gamma$$
$$^{14}N + p \rightarrow {}^{15}O + \gamma$$
$$^{15}O \rightarrow {}^{15}N + e^+ + \nu$$
$$^{15}N + p \rightarrow {}^{12}C + {}^{4}He$$

$$\rightarrow {}^{16}O + \gamma$$
$$^{16}O + p \rightarrow {}^{17}F + \gamma$$
$$^{17}F \rightarrow {}^{17}O + e^+ + \nu$$
$$^{17}O + p \rightarrow {}^{14}N + {}^{4}He$$

a small probability ($<10^{-3}$)



Main cycle (CN cycle). Carbon is a catalyst in this cycle because it is not destroyed by its operation and it must be present in the original material of the star for the CNO cycle to operate. At high enough temperatures, $T \geq 1.5\times10^7$ K, all reactions in the cycle come into a steady state or "equilibrium" where the rate of production of each nucleus equals its rate of consumption. In this situation, the speed of the whole CNO cycle is controlled by the slowest reaction which is the capture of a proton by ^{14}N. As a result, most of ^{12}C is converted to ^{14}N before the cycle reaches equilibrium and this is the source of most of the nitrogen in the Universe.

Temperature dependence of PP chain and CNO Cycle

The rates of two reactions p+p → and ^{14}N + p → have very different temperature dependences:

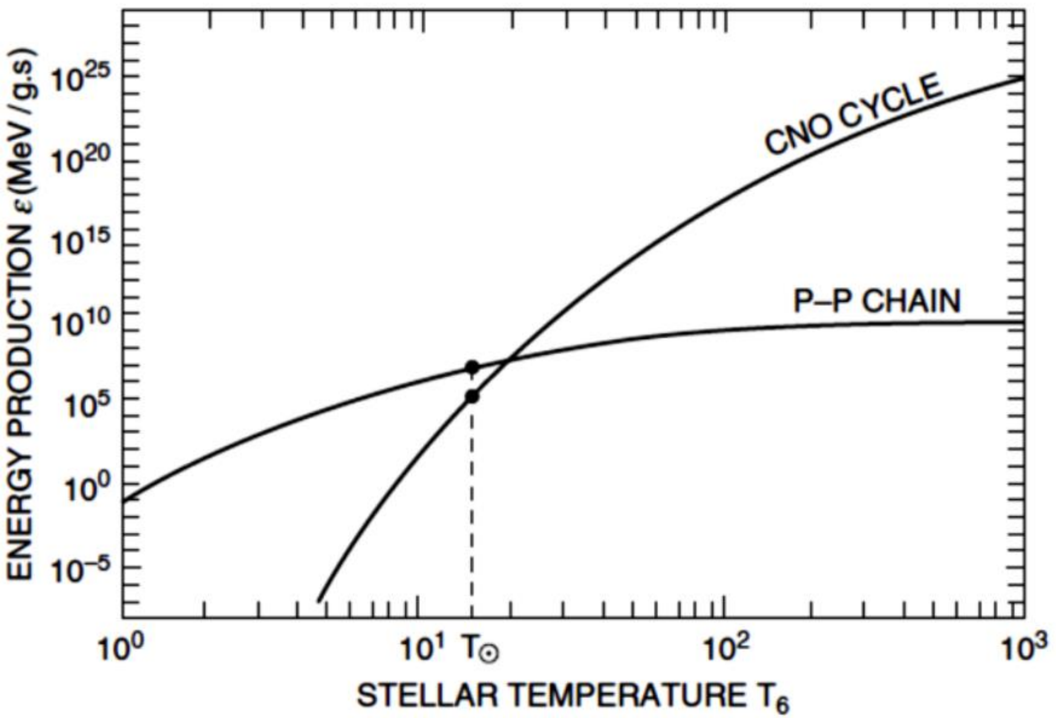
$$\varepsilon_{pp} \propto \left(\frac{T}{10^7 K}\right)^{4.53}$$

$$\varepsilon_{CNO} \propto \left(\frac{T}{2.5 \times 10^7 K}\right)^{16.7}$$

The rates are about equal at

$T \approx 1.7 \times 10^7$ K

Below this temperature the **pp** chain is most important, and above it the **CNO** cycle dominates. This occurs in stars slightly more massive than the Sun, $1.2 \div 1.5 M_\odot$.



The **pp** chain is the least temperature-sensitive of **all** nuclear burning cycles.

Helium Burning: the triple α-reaction

When there is no longer any hydrogen left to burn in the central regions of a star, gravity compresses the core until the temperature T_c reaches the point where helium burning reactions become possible.

Simplest reaction in a helium gas should be the fusion of two helium nuclei, e.g. $^4He + {}^4He \rightleftharpoons {}^8Be$

However, there is no stable configuration with A=8! For example, the beryllium isotope ^8Be has a lifetime of only 2.6×10^{-16} s and it rapidly decays to two ^4He nuclei again. While extremely short, this time is long enough to build up a very small equilibrium concentration of ^8Be, which increases with temperature and reaches about 10^{-9} at $T \approx 10^8$ K. Thus, a third helium nucleus can be added to ^8Be before it decays, forming ^{12}C by the so-called triple-alpha reaction:

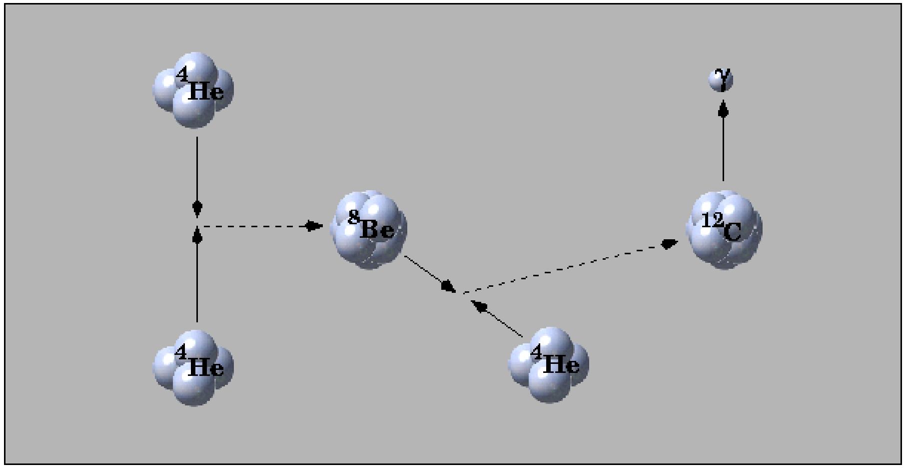
$^4He + {}^4He \rightarrow {}^8Be$

$^8Be + {}^4He \rightarrow {}^{12}C + \gamma$

Since the two reactions need to occur almost simultaneously, the 3α-reaction behaves as if it were a three-particle reaction:

$3\ {}^4He \rightarrow {}^{12}C + \gamma$

which has Q = 7.275MeV. The energy release per unit mass is $q = Q/m(^{12}C) = 5.9 \times 10^{17}$ erg/g, which is about 1/10 smaller than for H-burning.

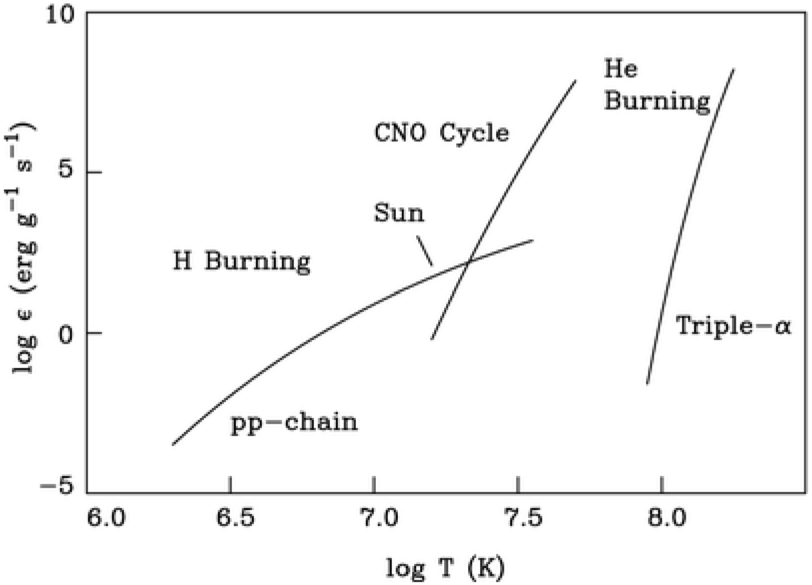


Helium Burning

In addition to the short-lived beryllium state, another factor that helps the 3α-reaction to go is the existence of a **resonance** in the ^{12}C nucleus that coincides closely in energy with that produced by colliding another helium nucleus with ^{8}Be.

This greatly enhances the rate at which the second step in the reaction chain takes place.

The temperature sensitivity of the 3α-reaction rate is extremely high, $\varepsilon_{3\alpha} \propto T^{40}$.



When a sufficient amount of ^{12}C has been created by the 3α-reaction, it can capture a further α-particle to form ^{16}O: $^{12}C + {}^{4}He \rightarrow {}^{16}O + \gamma$

It has $Q = 7.162$ MeV, or $q_{\alpha C} = 4.32 \times 10^{17}$ erg per gram of produced ^{16}O.

In principle further captures on ^{16}O are also possible, forming ^{20}Ne, but during normal helium burning become increasingly unlikely due to the increasing Coulomb barrier.

Thus, stars in which the triple-α process takes place wind up containing a mixture of carbon and oxygen, with the exact ratio depending on their age, density, and temperature.

Carbon burning and beyond

At even higher temperatures, the Coulomb barrier for oxygen and carbon can be overcome, creating yet heavier nuclei. Carbon burning (fusion of 2 carbon nuclei) requires temperatures above 5×10^{8} K, and oxygen burning in excess of 10^{9} K.

Many reaction paths are possible. We will not discuss these reactions here as the majority of the possible energy release by nuclear fusion reactions has occurred by the time that hydrogen and helium have been burnt.

However, examples of these reactions can be found in textbooks.

Silicon to Iron, photodisintegration

- At still higher temperatures, around 3×10^9 K, the typical photon becomes energetic enough that it can disrupt nuclei, knocking pieces off them in a process known as **photodisintegration**. The chemical balance in the star is then determined by a competition between this process and reactions between nuclei.
- However, as we might expect, the net effect is to drive the chemical balance ever further toward the most stable nucleus, iron. Once the temperature around 3×10^9 K, more and more nuclei begin to convert to ^{56}Fe, and its close neighbor's cobalt and nickel.
- Things stay in this state until the temperature is greater than about 7×10^9 K, at which point photons have enough energy to destroy even iron, and the entire process reverses: all elements are converted back into its constituent protons and neutrons, and photons reign supreme.

$$^{56}\mathrm{Fe} \rightarrow 13\ ^{4}\mathrm{He} + 4\ \mathrm{n}$$

Major nuclear burning processes

Common feature is release of energy by consumption of nuclear fuel. Rates of energy release vary enormously. Nuclear processes can also absorb energy from radiation field, we shall see consequences can be catastrophic.

Nuclear Fuel	Process	$T_{threshold}$ 10^6K	Products	Energy per nucleon (MeV)
H	PP	~4	He	6.55
H	CNO	15	He, N	6.25
He	3α	100	C,O	0.61
C	C+C	600	O,Ne,Na,Mg	0.54
O	O+O	1000	Mg,S,P,Si	~0.3
Si	Nuc eq.	3000	Co,Fe,Ni	<0.18

Burning times of burning phases:

H: 10^{10} (yrs)

He: 10^8

C: 10^4

…

Si: hrs

The r- and s-processes (1)

We have already seen how elements up through iron are built, but we have not yet mentioned how even heavier elements can be created. The answer is that they are not made in stars under normal circumstances, because when the only forces at work at electromagnetism and nuclear forces, it is never energetically favorable to create such elements in any significant number.
Creating such elements requires the intervention of another force: gravity.

When stars are in the process of being crushed by gravity, right before they explode as supernovae (which we will discuss toward the end of the course), gravity drives a process that converts most of the protons to neutrons. This creates a neutron-rich environment unlike any found at earlier stages of stellar evolution, when the lack of neutrons was often the rate-limiting step.

Neutrons capture by heavy nuclei is not limited by the Coulomb barrier – so could proceed at relatively low temperatures. In a neutron-rich environment, it becomes possible to create heavy nuclei via the absorption of neutrons by existing nuclei. Since the neutrons are neutral, there is no Coulomb barrier to overcome, and the reaction proceeds as quickly as the neutron supply allows. Reactions look like this:

$$\begin{aligned} I_0(\mathcal{A},\mathcal{Z})+n &\rightarrow I_1(\mathcal{A}+1,\mathcal{Z}) \\ I_1(\mathcal{A}+1,\mathcal{Z})+n &\rightarrow I_2(\mathcal{A}+2,\mathcal{Z}) \\ I_2(\mathcal{A}+2,\mathcal{Z})+n &\rightarrow I_3(\mathcal{A}+3,\mathcal{Z}) \\ &\text{etc.} \end{aligned}$$

The r- and s-processes (2)

This continues until it produces a nucleus that is unstable and undergoes β decay, converting one of the neutrons back into a proton:

$$I_N(\mathcal{A}+N,\mathcal{Z}) \;\rightarrow\; J(\mathcal{A}+N,\mathcal{Z}+1)+e^-+\overline{\nu}.$$

anti-neutrino

If the new element produced in this way is stable, it will begin neutron capturing again. If not, it will keep undergoing β decays until it becomes stable:

$$\begin{aligned} J(\mathcal{A}+N,\mathcal{Z}+1) &\;\rightarrow\; K(\mathcal{A}+N,\mathcal{Z}+2)+e^-+\overline{\nu} \\ K(\mathcal{A}+N,\mathcal{Z}+2) &\;\rightarrow\; L(\mathcal{A}+N,\mathcal{Z}+3)+e^-+\overline{\nu} \\ &\text{etc.} \end{aligned}$$

β decays therefore increase Z at constant A

These processes together lead to the build-up of elements heavier than iron. The chain **stops** if at any point it reaches a nucleus that is stable against β decay, and is also not able to capture neutrons.

The neutron capture reactions may proceed more **slowly** or more **rapidly** than the competing β decays. Elements that are build up by reaction chains in which β decays occur faster are called r-process, for rapid. Elements where β decays are slower are called s-process, for slow.

Knowing which process produces which elements requires knowing the stability, binding energy, and β decay lifetimes of the various elements, which must be determined experimentally.

Abundance changes (1)

The rate of change in the number density n_i of nuclei of type i owing to reactions with nuclei of type j is

$$\left(\frac{dn_i}{dt}\right)_j = -(1+\delta_{ij})r_{ij} = -n_i n_j \langle\sigma v\rangle_{ij}$$

One can define the nuclear lifetime of a species i owing to reactions with j as

$$\tau_{ij} = \frac{n_i}{\left|(dn_i/dt)_j\right|} = \frac{1}{n_j\langle\sigma v\rangle_{ij}}$$

which is the timescale on which the abundance of i changes as a result of this reaction.

The overall change in the number n_i of nuclei of type i in a unit volume can generally be the result of different nuclear reactions. Some reactions (with rate r_{ij} as defined above) consume i while other reactions, e.g. between nuclei k and l, may produce i. If we denote the rate of reactions of the latter type as $r_{kl,i}$, we can write for the total rate of change of n_i:

$$\frac{dn_i}{dt} = -\sum_j (1+\delta_{ij})r_{ij} + \sum_{k,l} r_{kl,i}$$

Abundance changes (2)

The number density n_i is related to the mass fraction X_i as $n_i = X_i\rho/(A_i m_H)$, so that we can write the rate of change of the mass fraction due to nuclear reactions as

$$\frac{dX_i}{dt} = A_i \frac{m_H}{\rho}\left(-\sum_j (1+\delta_{ij}) r_{ij} + \sum_{k,l} r_{kl,i}\right)$$

For each nuclear species i one can write such an equation, describing the composition change at a particular mass shell inside the star (with density ρ and temperature T) resulting from nuclear reactions. In the presence of internal mixing (in particular of convection) the redistribution of composition between different mass shells should also be taken into account.

Note the similarity between the expressions for the nuclear energy generation rate and the equation for composition changes, both of which are proportional to r_{ij}. Combining them together, we can obtain a useful expression for a simple case where only one reaction occurs, or a reaction chain in which one reaction determines the overall rate.

$$\varepsilon_{ij} = \frac{Q_{ij} r_{ij}}{\rho} = \frac{Q_{ij}}{(1+\delta_{ij}) A_i A_j m_H^2} \rho X_i X_j \langle\sigma v\rangle_{ij}$$

An example is the fusion of $4\ {}^1H$ into ${}^4He$. One can show that

$$\frac{dY}{dt} = -\frac{dX}{dt} = \frac{\varepsilon_H}{q_H}$$

where ε_H is the energy generation rate by the complete chain of H-burning reactions, and q_H is amount of energy produced by converting 1 gram of $4\ {}^1H$ into ${}^4He$.

Summary of lectures on nuclear reactions

Recalling that we wanted to determine the physics governing ε will be defined by the nuclear energy source in the interiors. We needed to develop a theory and understanding of nuclear physics and reactions.

- We have covered the basic principles of energy production by fusion.
- The PP chain and CNO cycle have been described.
- He burning by the triple-alpha reaction was introduced.
- Later burning stages of the heavier elements (C,O, Si) were shortly discussed.
- The *r*- and *s*-processes – origin of the elements heavier than Fe.

Solution of the Equations of Stellar Structure

THE STELLAR STRUCTURE EQUATIONS AND HOW TO SOLVE THEM?

SIMPLE STELLAR MODELS

POLYTROPIC MODELS

LANE-EMDEN EQUATION

RELATIONSHIPS FOR POLYTROPIC STARS

CHANDRASEKHAR MASS

DYNAMICAL STABILITY OF STARS

Introduction

We have now derived all the four differential equations and the three additional functions that, together with boundary conditions, define uniquely the equilibrium properties of a star of a given mass and composition.

- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$
- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$
- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$
- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma r^2}\frac{\rho(r)\kappa_R(r)}{T^3(r)}L(r)$
- $\frac{P}{T}\frac{dT}{dP} = \frac{\gamma-1}{\gamma}$

Energy transport

by radiation

by convection

- r = radius
- P = pressure at r
- m = mass of material within r
- ρ = density at r
- L = luminosity at r
- T = temperature at r
- κ_R = Rosseland mean opacity at r
- ε = energy release

Three supplement equations:

$P = P(\rho, T,$ chemical composition$)$ — **EOS**

$\kappa_R = \kappa_R(\rho, T,$ chemical composition$)$

$\varepsilon = \varepsilon(\rho, T,$ chemical composition$)$

Plus, the equation of composition changes:

$$\frac{dX_i}{dt} = A_i \frac{m_H}{\rho}\left(-\sum_j (1+\delta_{ij}) r_{ij} + \sum_{k,l} r_{kl,i}\right)$$

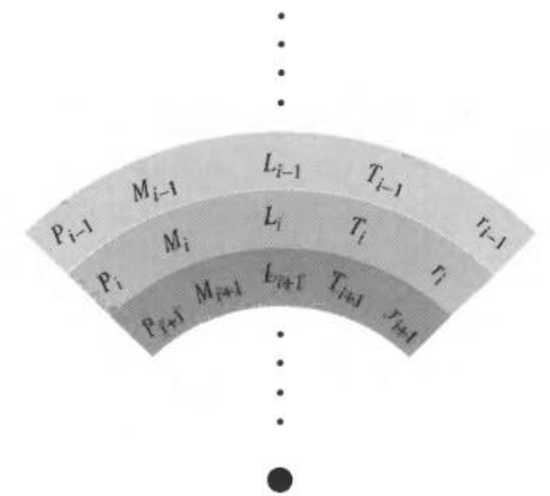
How to solve the Stellar Structure equations?

- "Solving" this system of coupled equations means finding the functions $P(r)$, $T(r)$, and $\rho(r)$, which are the ones that are usually considered to describe the structure of the star.
- **The Vogt-Russell theorem:**
  "The mass and the composition structure throughout a star **uniquely** determine its radius, luminosity, and internal structure, as well as its subsequent evolution."
  - This "theorem" has not been proven and is not even rigorously true; there are known exceptions. However, an actual star would probably adopt one unique structure as a consequence of its evolutionary history. In this sense, the Vogt-Russell "theorem" should be considered a general rule rather than a rigorous law.
- Unfortunately, unless some unrealistic assumptions are made, there is no analytic solution to the equations, given the complicated nature of the functions P, κ, and ε when all relevant processes are included. Because the complete set of equations with two-point boundary values is highly non-linear and time-dependent, their full solution requires a complicated numerical procedure. This is what is done in detailed stellar evolution codes, the results of which we will discuss in the following few lectures.

Numerical modeling of the equations

We will not go into any detail about the numerical methods commonly used in such codes, but shortly they can be described as the following:

- In a numerical solution, the differentials in the equations are replaced by differences. For instance, by replacing dP/dr by $\Delta P/\Delta r$. The star is then imagined to be constructed of spherically symmetric shells.
- The numerical integration of the stellar structure equations may be carried out shell by shell from the surface toward the center, from the center toward the surface, or, as is often done, in both directions simultaneously.
- If the integration is carried out in both directions, the solutions will meet at some fitting point where the variables must vary smoothly from one solution to the other.
- Simultaneously matching the surface and central boundary conditions for a desired stellar model usually requires several iterations before a satisfactory solution is obtained. If the surface-to-center and center-to-surface integrations do not agree at the fitting point, the starting conditions must be changed. The initial conditions of the next integration are estimated from the outcome of the previous integration.



Simple stellar models

- Although today the standard approach is to hand the problem to a computer, insight into the structure of stars may be gained both by analyzing the equations, without actually solving them, and by seeking simple solutions based on additional simplifying assumptions.
- The main purpose of this lecture is to briefly analyze the differential equations of stellar evolution and their boundary conditions, and to see how the full set of equations can be simplified in some cases to allow simple or approximate solutions – so-called simple stellar models. We will concentrate on polytropic models.
- Although nowadays their practical use has mostly been superseded by more realistic stellar models, due to their simplicity polytropic models still give useful insight into several important properties of stars. Moreover, in some cases the polytropic relation is a good approximation to the real equation of state
- As the very first simplification, we assume that a star is in both **hydrostatic** and **thermal** equilibrium. In this case, the four partial differential equations for stellar structure reduce to ordinary, time-independent differential equations.

Simplifying assumptions

- The four equations of stellar structure divide naturally into two groups:
  - one describing the mechanical structure of the star
  - and the other giving the thermal structure.

$$\frac{dm}{dr} = 4\pi r^2 \rho(r); \frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r); \frac{P}{T}\frac{dT}{dP} = \frac{\gamma-1}{\gamma}$$

- However, the only contact between the mechanical variables and thermal equations is through the temperature dependence of the equation of state.
- If we can write the pressure in terms of the density alone, **without reference to the temperature**, then we can separate these two equations from the others and solve them by themselves. Solving two differential equations (plus one algebraic equation relating P and ρ is much easier than solving seven equations.
- We have already seen, that under certain circumstances, the pressure can indeed become independent of temperature, and only depend on density, i.e.,
- In the above examples we derived a relation of the form $P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}}$ where K and γ are constants; this is called a polytropic relation, and the resulting models are called polytropic models.

Polytropic models

When the equation of state can be written in this form, the temperature does not enter at all into the equations and the calculations of stellar structure simplify enormously. There are even analytical solutions for certain values of γ.

- If we then take the equation for hydrostatic support, multiply it by r^2/ρ, differentiate with respect to r, and then divide by r^2, we get:

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho\,;\frac{dm}{dr} = 4\pi r^2\rho$$

$$\frac{r^2}{\rho}\frac{dP}{dr} = -Gm \quad \rightarrow \quad \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr}$$

$$\frac{dm}{dr} = 4\pi r^2\rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

What we have done is exact. Now we make our approximation. We approximate that the pressure and density are related by a power-law $P = K\rho^\gamma = K\rho^{1+\frac{1}{n}}$ (it customary to adopt γ=1+1/n, or n=1/(1- γ), where n is the polytropic index):

$$\frac{K(n+1)}{r^2 n}\frac{d}{dr}\left(\frac{r^2}{\rho}\rho^{1/n}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

Lane-Emden equation

$$\frac{K(n+1)}{4\pi G n}\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho^{1/n-1}\frac{d\rho}{dr}\right) = -\rho$$

The solution $\rho(r)$ for $0 \le r \le R$ is called a polytrope and requires two boundary conditions.

Hence a polytrope is uniquely defined by three parameters: K, n, and R. This enables calculation of additional quantities as a function of radius, such as pressure, mass or gravitational acceleration.

Let's define a dimensionless variable θ in the range $0 \le \theta \le 1$ by $\rho=\rho_c\theta^n$, where ρ_c is the central density. Then the equation becomes

$$\frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

To simplify the equation further, we introduce the dimensionless radius $\xi=r/\alpha$, where

$$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}$$

constant having the dimension of length squared!

The equation finally becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

This equation is called the *Lane-Emden equation*, and the solution $\theta = \theta_n(\xi)$ is called the *Lane-Emden function*.

Solving the Lane-Emden equation

$\xi = r/\alpha$

$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

$P = K\rho^\gamma = K\rho^{1+\frac{1}{n}}$

$\rho = \rho_c\theta^n$

Since it is a second order differential equation, we need two boundary conditions.
The 1st is at the center: from spherical symmetry, the pressure gradient at the center ($\theta = 1$) must be zero.
The 2nd condition comes from the surface, $\xi = \xi_1$, where the density should go to zero.
So, our boundary conditions are $\frac{d\theta}{d\xi} = 0,\ \theta = 1$ at $\xi = 0$ (the center), and $\theta = 0$ at $\xi = \xi_1$ (the surface).

Solving the equation for the dimensionless function $\theta_n(\xi)$ in terms of ξ for a specific polytropic index n leads directly to the profile of density with radius $\rho_n(r)$. The polytropic equation of state provides the pressure profile. In addition, if the ideal gas law and radiation pressure are assumed for constant composition, then the temperature profile, $T(r)$, is also obtained.

$$P = \frac{\Re T\rho}{\mu} + \frac{aT^4}{3}$$

Unfortunately, the Lane-Emden equation does not have an analytic solution for arbitrary values of n.
In fact, there are only three analytic solutions, namely n= 0, 1, and 5:

$$n = 0, \theta = 1 - \frac{\xi^2}{6} \qquad \xi_1 = \sqrt{6}$$

$$n = 1, \theta = \frac{\sin\xi}{\xi} \qquad \xi_1 = \pi$$

$$n = 5, \theta = \left(1 + \frac{\xi^2}{3}\right)^{-1/2} \qquad \xi_1 \to \infty$$

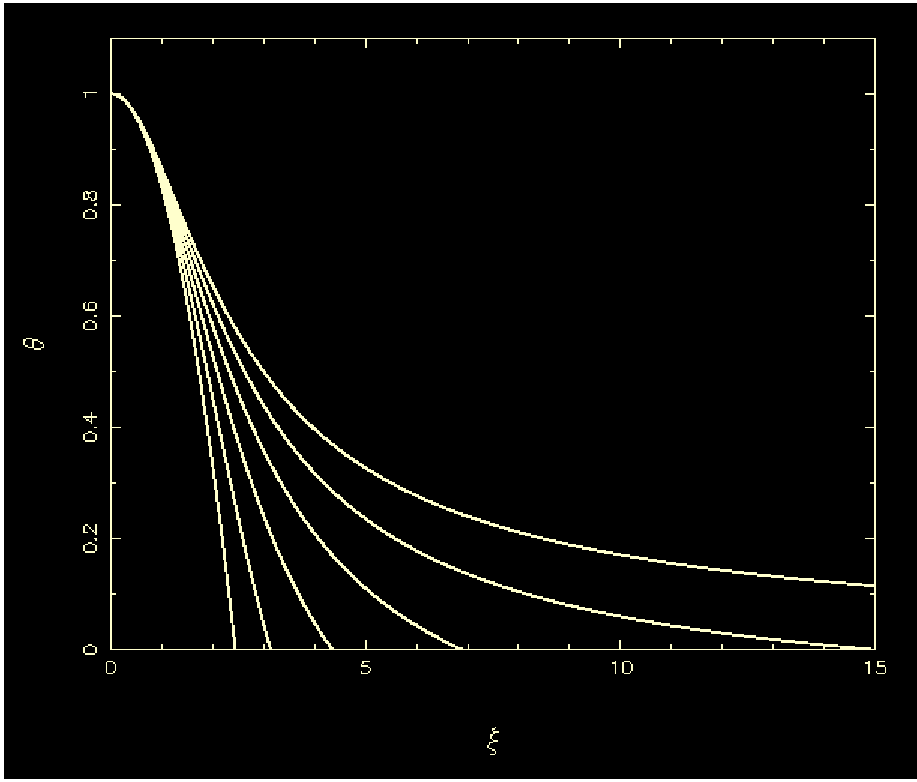
Solutions for all other values of n must be solved numerically.

1. Solve the Eqn for n=0

2. Find the dimensionless radius of these polytropic stars

Solutions of the Lane-Emden equation

Numerical solutions to the Lane-Emden equation for (left-to-right) n = 0,1,2,3,4,5. Some key values resulting from the integration are shown in the table.



n	ξ_1	$-\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}$	$\rho_c/\bar{\rho}$
0.0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800.
5.0	∞	1.73205	∞

Solutions decrease monotonically and have θ=0 at $\xi = \xi_1$ (i.e. the stellar radius).
With increasing polytropic index, the star becomes more centrally condensed.

Polytropic stars

$\xi = r/\alpha$

$\rho = \rho_c \theta^n$

$$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}$$

The surface radius of the polytropic model is

$$R = \left(\frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}\right)^{1/2} \xi_1$$

The total mass M of a polytropic star is given by

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi\alpha^3\rho_c \int_0^{\xi_1} \xi^2\theta^n d\xi = -4\pi\alpha^3\rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = -4\pi\alpha^3\rho_c \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}$$

From a polytropic model, we can derive other useful numbers and relationships.
As one example, it is often convenient to know how centrally concentrated a star is,
i.e. how much larger its central density is than its mean density. We define this quantity as

$$D_N \equiv \frac{\rho_c}{\bar{\rho}} = \frac{\rho_c 4\pi R^3}{3M} = \frac{4\pi}{3}\rho_c(\alpha\xi_1)^3 \left[-4\pi\alpha^3\rho_c\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{-1} = \left[-\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{-1}$$

Values in Table in slide 266

Mass-Radius relationship for polytropic stars

Another useful relationship is between mass and radius. We start by expressing the central density ρ_c in terms of the other constants and our length scale α:

$$\rho_c = \left[\frac{K(n+1)}{4\pi G\alpha^2}\right]^{n/(n-1)}$$

$$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}$$

Substitute this into the equation for the mass:

$$M = -4\pi\alpha^3\rho_c\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1} = -4\pi\alpha^3\left[\frac{K(n+1)}{4\pi G\alpha^2}\right]^{n/(n-1)}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}$$

Making the substitution $\alpha = R/\xi_1$ and re-arranging, we arrive at

$$\left[\frac{GM}{-\xi_1^2(d\theta/d\xi)_{\xi_1}}\right]^{(n-1)}\left(\frac{R}{\xi_1}\right)^{3-n} = \frac{[K(n+1)]^n}{4\pi G}$$

n = 1 is a special case, for which the radius is independent of mass and is uniquely determined by K:

$$R = \xi_1\left(\frac{K}{2\pi G}\right)^{1/2}$$

Another important polytropic index is n=3, for which the R dependence disappears.
We find that

$$M = -\frac{4}{\sqrt{\pi}}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\left(\frac{K}{G}\right)^{3/2}$$

M-R relationship for polytropic stars

For other n, mass and radius are related by $M \sim R^{(n\text{-}3)/(n\text{-}1)}$.

Note what this means: for a polytropic index of n=1.5 (the $\gamma = 5/3$ case), $R \sim M^{-1/3}$. Thus, for a set of stars with the same K and n (i.e., white dwarfs), the stellar radius is inversely proportional to the mass. We will use it a few slides later.

Eddington standard model (1)

Another important polytropic index is n=3, "Eddington standard model" associated with a star in radiative equilibrium (slide 208). The contribution to the total pressure at a certain location in the star due to an ideal gas is given by

$$P = \frac{\Re T\rho}{\mu} + \frac{aT^4}{3}$$

$$P_g = \frac{\Re T\rho}{\mu} = \beta P$$

$$\Re = \frac{k}{m_p}$$

Then the contribution due to radiation pressure is

$$P_r = \frac{aT^4}{3} = (1-\beta)P$$

Combining both equations to eliminate T we get:

$$P^3 = \frac{3(1-\beta)}{a}\left(\frac{\Re\rho}{\mu\beta}\right)^4$$

This leads immediately to an expression for the total pressure in terms of the density, namely

$$P = K\rho^\gamma \quad \text{where} \quad K \equiv \left[\frac{3(1-\beta)}{a}\right]^{1/3}\left(\frac{\Re}{\mu\beta}\right)^{4/3}, \qquad \gamma = 4/3, \qquad \text{and} \quad n = 3$$

Thus, we have obtained a polytropic equation of state of index 3, which implies a unique relation between K and M. The *Eddington quartic equation* is

$$1-\beta = 0.003\left(\frac{M}{M_{sun}}\right)^2 \mu^4\beta^4$$

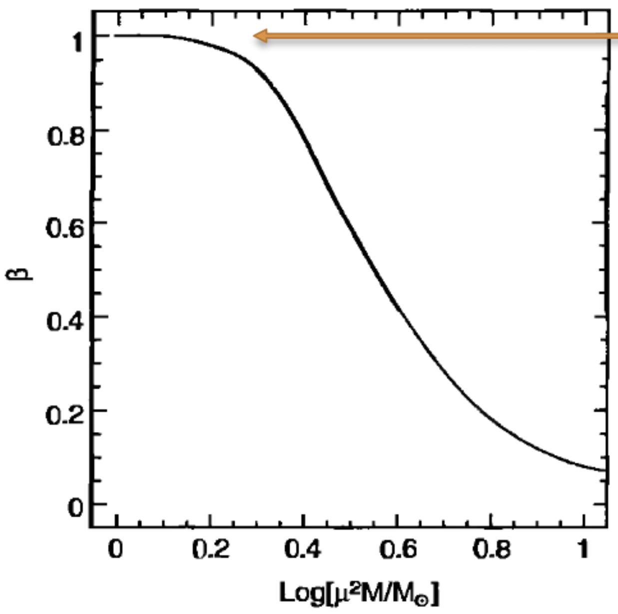
Eddington standard model (2)

$$P = K\rho^{\gamma} \quad \text{where} \quad K \equiv \left[\frac{3(1-\beta)}{a}\right]^{1/3}\left(\frac{\Re}{\mu\beta}\right)^{4/3}, \qquad \gamma = 4/3, \qquad \text{and} \quad n = 3$$

Thus, we have obtained a polytropic equation of state of index 3, which implies a unique relation between K and M. The *Eddington quartic equation* is

$$1-\beta = 0.003\left(\frac{M}{M_{sun}}\right)^2 \mu^4\beta^4$$

$$M = -\frac{4}{\sqrt{\pi}}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\left(\frac{K}{G}\right)^{3/2}$$



pure gas pressure

pure radiation pressure

The central pressure in polytropic stars

Another important relation is obtained between the central pressure and the central density.
Substitute K from the mass-radius relation:

$$P_c = K\rho_c^{1+\frac{1}{n}} \quad \rightarrow \quad \left[\frac{GM}{-\xi_1^2(d\theta/d\xi)_{\xi_1}}\right]^{(n-1)} \left(\frac{R}{\xi_1}\right)^{3-n} = \frac{[K(n+1)]^n}{4\pi G}$$

We obtain

$$P_c = \frac{(4\pi G)^{1/n}}{(n+1)} \left[\frac{GM}{M_n}\right]^{\frac{n-1}{n}} \left(\frac{R}{R_n}\right)^{\frac{3-n}{n}} \rho_c^{\frac{n+1}{n}}$$

where $M_n = -\xi_1^2(d\theta/d\xi)_{\xi_1}$ and $R_n = \xi_1$.

Now eliminating R, using $D_N \equiv \frac{\rho_c}{\bar{\rho}} = \frac{\rho_c 4\pi R^3}{3M}$, and assembling all n-dependent coefficients into one constant B_n, we get

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3}$$

The remarkable property of this relation is that it depends on the polytropic equation of state only through the value of B_n, which varies very slowly with n.

It therefore constitutes an almost universal relation!

n	D_n	M_n	R_n	B_n
1.0	3.290	3.14	3.14	0.233
1.5	5.991	2.71	3.65	0.206
2.0	11.40	2.41	4.35	0.185
2.5	23.41	2.19	5.36	0.170
3.0	54.18	2.02	6.90	0.157
3.5	152.9	1.89	9.54	0.145

The degeneracy pressure in polytropic stars

As a star contracts, the density may become so high that the electrons become degenerate and exert a (much) higher pressure than they would if they behaved classically. Stars that are so compact and dense that their interior pressure is dominated by degenerate electrons are known observationally as white dwarfs. They are the remnants of stellar cores in which hydrogen has been completely converted into helium. In most cases, also helium has been fused into carbon and oxygen.

We discussed the degeneracy pressure in Lecture 7. Let's now add a bit more detail.

The pressure of a completely degenerate electron gas in the non-relativistic limit is

$$P_e = K_{NR}\left(\frac{\rho}{\mu_e}\right)^{5/3} \quad \text{with} \quad K_{NR} = \frac{h^2}{20 m_e m_H^{5/3}}\left(\frac{3}{\pi}\right)^{2/3} = 1.0036 \times 10^{13}\ [\text{cgs}]$$

This corresponds to a polytropic relation with n=1.5 (the $\gamma = 5/3$ case). Since in the limit of strong degeneracy the pressure no longer depends on the **temperature**, this degeneracy pressure can hold the star up against gravity, regardless of the temperature. Therefore, a degenerate star does **not** have to be **hot** to be in hydrostatic equilibrium, and it can remain in this state forever even when it cools down. This is the situation in white dwarfs.

A few slides ago we obtained that for n=1.5, $R \sim M^{-1/3}$, i.e. the stellar radius is inversely proportional to the mass.

The relativistic degeneracy in polytropic stars

More massive white dwarfs are thus more compact, and therefore have a higher density. Above a certain density the electrons will become relativistic as they are pushed up to higher momenta by the Pauli exclusion principle. The degree of relativity increases with density, and therefore with the mass of the white dwarf, until at a certain mass all the electrons become extremely relativistic, i.e., their speed $v_e \to c$. In this limit the equation of state has changed to (the pressure increases less steeply with density)

$$P_e = K_{ER}\left(\frac{\rho}{\mu_e}\right)^{4/3} \quad \text{with} \quad K_{ER} = \frac{hc}{8m_H^{4/3}}\left(\frac{3}{\pi}\right)^{1/3} = 1.2435 \times 10^{15}\ [\text{cgs}]$$

which is also a polytropic relation but with $n = 3$.

We have already seen above that an n=3 polytrope is special in the sense that it has **a unique mass**, which is determined by K and is independent of the radius:

$$M = -\frac{4}{\sqrt{\pi}}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\left(\frac{K}{G}\right)^{3/2}$$

This value corresponds to an upper limit to the mass of a gas sphere in hydrostatic equilibrium that can be supported by degenerate electrons, and thus to the **maximum possible mass** for a white dwarf. Its existence was first found by Subrahmanyan Chandrasekhar in 1931, after whom this limiting mass was named.

Chandrasekhar mass

-2.01824

$K_{ER} = 1.2435 \times 10^{15}$ [cgs]

A relativistic electron gas has $K = K_{ER}/\mu_e^{4/3}$

Substituting it and other proper numerical values into

$$M = -\frac{4}{\sqrt{\pi}} \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1} \left(\frac{K}{G}\right)^{3/2}$$

$\mu_e \approx \frac{2}{1+X}$

we obtain the Chandrasekhar mass

$$M = M_{Ch} = \frac{5.826}{\mu_e^2} M_\odot$$

Thus, for a highly relativistic electron gas, there is only a **single** possible mass which can be in hydrostatic equilibrium.

White dwarfs are typically formed of helium, carbon or oxygen, for which $\mu_e = 2$ and therefore $M_{Ch} = 1.456\ M_\odot$.

This quantity is called the Chandresekhar mass, after Subrahmanyan Chandresekhar, who first derived it. He did the calculation while on his first trip out of India, to start graduate school at Cambridge at age 20… This work earned Chandrasekhar the 1983 Nobel Prize for Physics (which he shared with Fowler for their contributions to the understanding of stellar evolution).

A further increase of the mass (e.g., due to accretion from a companion star) leads to the loss of stability and collapse. This is the cause of supernovae type Ia explosions.

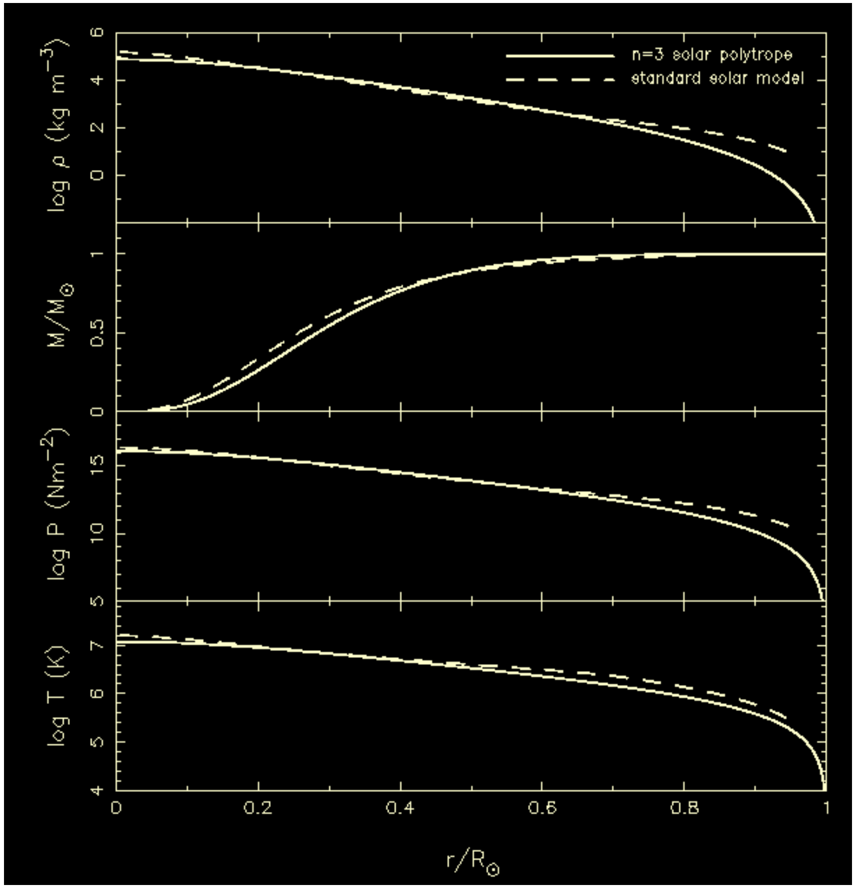
Find the value

Comparison with real models

- How do these polytropic models, compare to the results of a detailed solution of the equations of stellar structure? To make this comparison we will take an n=3 polytropic model of the Sun (often known as the Eddington Standard Model, a model with the constant fraction of radiation pressure and μ=const), with the co-called Standard Solar Model (SSM - Bahcall 1998, Physics Letters B, 433, 1).
- For this, we need to convert the dimensionless radius ξ and density θ to actual radius (in cm) and density (in g cm^{-3}).
- Polytrope does remarkably well (particularly at the core) considering how simple the physics is.

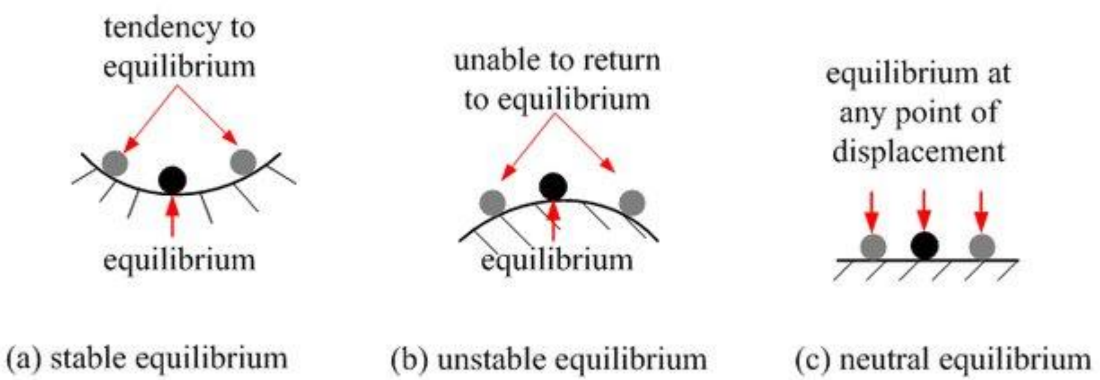
Property	n=3 polytrope	SSM
ρ_c	7.65×10^1 g cm^{-3}	1.52×10^2 g cm^{-3}
P_c	1.25×10^{17} dyn cm^{-2}	2.34×10^{17} dyn cm^{-2}
T_c	1.18×10^7 K	1.57×10^7 K

Comparison of numerical solution for an $n = 3$ polytrope of the Sun versus the Standard Solar Model.



The stability of stars

We have so far considered stars in both hydrostatic and thermal equilibrium (HE & TE). But an important question is whether these equilibria are stable?



A rigorous treatment of this problem is very complicated, so we will only look at a very simplified example to illustrate the principles.

Dynamical stability of stars

- Suppose a star in hydrostatic equilibrium is compressed on a short timescale, $\tau << \tau_{KH}$, so that the compression can be considered as adiabatic.
- It can be shown that a star that has $\gamma_{ad}>4/3$ everywhere is dynamically stable, and if $\gamma_{ad}=4/3$, it is neutrally stable.
  However, the situation when $\gamma_{ad}<4/3$ in some part of the star requires further investigation.
- If $\gamma_{ad}<4/3$ in a sufficiently large core, where P/ρ is high, the star becomes unstable. However if $\gamma_{ad}<4/3$ in the outer layers where P/ρ is small, the star as a whole need **not** become unstable.
- Stars dominated by an ideal gas or by non-relativistic degenerate electrons have $\gamma_{ad}=5/3$ and are therefore dynamically stable. However, we have seen that for relativistic particles $\gamma_{ad}\rightarrow 4/3$ and stars dominated by such particles tend towards a neutrally stable state.
- A small disturbance of such a star could either lead to a collapse or an explosion. This is the case if radiation pressure dominates (at high T and low ρ), or the pressure of relativistically degenerate electrons (at very high ρ).

Overall, if the configuration of a star is to be approximately described by a polytrope (in which case γ and γ_{ad} are identical), the index n may only vary between 1.5 and 3, or

$$\frac{4}{3} < \gamma_{ad} \leq \frac{5}{3}$$

Cases of dynamical instability

- We have seen earlier (Lectures 3 and 7) that the contribution of radiation pressure increases with mass and becomes dominant for $M \gtrsim 100\ M_\odot$. A gas dominated by radiation pressure has an adiabatic index $\gamma_{ad}=4/3$, or $n=3$, which means that hydrostatic equilibrium in such stars becomes marginally unstable. Therefore, stars much more massive than 100 $M_\odot$ should be very unstable, and indeed none are known to exist (while those with $M > 50\ M_\odot$ indeed show signs of being close to instability, e.g. they lose mass very readily).
- A process that can lead to $\gamma_{ad}<4/3$ is partial ionization (e.g. $H \leftrightarrow H^+ + e^-$). Since this normally occurs in the very outer layers, where P/ρ is small, it does not lead to overall dynamical instability of the star. However, partial ionization is connected to driving oscillations in some kinds of star.
- At very high temperatures two other processes can occur that have a similar effect to ionization:
  - These are pair creation ($\gamma + \gamma \leftrightarrow e^+ + e^-$) and photo-disintegration of nuclei (e.g. γ + Fe $\leftrightarrow \alpha$). These processes, that may occur in massive stars in late stages of evolution, also lead to $\gamma_{ad}<4/3$ but now in the core of the star. These processes can lead to a stellar explosion or collapse.

Summary

- We discussed methods of finding the solution of the equations of stellar structure.
- We have defined a method to relate the internal pressure and density as a function of radius – the polytropic equation of state.
- We derived the Lane-Emden equation.
- We saw how this equation could be numerically integrated in general.
- We derived a number useful relations between stellar parameters.
- There is a theoretical upper limit to the mass of a white dwarf (Chandrasekhar limit). It is confirmed by observations, we do not see WDs with masses >$1.4M_{\odot}$.
- Further increase of the WD mass e.g. as a result of accretion from the companion, will lead to the loss of stability and collapse, causing supernovae type Ia explosions.
- We compared the n=3 polytrope with the Standard Solar model, finding quite good agreement considering how simple the input physics was.
- Finally, we discussed cases of dynamical instability of stars.

Stellar evolution codes

- A stellar evolution code — a piece of software that can construct a model for the interior of a star, and then evolve it over time.
- Stellar evolution codes are often complicated to use. Rich Townsend from the University of Wisconsin-Madison created EZ-Web, a simple, web-based interface to a code that can be used to calculate models over a wide range of masses and metallicities: http://ftp.astro.wisc.edu/~townsend/static.php?ref=ez-web
- Read **carefully** the description of the program on its webpage and play with it.
- To construct and evolve a model, enter parameters into the form, and then submit the calculation request to the server.

Submit a Calculation

Initial Mass 1.0
Metallicity 0.02
Maximum Age 0
Maximum Number of Steps 0
Create Detailed Structure Files? ☑
Use CGS units? ☑
Email Address vitaly.neustroev@oulu.fi

Submit!

EZ-Web: Output File Formats

Detailed structure files are text (ASCII) files containing one line for each grid point of the model. Each line is divided into 36 columns, containing the following data:

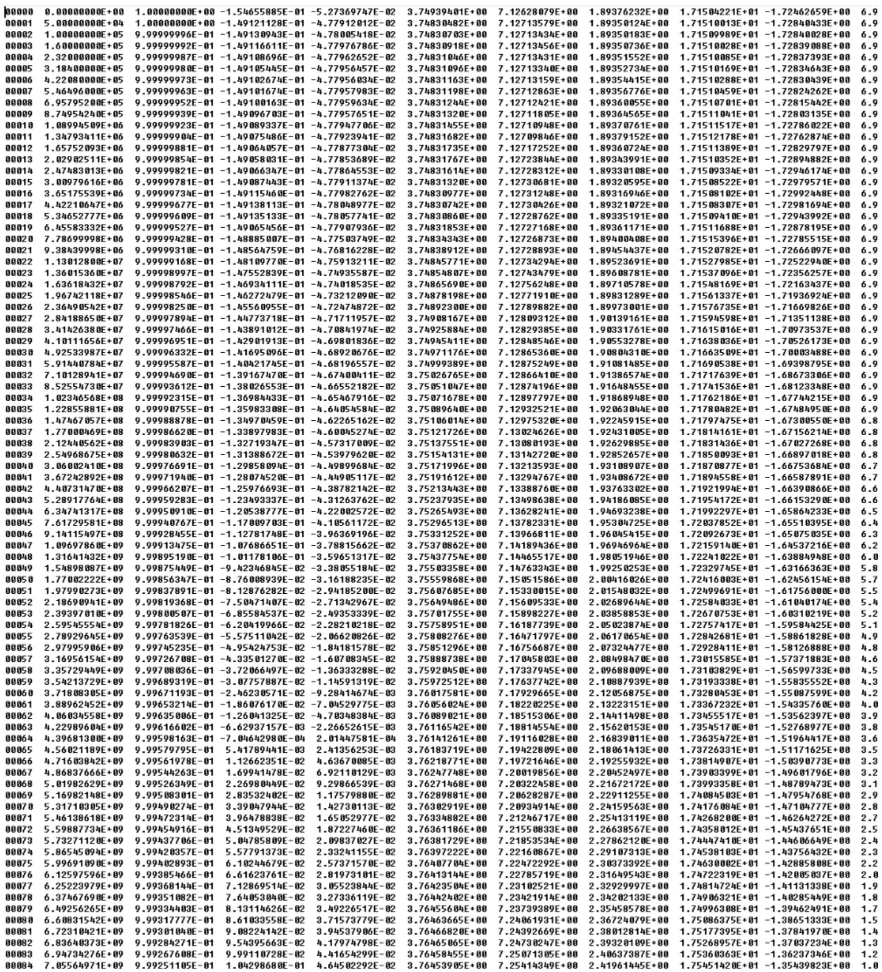
Column Number	Datum	Description
1	M_r	Lagrangian mass coordinate ($M_\odot$)
2	r	Radius coordinate ($R_\odot$)
3	L_r	Luminosity ($L_\odot$)
4	P	Total pressure (N m^{-2})
5	ρ	Density (kg m^{-3})
6	T	Temperature (K)
7	U	Specific internal energy (J kg^{-1})
8	S	Specific entropy (J K^{-1} kg^{-1})
9	c_P	Specific heat at constant pressure (J K^{-1} kg^{-1})
10	Γ_1	First adiabatic exponent
11	∇_{ad}	Adiabatic temperature gradient
12	μ	Mean molecular weight (see note below)
13	n_e	Electron number density (m^{-3})
14	P_e	Electron pressure (N m^{-2})
15	P_r	Radiation pressure (N m^{-2})
16	∇_{rad}	Radiative temperature gradient
17	∇	Material temperature gradient
18	v_c	Convective velocity (m s^{-1})
19	κ	Rosseland mean opacity (m^2 kg^{-1})
20	ε_{nuc}	Power per unit mass from all nuclear reactions, excluding neutrino losses (W kg^{-1})
21	ε_{pp}	Power per unit mass from PP chain (W kg^{-1})
22	ε_{CNO}	Power per unit mass from CNO cycle (W kg^{-1})
23	$\varepsilon_{3\alpha}$	Power per unit mass from triple-alpha reaction (W kg^{-1})
24	$\varepsilon_{\nu,nuc}$	Power loss per unit mass in nuclear neutrinos (W kg^{-1})
25	ε_{ν}	Power loss per unit mass in non-nuclear neutrinos (W kg^{-1})
26	ε_{grav}	Power per unit mass from gravitational contraction (W kg^{-1})
27	X	Hydrogen mass fraction (all ionization stages)
28	—	Molecular hydrogen mass fraction
29	X^+	Singly-ionized hydrogen mass fraction
30	Y	Helium mass fraction (all ionization stages)
31	Y^+	Singly-ionized helium mass fraction
32	Y^{++}	Doubly-ionized helium mass fraction
33	X_C	Carbon mass fraction
34	X_N	Nitrogen mass fraction
35	X_O	Oxygen mass fraction
36	ψ	Electron degeneracy parameter

Summary files have the filename 'summary.txt'. They are text (ASCII) files containing one line for each time step. Each line is divided into 23 columns, containing the following data:

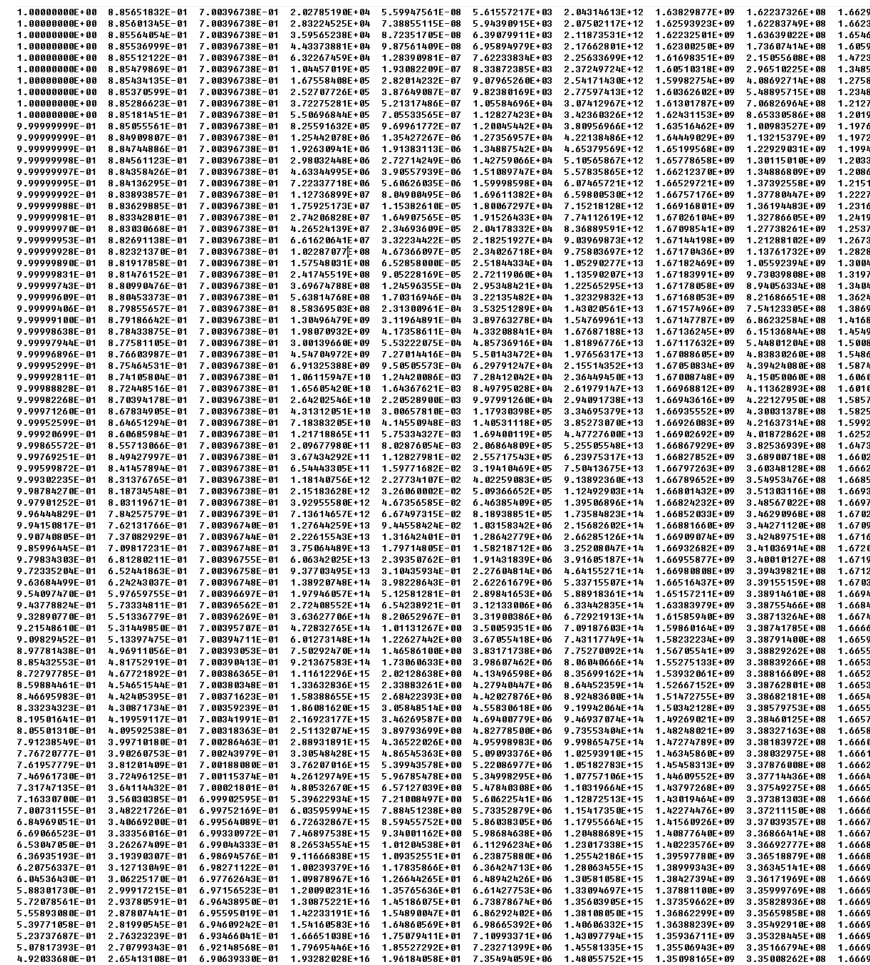
Column Number	Datum	Description
1	i	Step number
2	t	Age (years)
3	M	Mass ($M_\odot$)
4	$\log_{10} L$	Luminosity ($L_\odot$)
5	$\log_{10} R$	Radius ($R_\odot$)
6	$\log_{10} T_s$	Surface temperature (K)
7	$\log_{10} T_c$	Central temperature (K)
8	$\log_{10} \rho_c$	Central density (kg m^{-3})
9	$\log_{10} P_c$	Central pressure (N m^{-2})
10	ψ_c	Central electron degeneracy parameter
11	X_c	Central hydrogen mass fraction
12	Y_c	Central helium mass fraction
13	$X_{C,c}$	Central carbon mass fraction
14	$X_{N,c}$	Central nitrogen mass fraction
15	$X_{O,c}$	Central oxygen mass fraction
16	τ_{dyn}	Dynamical timescale (seconds)
17	τ_{KH}	Kelvin-Helmholtz timescale (years)
18	τ_{nuc}	Nuclear timescale (years)
19	L_{pp}	Luminosity from PP chain ($L_\odot$)
20	L_{CNO}	Luminosity from CNO cycle ($L_\odot$)
21	$L_{3\alpha}$	Luminosity from triple-alpha reactions ($L_\odot$)
22	L_Z	Luminosity from metal burning ($L_\odot$)
23	L_ν	Luminosity of neutrino losses ($L_\odot$)
24	M_{He}	Mass of helium core ($M_\odot$)
25	M_C	Mass of carbon core ($M_\odot$)
26	M_O	Mass of oxygen core ($M_\odot$)
27	R_{He}	Radius of helium core ($R_\odot$)
28	R_C	Radius of carbon core ($R_\odot$)
29	R_O	Radius of oxygen core ($R_\odot$)

EZ-Web: Output Files

summary.txt



structure_00000.txt



EZ-Web: Limitations

- EZ-Web and the underlying EZ code have a number of limitations which restrict their validity. In some cases, the results can be misleading or inaccurate, and users should be aware of this if using EZ-Web for research purposes.

- As an alternative to EZ-Web, consider using MESA-Web — a web-based interface to the fully-featured MESA stellar evolution code. MESA-Web can produce models which are suitable for detailed scientific investigations: http://user.astro.wisc.edu/~townsend/static.php?ref=mesa-web-submit

Schematic stellar evolution

THE (TEMPERATURE, DENSITY) DIAGRAM
ZONES OF THE EQUATION OF STATE
ZONES OF NUCLEAR BURNING
EVOLUTION OF A STAR IN THE ($\log P$, $\log \rho$) PLANE

Introduction

- We have derived all the differential equations that define uniquely the equilibrium properties of a star of a given mass and composition. We know how to solve them.
- Our task now is to combine the knowledge acquired so far into a general picture of the evolution of stars.
- We will consider the schematic evolution of a star, as seen from its centre. The centre is the point with the highest pressure and density, and (usually) the highest temperature, where nuclear burning proceeds fastest. Therefore, the centre is the most evolved part of the star, and it sets the pace of evolution, with the outer layers lagging behind.
- The stellar centre is characterized by the central density ρ_c, pressure P_c and temperature T_c and the composition (usually expressed in terms of μ and/or μ_e). These quantities are related by the equation of state (EOS).
- We can thus represent the evolution of a star by an evolutionary track in the (P_c, ρ_c) diagram or the (T_c, ρ_c) diagram.
- Since the only property that distinguishes the evolutionary track of a star from that of any other star of the same composition is its **mass**, we may expect to obtain different lines in the (T_c, ρ_c) plane for different masses.
- The (T_c, ρ_c) plane will be divided into zones dominated by different equations of state and different nuclear processes.

Zones of the equation of state

- As the ranges of density and temperature typical of stellar interiors span many orders of magnitude, logarithmic scales will be used for both.
- By considering the EOS we can derive the evolution of the central temperature. This is obviously crucial for the evolution track of a star because nuclear burning requires T_c to reach certain (high) values.
- We have previously encountered various regimes for the EOS:
  - The most common EOS is that of an ideal gas: $P = \frac{\Re T \rho}{\mu} = K_0 \rho T$
  - If radiation pressure is dominant, then the equation of state changes to $P = \frac{aT^4}{3}$
  - At high densities and relatively low temperatures, the electrons become degenerate, and since their contribution to the pressure is dominant, the EOS is replaced by $P = K_1 \rho^{5/3}$. This is independent of temperature. More accurately: the complete degeneracy implied by this relation is only achieved when $T_c \rightarrow 0$.
  - For still higher densities, when relativistic effects play an important role, the EOS changes to the form $P = K_2 \rho^{4/3}$.

EOS in the ($\log P$, $\log \rho$) plane

- The transition from one state to the other is, of course, gradual with the change in density and temperature, but an approximate boundary may be traced in the ($\log P$, $\log \rho$) plane.
- The boundaries may be defined by the requirement that the pressure obtained from a one EOS be equal to that obtained another. For example, the boundary between the ideal gas zone and the non-relativistic-degeneracy zone may be obtained, $K_0 \rho T = K_1 \rho^{5/3}$, which defines a straight line with a slope of 1.5:

$$\log \rho = 1.5 \log T + \text{constant}.$$

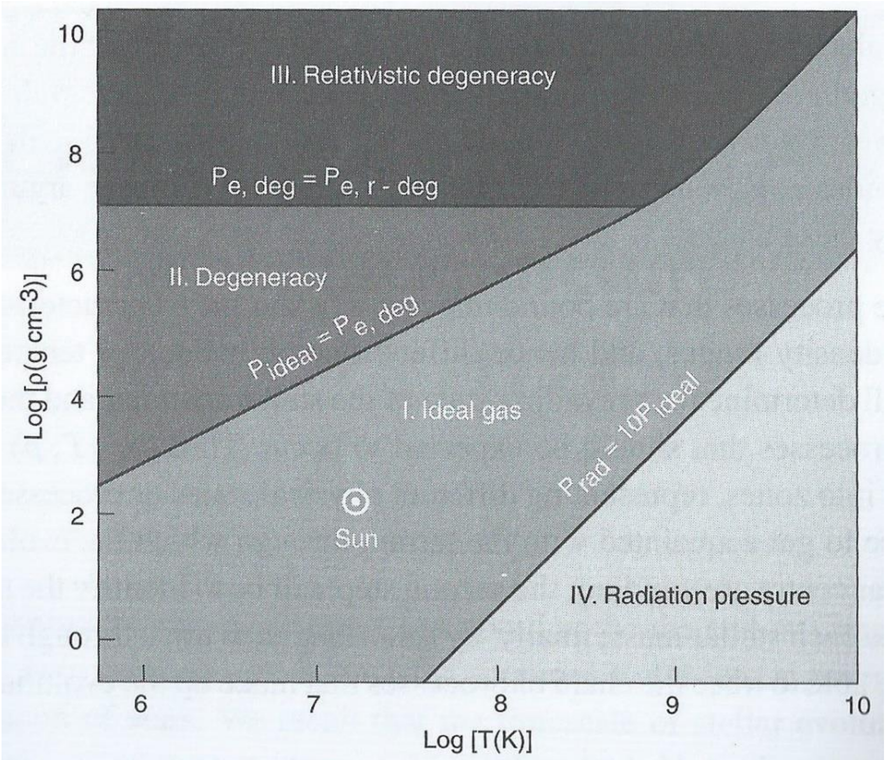


Figure from Prialnik

Zones of nuclear burning

The nuclear energy generation rate is a sensitive function of the temperature, which can be written as

$$\varepsilon \approx \varepsilon_0 \rho^{\lambda} T^{n}$$

where for most nuclear reactions (those involving two nuclei) $\lambda=1$, while n depends mainly on the masses and charges of the nuclei involved and usually $n>>1$.

For H-burning by the pp-chain, $n\approx 4$ and for the CNO-cycle which dominates at somewhat higher temperature, $n\approx 18$.

For He-burning by the triple-alpha reaction, $n\approx 40$ (and $\lambda=2$ because three particles are involved). For C-burning and O-burning reactions n is even larger.

As discussed in previous lectures, the consequences of this strong temperature sensitivity are that

- each nuclear reaction takes place at a particular, nearly constant temperature, and
- nuclear burning cycles of subsequent heavier elements are well separated in temperature

The threshold given by $\varepsilon \approx \varepsilon_{min}$ is

$$\log \rho = -\frac{n}{\lambda} \log T + \frac{1}{\lambda} \log \frac{\varepsilon_{min}}{\varepsilon_0}$$

On one side of the threshold the rate of nuclear burning may be assumed negligible, and on the other side – considerable.

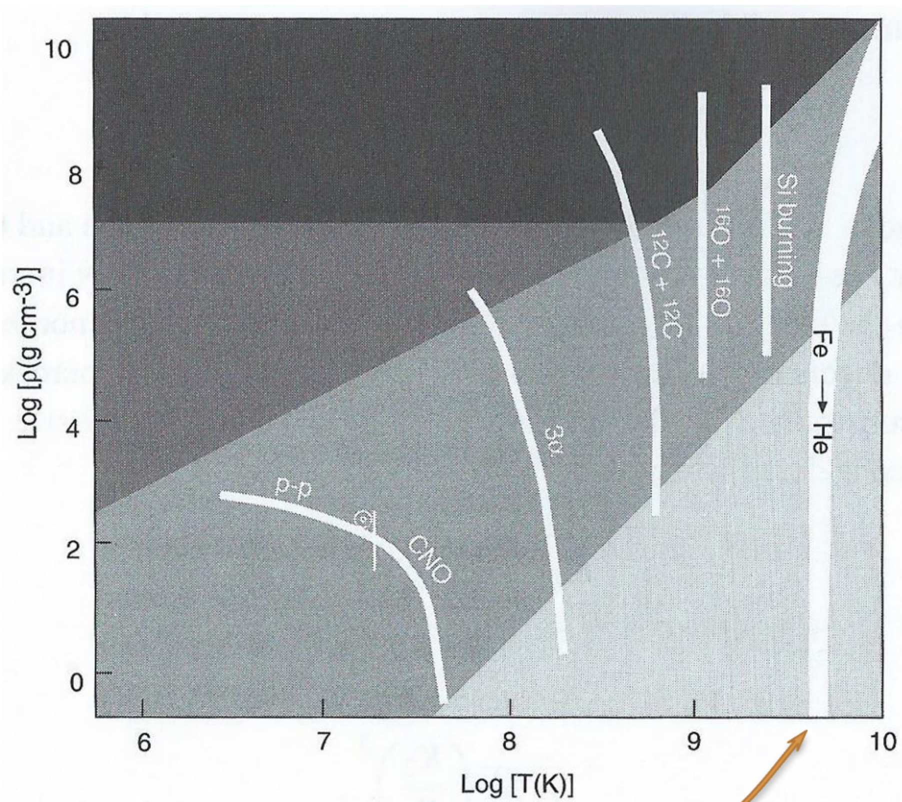
Nuclear burning in the (log P, log ρ) plane

The transformation of hydrogen into the iron group elements comprises five major stages:

- hydrogen burning into helium either by the p-p chain or by the CNO cycle;
- helium burning into carbon by the 3a reaction;
- carbon burning;
- oxygen burning;
- silicon burning.

Nucleosynthesis ends with iron.

Iron nuclei heated to very high temperatures are disintegrated by energetic photons into helium nuclei. This energy **absorbing** process reaches equilibrium, with the relative abundance of iron to helium nuclei determined by the values of temperature and density. A threshold may be defined for the process of iron photodisintegration, as a strip in the (log P, log ρ) plane, by the requirement that the number of helium and iron nuclei be approximately equal.



The evolutionary path of the central point

Are the centre of a star of given mass M may assume any combination of temperature and density values, or these values are in some way constrained by M ?

We now regard the ($\log P$, $\log \rho$) plane as a ($\log P_c$, $\log \rho_c$) plane, referring to the stellar centre.

Assuming a polytropic configuration for a star in hydrostatic equilibrium, the central density is related to the central pressure by equation

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3}$$

This relation is only weakly dependent on the polytropic index n, especially for stable configurations, for which n varies between 1.5 and 3 and the coefficient B_n between 0.157 and 0.206 (see the table above), and it is independent of K. As we noted before, this relation provides a good approximation to hydrostatic equilibrium for any configuration.

Additionally, P_c is related to ρ_c and T_c by the EOS (we have different ones). Combining each of them with the above relation, we can eliminate P_c to obtain a relation between ρ_c and T_c.

For example, for a star of mass M, whose central point is found in the ideal gas zone I, we obtain the relation between ρ_c and T_c

$$\rho_c = \frac{K_0^3}{4\pi B_n^3 G^3} \frac{T_c^3}{M^2}$$

For a star of given mass, the central density varies as the central temperature cubed.

The central point in the (log P, log ρ) plane

- For a star of mass M, whose central point is found in the ideal gas zone I, we obtain the relation

$$\rho_c = \frac{K_0^3}{4\pi B_n^3 G^3} \frac{T_c^3}{M^2}$$

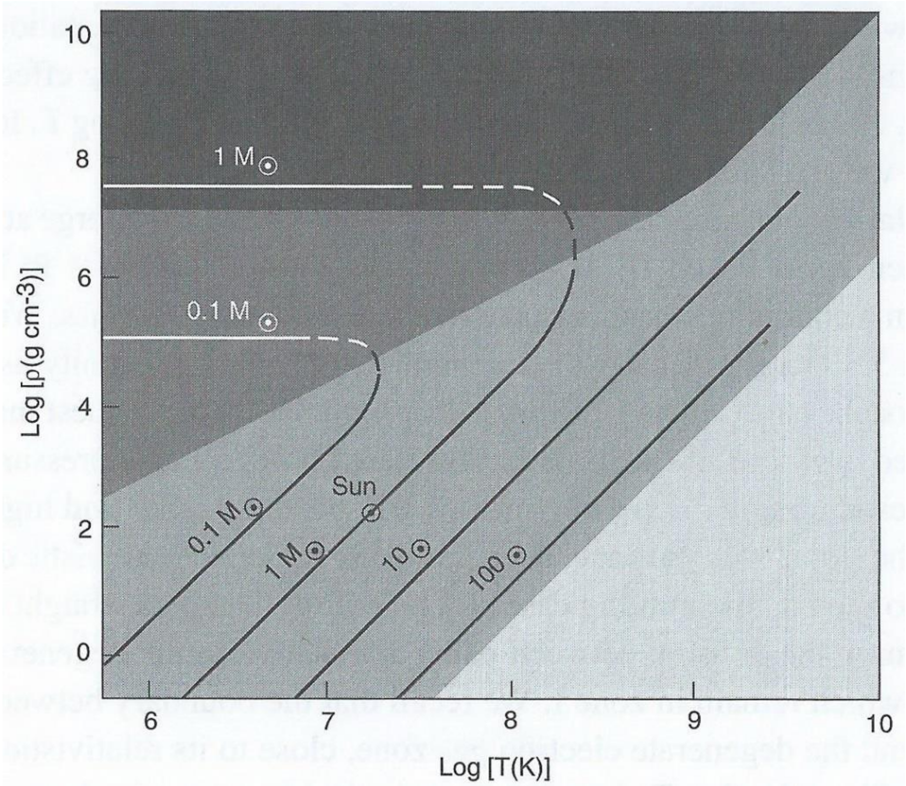
On logarithmic scales, it becomes a straight line with a slope of 3. Different masses define different parallel lines.

- If at the centre of a star the electrons are strongly but non-relativistically degenerate, the central point is found in zone II . Then the relation is

$$\rho_c = \left(\frac{B_{1.5} G}{K_1}\right)^3 M^2$$

which replaces the ideal gas relation. Here ρ_c is independent of T_c, and the corresponding line in the (log P_c, log ρ_c) plane is horizontal and increases with mass M.

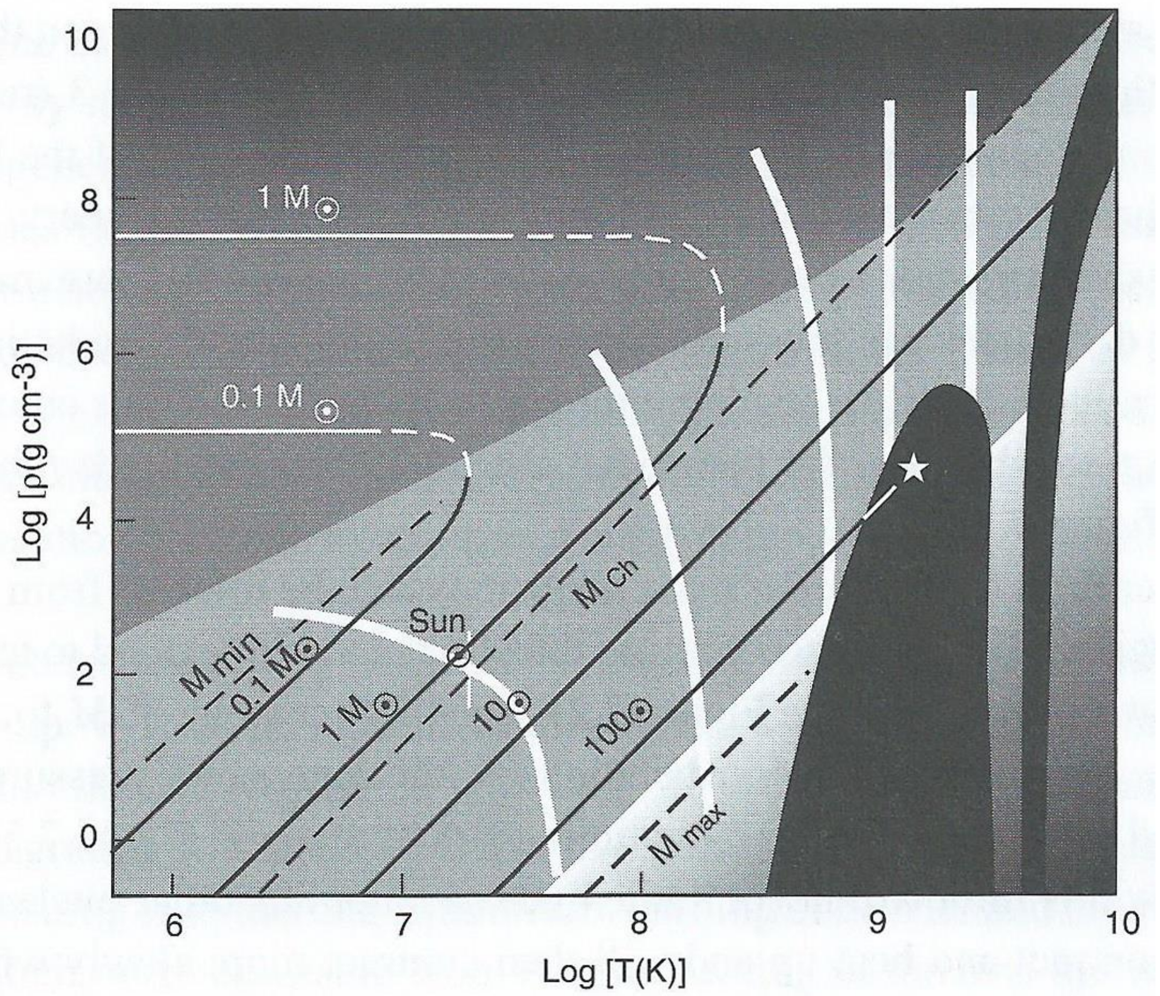
- Zones I and II are the only stable regions in the (log P, log ρ) plane. Hence, there is no need to consider the others.



For relatively low masses, the relations will merge at the boundary between zones I and II, resulting in a continuous bending path characteristic of each mass.

Evolution of a star in the (log P, log ρ) plane

Stars are limited to a rather narrow mass range of 0.1 $M_\odot$ to ~100 $M_\odot$. The lower limit is set by the minimum temperature required for nuclear burning, and the upper limit by the requirement of dynamical stability.



Star formation

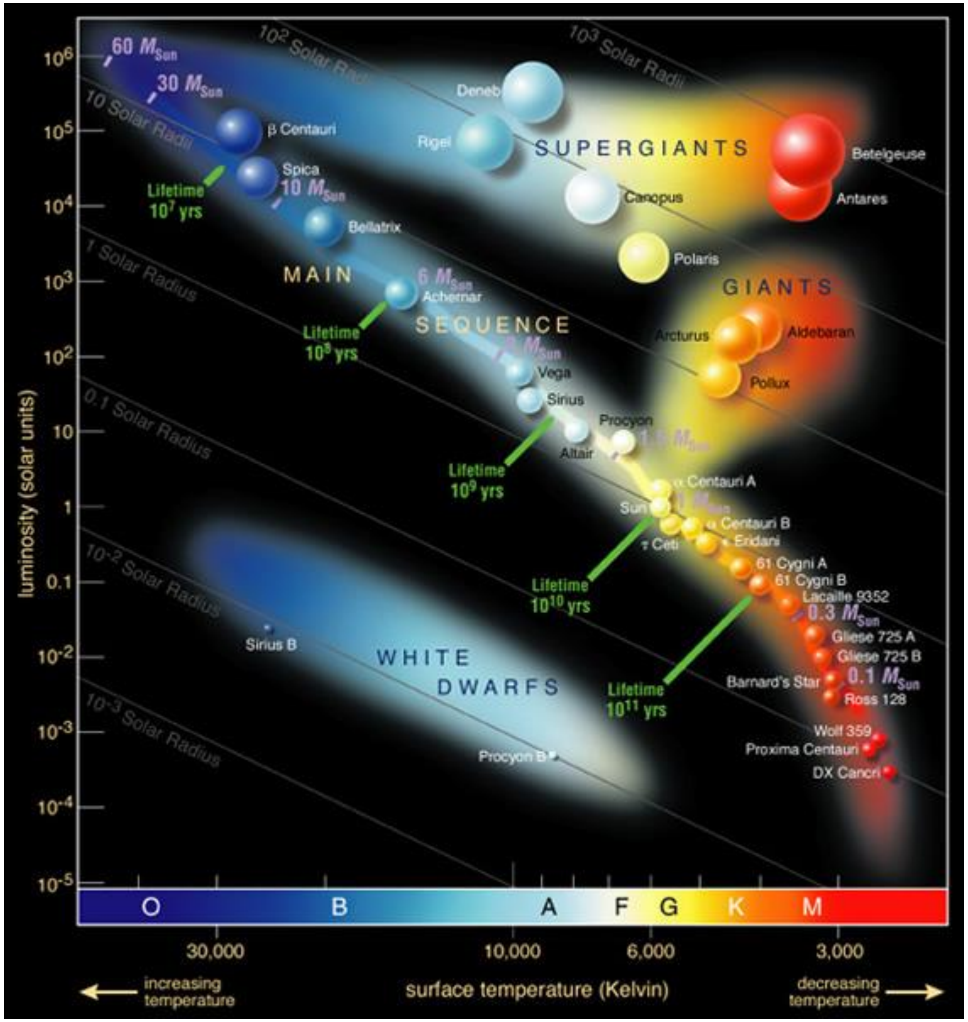
BASIC PHYSICS OF STAR FORMATION
JEANS INSTABILITY
THE JEANS MASS, LENGTH, DENSITY
STEPS OF STAR FORMATION

Introduction

- Today we will deal with early phases in the evolution of stars, as they evolve towards and during the main-sequence phase.

- We start with a very brief (and incomplete) overview of the formation of stars.

The Hertzsprung–Russell diagram



Stars are like people in that they are born, grow up, mature, and die.
A star's mass determines what life path it will take.

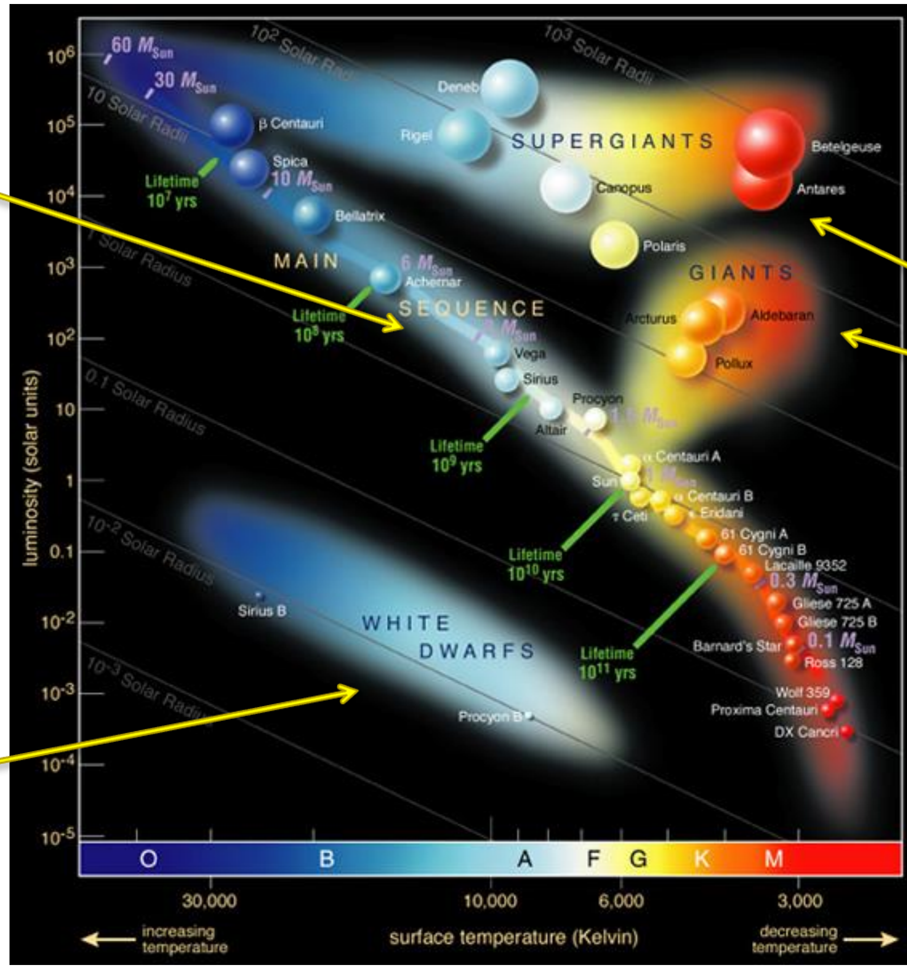
The HR diagram shows the relationship between the stars' luminosities versus their effective temperatures.

Different evolutionary stages correspond to different positions at HR diagram.

90% of a lifetime a star spends at the Main Sequence, but before it, a star must be formed and have arrived there.

The HR diagram

Most of the stars lie on the Main Sequence, with increasing L as T increases



A relatively cool star can be quite luminous if it has a large enough radius (10-100 $R_\odot$): Red Giants and Supergiants

A relatively hot star can have very low luminosity, if its radius is very small (0.01 $R_\odot$): White Dwarfs

Star formation

- Observations indicate that stars are formed in giant molecular clouds with masses of order $10^2 - 10^5\ M_\odot$. These clouds have typical dimensions of $\sim$100 parsec, temperatures of $10\text{–}100$ K and densities of $10^2\text{–}10^4$ cm^{-3} (where the lowest temperatures pertain to the densest parts of the cloud).
- A certain fraction, about 1%, of the cloud material is in the form of dust which makes the clouds very opaque to visual wavelengths.
- While the densities of molecular clouds are among the highest encountered in the Interstellar Medium (ISM), even this gas is extremely rarified compared to gas at an atmospheric density of $\sim 10^{19}$ cm^{-3}. In fact, the densities of molecular clouds are many orders of magnitude lower than the density of the best vacuum achievable in the laboratory.
- As we have seen, the mean mass densities inside stars are ~ 1 g cm^{-3}, i.e., particle densities of $\sim 10^{24}$ cm^{-3}. Thus, to form new stars, some regions of a molecular cloud must be compressed by many orders of magnitude.

The nearest giant molecular clouds are in the Orion star-forming region, at a distance of about 500 pc.

In spiral galaxies, star formation is concentrated along spiral arms.

Spiral arms are places where gas is compressed, probably the first step toward star formation.

What we know from observations?

The details of the process of star formation are not understood yet. We can outline, however, some of the general criteria under which gravitational contraction of a gas cloud can proceed, and potentially lead to the conditions required for star formation. This is what we know from observations:

- Stars form out of molecular gas which is assembled into dense molecular clouds in spiral arms.
- Molecular clouds have a complex, often filamentary structure. Individual stars, or small groups, form from the smallest scale structures, cloud cores of size ~0.1 pc.
- Molecular clouds probably have lifetimes of 10^6 to 10^7 yr, which is only a few dynamical times. Star formation is a fairly rapid process once molecular clouds have formed.
- If massive stars form within a young cluster, their ionizing radiation / stellar winds / supernovae destroy the molecular cloud on a short time scale.
- Most stars (~80%) form in clusters at least as rich as Orion.

Basic physics of star formation

Consider the forces acting on a "star forming unit" within a molecular cloud - a molecular cloud core:

- **Gravity** - act to collapse the cloud
- **Pressure**
- **Magnetic fields**
- **"Bulk motions"**

(Pressure, magnetic fields and "bulk motions":) sources of support against collapse to form a star

If somehow we form a core in which gravity dominates over all other forces, collapse will occur on the dynamical or free-fall time:

$$t_{dyn} = \frac{\pi}{2\sqrt{2}}\left(\frac{R^3}{GM}\right)^{1/2} = \left(\frac{3\pi}{32}\right)^{1/2}\frac{1}{\sqrt{G\bar{\rho}}}$$

... for a cloud of mass M, radius R, and mean density $\bar{\rho}$.

The Jeans Mass

Ignore for now magnetic fields and bulk motions. The **Jeans mass** is the minimum mass a cloud must have if gravity is to overwhelm pressure and initiate collapse.

Borderline case is the one where the cloud is in hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

To derive an estimate of the Jeans mass, consider a cloud of mass M and radius R:

- approximate derivative dP/dr by $-P/R$
- assume pressure is that of an ideal gas: $P = \frac{\Re \rho T}{\mu}$

Substitute:

$$-\frac{\Re \rho T}{\mu\, R} = -\frac{GM}{R^2}\rho \quad \rightarrow \quad M = \frac{\Re}{\mu G} T R$$

Can eliminate R in favor of the density ρ using $M = \frac{4}{3}\pi R^3 \rho$ and we get a final expression for **the Jeans mass**:

$$M_J = \left(\frac{\Re}{\mu G}\right)^{3/2} \left(\frac{3}{4\pi}\right)^{1/2} T^{3/2} \rho^{-1/2}$$

This is a basic formula for star formation. Numerical constants can vary depending on the details of the derivation.

Mass scale of star formation

$$M_J = \left(\frac{\Re}{\mu G}\right)^{3/2} \left(\frac{3}{4\pi}\right)^{1/2} T^{3/2} \rho^{-1/2}$$

Observationally, stars form from cold dense **molecular** gas with typical density $\rho \sim 10^{-19}$ g cm^{-3} and temperature $T \sim 10$ K, take $\mu = 2$ for molecular hydrogen **What ?**
Put all these numbers in the Jeans mass formula and get

$\Re = 8.314 \times 10^7$ erg mol^{-1}K^{-1}

$$M_J = 7.8 \times 10^{32} \text{ g} \approx 0.4\, M_\odot$$

...which matches the typical mass of stars in the Galaxy!

Level of agreement here is "too good to be true", however we can conclude that the Jeans mass in these conditions is about a Solar mass and sets the basic mass scale for star formation.

The Jeans length

We can likewise define a characteristic length scale — **the Jeans length** — by eliminating mass rather than radius from the previous expression:

$$M = \frac{\Re}{\mu G} TR \quad \rightarrow \quad \frac{4}{3}\pi R^3 \rho = \frac{\Re}{\mu G} TR$$

$$R_J = \left(\frac{\Re}{\mu G}\right)^{1/2} \left(\frac{3}{4\pi}\right)^{1/2} T^{1/2} \rho^{-1/2}$$

For the same density / temperature as before, $R_J \sim 1.2 \times 10^{17}$ cm = 10^4 AU

Free-fall timescale for a cloud of this density is:

$$t_{dyn} \sim \frac{1}{\sqrt{G\bar{\rho}}} \sim 10^{13}\ \text{s} \sim 10^5\ \text{yr}$$

Star formation in these conditions should create solar mass stars within a few hundred thousand years.

The Jeans density

The above condition can also be stated as a condition on the density, which must be larger than **the Jeans density**:

$$M_J = \left(\frac{\Re}{\mu G}\right)^{3/2} \left(\frac{3}{4\pi}\right)^{1/2} T^{3/2} \rho^{-1/2} \quad \rightarrow \quad \rho_J = \left(\frac{\Re}{\mu G}\right)^{3} \frac{3T^3}{4\pi M^2}$$

For a typical cloud mass of $1000\ M_\odot$ and temperature $50\ K$,

$$\rho_J \sim 1.8\times10^{-24}\ \mathrm{g\ cm^{-3}}$$

corresponding to a number density of

$$n_J = \frac{\rho_J}{\mu m_p} = \frac{1.8\times10^{-24}}{2\times1.67\times10^{-24}} \approx 0.5\ \mathrm{cm^{-3}}$$

Thus, the typical observed density of molecular clouds, 10^2–10^4 $\mathrm{cm^{-3}}$, is several orders of magnitude higher than the Jeans density and, according to the criterion we have just formulated, the clouds should be unstable to gravitational collapse.

Since the clouds exist and appear to be long lived, another source of pressure, other than thermal pressure, must be present. It is currently believed that the dominant pressure is provided by turbulence, magnetic fields, or both.

Interstellar cloud collapse and fragmentation

What happens during the collapse?

Jeans mass formula:

$$M_J \propto T^{3/2}\rho^{-1/2}$$

Initially: gas is optically thin, the cloud is transparent to far-infrared radiation and thus cools efficiently, so that the early stages of the collapse are isothermal (T constant).

If T stays constant, the density of the collapsing cloud increases, its Jeans mass decreases.

- Gravity becomes even more dominant over pressure
- The stability criterion within the cloud may now also be violated:
  M_J drops - allows for possibility that the cloud might break up into smaller fragments,
  so that the cloud starts to fragment into smaller pieces, each of which continues to collapse.
- The fragmentation process probably continues until the mass of the smallest fragments (dictated by the decreasing Jeans mass) is less than $0.1\ M_\odot$.

Formation of a proto-stellar core

The increasing density of the collapsing cloud fragment eventually makes the gas opaque to infrared photons. As a result, radiation is trapped within the central part of the cloud, leading to heating the cloud and an increase in gas pressure. It changes the nature of the collapse from an "isothermal" phase to an "adiabatic" phase. By the definition of an adiabat,

$$P \propto \rho^{\gamma} \rightarrow \quad T \propto \rho^{\gamma-1}$$

Substituting this into the Jeans equation we obtain

$$M_J \propto T^{3/2}\rho^{-1/2} = \rho^{\frac{3}{2}(\gamma-1)}\rho^{-1/2} = \rho^{\frac{(3\gamma-4)}{2}}$$

For γ=5/3, $M_J \propto \rho^{1/2}$.

In other words, the Jeans mass no longer decreases with increasing density!
Thus, the cloud is no longer unstable to fragmentation. As a result, the cloud core comes into hydrostatic equilibrium and the dynamical collapse is slowed to a quasistatic contraction. At this stage we may start to speak of a protostar.

The contraction will now proceed slowly, at a pace determined by the rate at which thermal energy is radiated away. The gravitational energy is converted to dissociation of H_2, which uses up 4.5 eV per molecule, and ionization of hydrogen, which takes 13.6 eV per atom (see below).

Accretion

The surrounding gas keeps falling onto the protostellar core, so that the next phase is dominated by accretion. Since the contracting clouds contain a substantial amount of angular momentum, the infalling gas forms an accretion disc around the protostar. These accretion discs are a ubiquitous feature of the star formation process and are observed around most very young stars, mostly at infrared and sub-millimeter wavelengths.

The accretion of gas generates gravitational energy. As with other accretion, half the energy goes into further heating of the core, and half is radiated away, providing the luminosity of the protostar:

$$L_{acc} = \frac{GM\dot{M}}{2R}$$

where M and R are the mass and radius of the core and $\dot{M}$ is the mass accretion rate. Meanwhile, the core heats up almost adiabatically since the accretion timescale $\tau_{\mathrm{acc}} = M/\dot{M}$ is much smaller than the thermal timescale τ_{KH}.

Dissociation and ionization

- The gas initially consists of molecular hydrogen and behaves like an ideal gas, such that $\gamma > 4/3$ and the protostellar core is dynamically stable:

$$M_J \propto \rho^{\frac{(3\gamma-4)}{2}}$$

- Eventually, the core reaches a temperature of ~ 2000 K and begins to dissociate the molecular hydrogen, which is analogous to ionization and leads to a strong increase of the specific heat and a decrease of γ_{ad} below the critical value of $4/3$.
- Hydrostatic equilibrium is no longer possible, and a renewed phase of dynamical collapse follows. The collapse releases energy, which further dissociates molecules without a significant rise in temperature. When H_2 is completely dissociated into atomic hydrogen, the star settles into a new hydrostatic equilibrium. Somewhat later, the same thing happens when the temperature rises enough to ionize hydrogen (and then helium, at $\sim 10^4$ K, and then helium again, at $\sim 8\times10^4$ K).
- When ionization of the protostar is complete it settles back into hydrostatic equilibrium at a **much reduced** radius.
- Note also that the temperature required to doubly ionize helium (~ 80000 K) is far less than that required to fuse hydrogen.

Pre-main sequence phase

- Finally, the accretion slows down and eventually stops and the protostar is revealed as a pre-main sequence star. Its luminosity is now provided by gravitational contraction and, according to the virial theorem, its internal temperature rises as

$$T \propto M^{2/3} \rho^{1/3}$$

Prove it!

- The surface cools and a temperature gradient builds up, transporting heat outwards.
- Further evolution takes place on the thermal timescale τ_{KH}.
- The temperature of this early-evolution proto-star is roughly 4000 K. A size of $R \sim 35\ R_{\odot}$. Then it implies a luminosity of $\sim 10^3\ L_{\odot}$. Note however, that observed pre-main sequence stars have radii substantially smaller than 35 $R_{\odot}$, indicating that additional energy must have been radiated (or otherwise lost) in the contraction process.
- At these low temperatures, the opacity is very high – the dominant source of opacity is H^- (**hydrogen with two electrons** rather than one) – rendering radiative transport inefficient and making the protostar convective throughout.
- The properties of such fully convective stars must be examined more closely.

Summary of star formation on large scales

If the internal gas pressure is not strong enough to prevent gravitational collapse of a region filled with matter, then the Jeans instability may occur, causing the collapse of interstellar gas clouds and subsequent star formation.

Jeans' mass - minimum mass of a gas cloud of temperature T and density ρ that will collapse under gravity:

$$M_J = \left(\frac{\Re}{\mu G}\right)^{3/2} \left(\frac{3}{4\pi}\right)^{1/2} T^{3/2} \rho^{-1/2}$$

Several stages of collapse:

- Initial isothermal collapse - still optically thin
- The creation of proto-stellar cores with surrounding discs
- Collapse slows or halts once gas becomes optically thick - heats up so pressure becomes important again
- Second phase of free-fall collapse as hydrogen molecules are broken up - absorbs energy and robs cloud of pressure support
- Finally forms protostar with radius of 5 - 10 $R_{\odot}$ and the remnants of a disc.

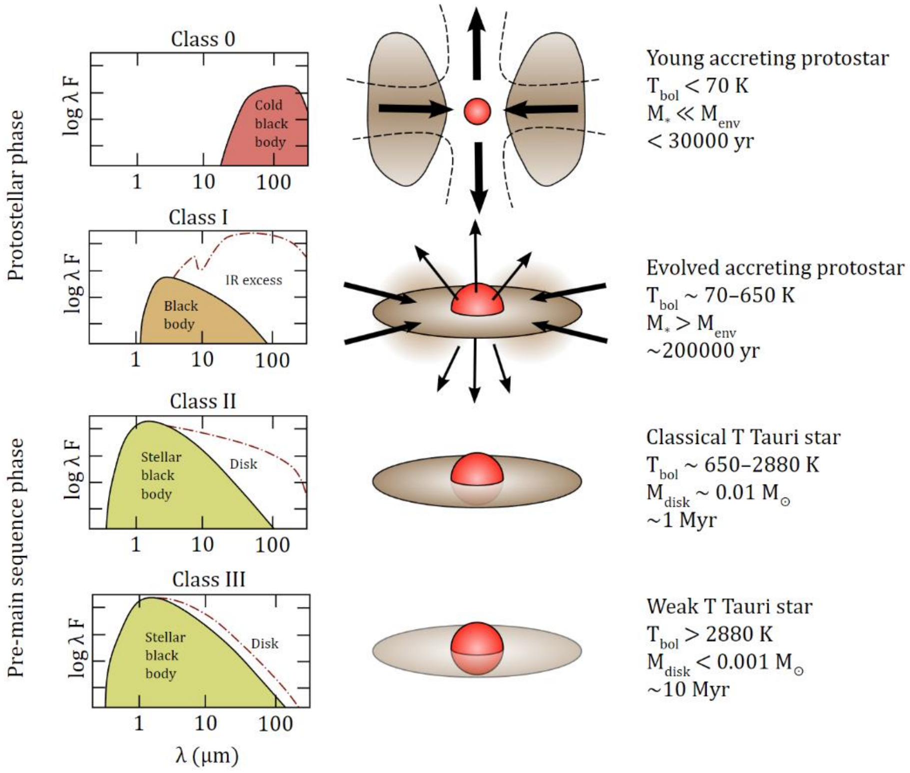
All this happens very rapidly – $t_{dyn} \sim 10^5$ yr – not easy to observe

Identification of Young Stars

Observationally, young stellar objects (YSOs) can be classified

- By infrared excess in their spectral energy distributions (SEDs): Young stars are built up by accretion, and the accretion disc can linger for some time after the nuclear reactions have started. The dust and material in the disk can reprocess some of the light into the IR.
- By X-ray identification: Conservation of angular momentum guarantees that young stars will be rotating quickly. If the star has a convective envelope, then differential rotation will cause an increased magnetic dynamo effect, leading to increased flare activity and X-rays.
- Hα emission: The same mechanism that can create X-rays may also result in Hα emission. In addition, there may still be residual Hα emission from the accretion disc.
- Lithium absorption: Lithium can be burned during the pre-main sequence phase. Stars with lithium absorption in their spectrum must be young (see below).

Four main classes of object have been identified.



On the left, the stellar flux is depicted (shaded area) and the contribution from the disc (dotted line). On the right the corresponding geometry of the object is shown.

Class 0 source

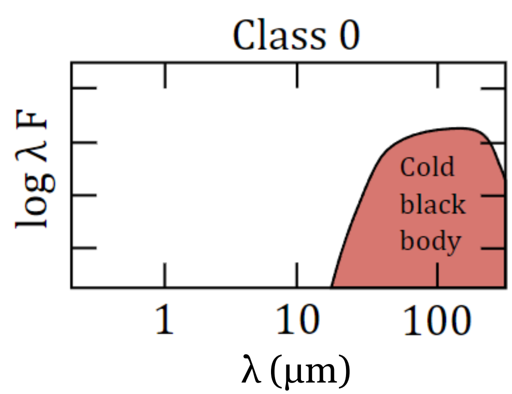
Young accreting protostars. Observationally, this is a source whose SED peaks in the far-infrared or mm part of the spectrum. No flux in the near-infrared (at a few microns). Effective temperature is several tens of degrees Kelvin.

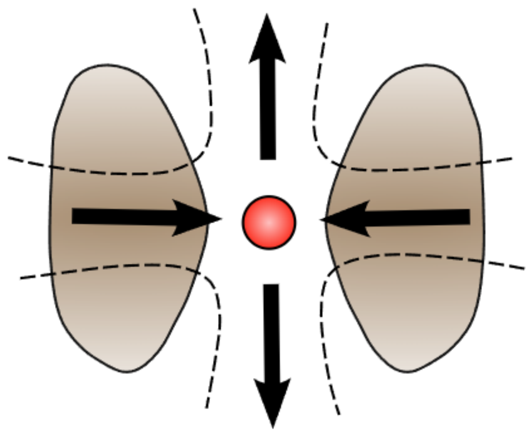
What are **Class 0** sources? Earliest observed stage of star formation…

- Still very cool - not much hotter than molecular cloud cores. Implies extreme youth.
- Deeply embedded in gas and dust, any shorter wavelength radiation is absorbed and re-radiated at longer wavelengths before escaping.
- Fairly small numbers - consistent with short duration of the initial collapse.
- Outflows are seen - suggests a protostar is forming.

$t < 30000$ yr

$T_{bol} < 70$ K
$M_* \ll M_{env}$

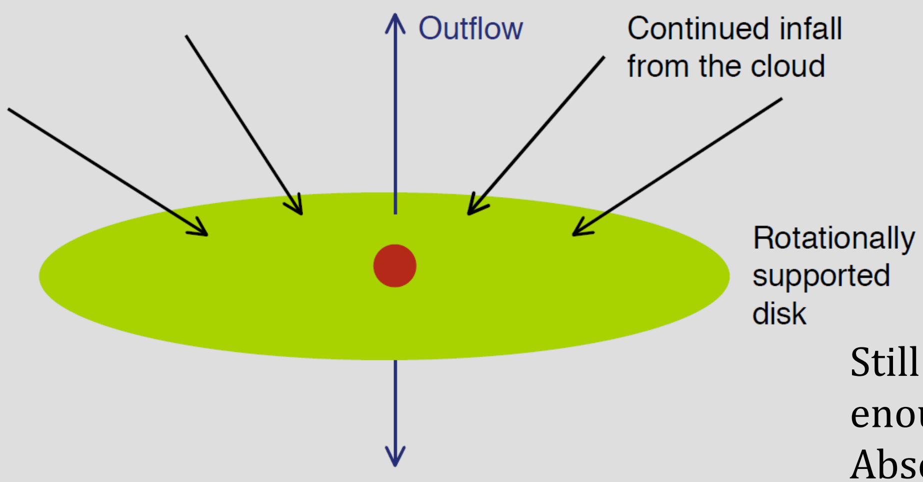


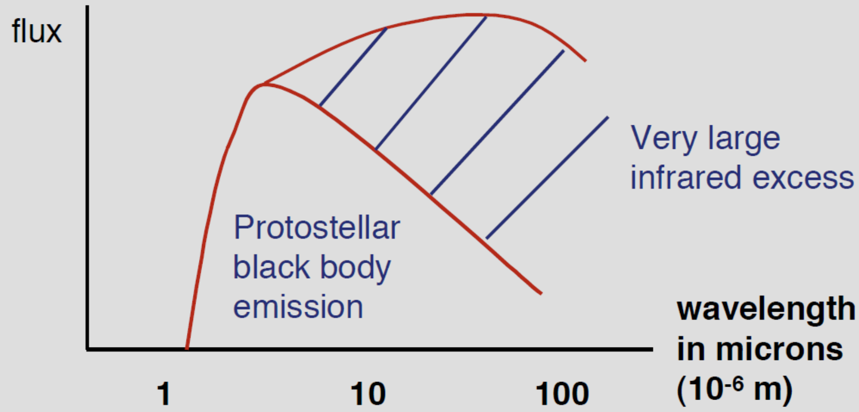


Class I source

Class I sources – evolved accreting protostars – also have SEDs that rise into the mid and far IR. But they differ from Class 0 in having detectable near infrared flux.
Still not seen at visible wavelengths.

$t \approx 200000$ yr
$M_* > M_{env}$
$T_{bol} \sim 70 - 650$ K



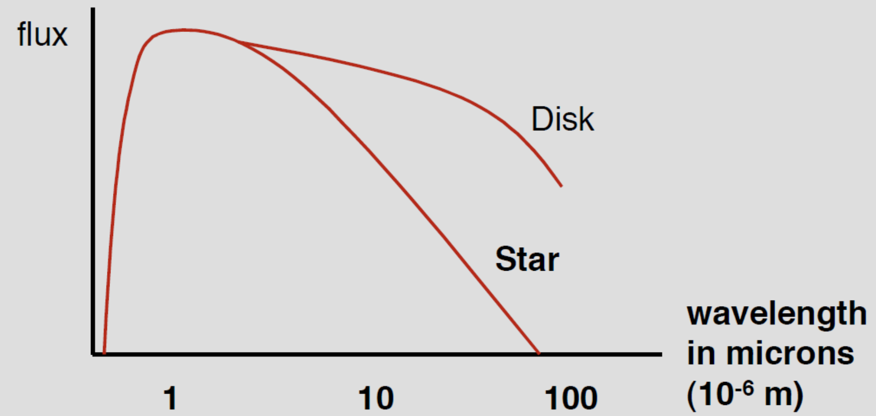


Still can't see the star itself, but dust has cleared enough to see the hot gas and dust close to the star. Absorption and re-radiation of this near-infrared flux by the dust in the envelope produces the far-infrared peak.

Class II source: classical T Tauri stars

Class II: Flat or falling SEDs in the mid-infrared. Optically visible pre-main-sequence stars.

Also called classical T Tauri stars, after the prototype star T Tauri in the Taurus star forming region.



- By this stage almost all of the collapsing cloud has settled onto the star or onto a disc surrounding the star.
- From most angles we can see the young star directly.
- Disk slowly drains onto the star over several million years.

$t \approx 1$ Myr

$M_{\rm disk} \sim 0.01\, M_\odot$

$T_{\rm bol} \sim 650 - 2880$ K

Class III source: weak-lined T Tauri stars

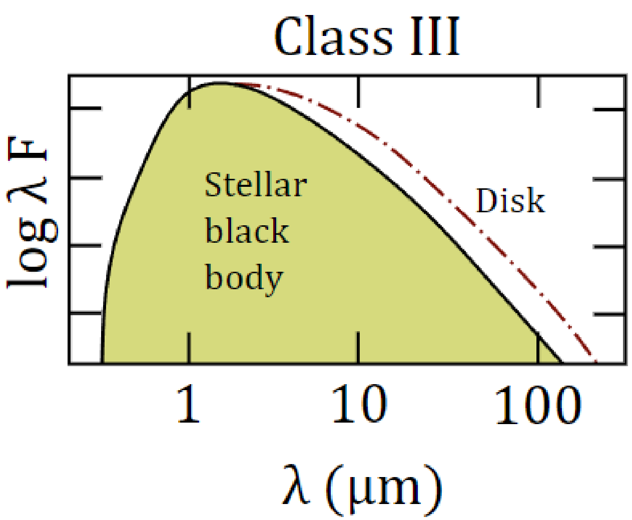
Class III sources: Fairly normal stellar SEDs, but more luminous than main-sequence stars of the same effective temperature (i.e. they lie above the main sequence).

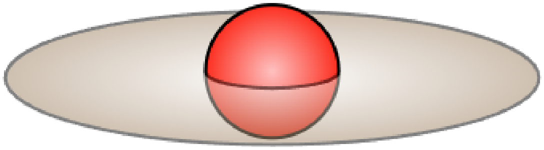
Also, they are more active (e.g. in X-rays) than ordinary main-sequence stars.

$t \approx 10$ Myr

$M_{disk} < 0.001\ M_\odot$

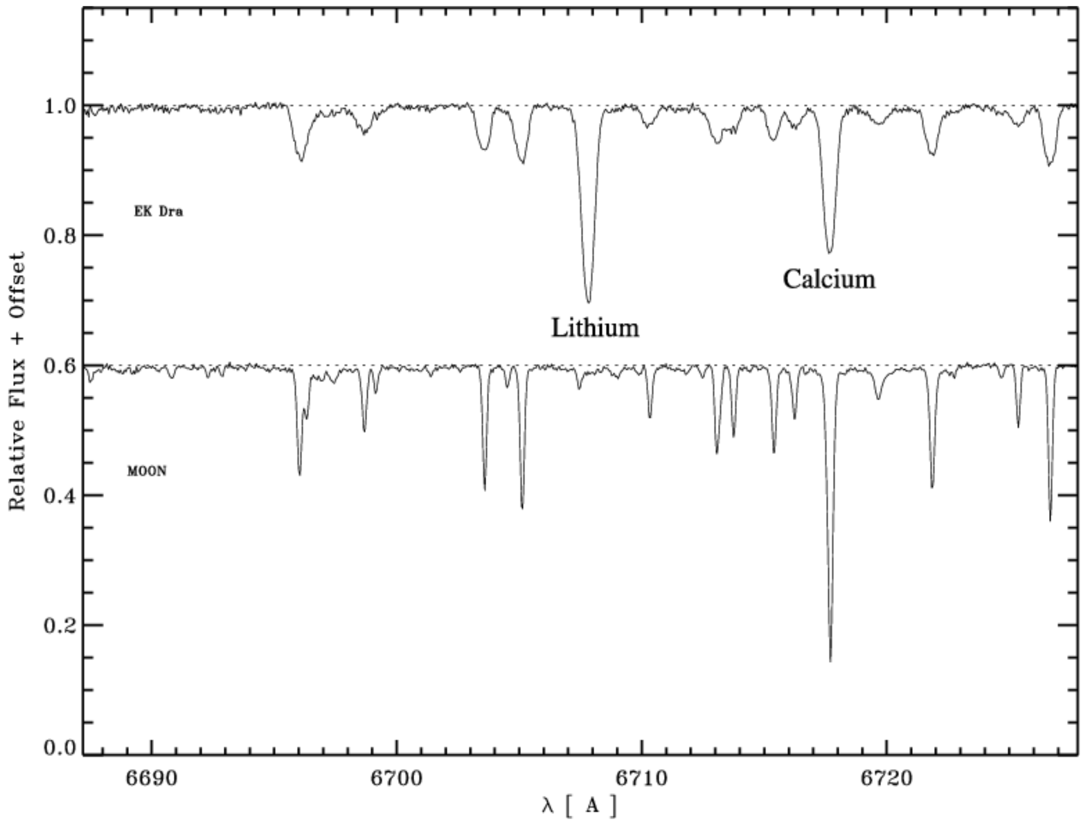
$T_{bol} > 2880$ K



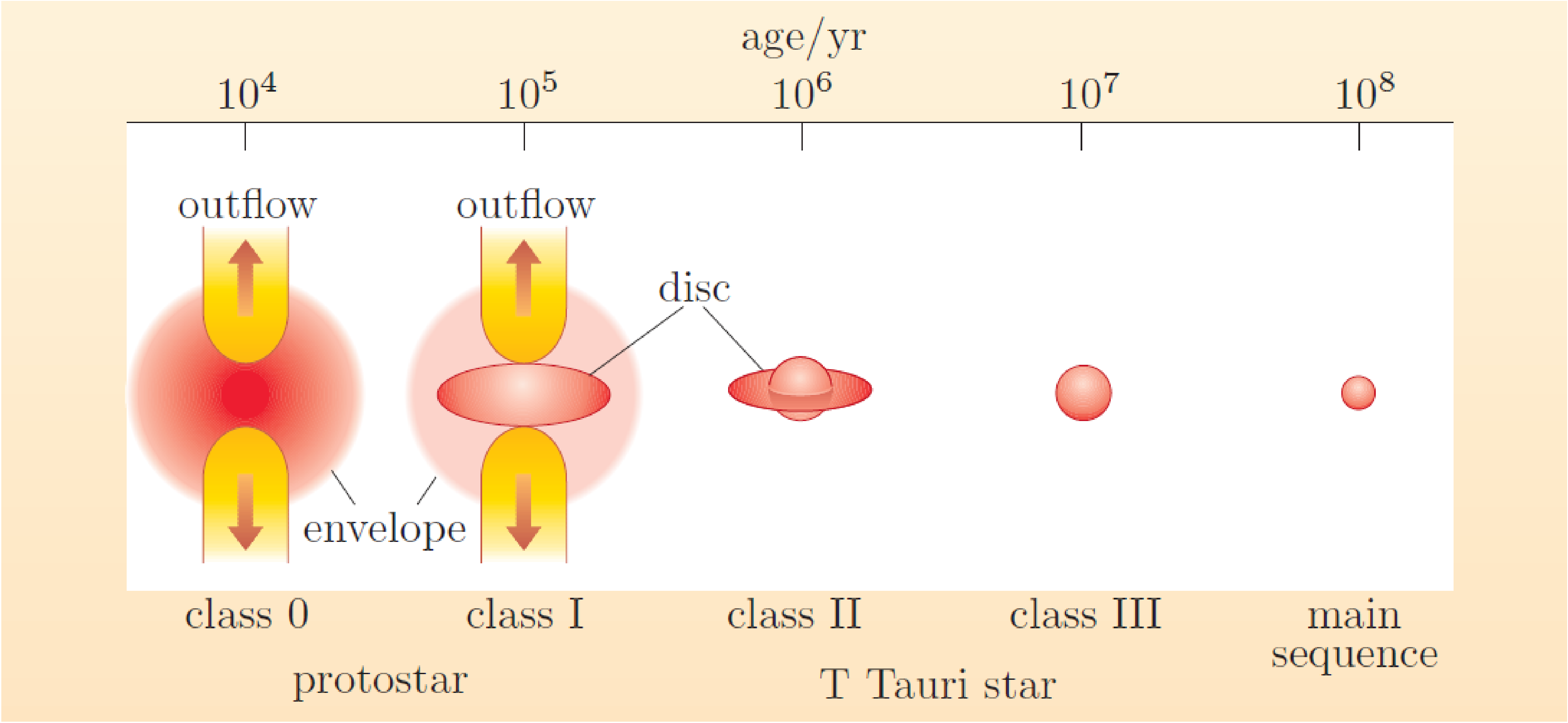


Lithium absorption

Lithium can be burned during the pre-main sequence phase. Moreover, in stars with convective envelopes, the temperature at the bottom of the convective layers is sufficient for lithium burning. Stars with lithium absorption in their spectrum must be young.



Protostars and pre-main-sequence stars



A schematic diagram (from Ryan & Norton) of the evolution of a pre-main-sequence star from the protostar stage, through the T Tauri stage, to the stage where the star becomes a genuine main-sequence object.

Pre-main sequence star evolution

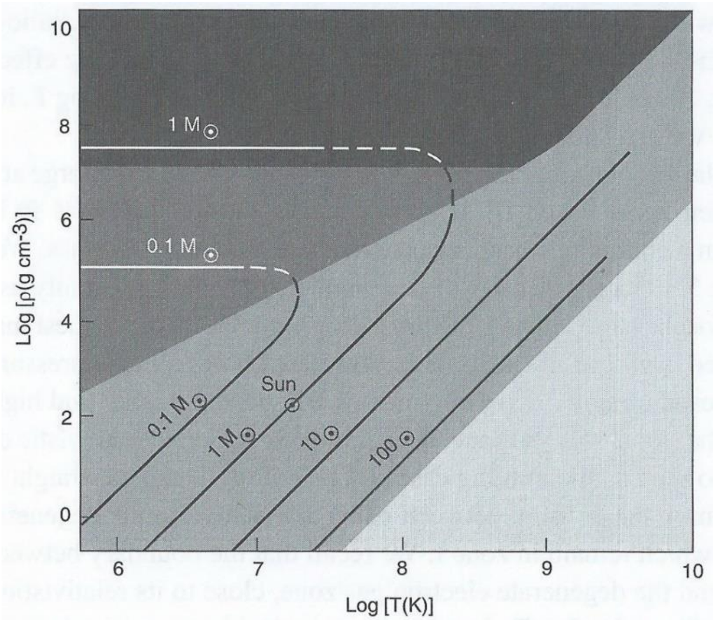
The birth-line for pre-main sequence stars is usually taken as a transition from Class I to Class II (T Tauri stars).

At this stage, in addition to the radiation emitted by accreting material as it strikes the stellar surface, the star itself also radiates. However, since the protostar is initially not hot enough to burn hydrogen, it has no internal source of nuclear energy to balance out this radiation, and it is forced to contract on a Kelvin-Helmholtz timescale.

(It can burn deuterium, but this all gets used up on a timescale well under the KH timescale.)

This contracting state represents the "initial condition" for a calculation of stellar evolution. In terms of the ($\log T$, $\log \rho$) plane describing the centre of the star, we already know what this configuration looks like: the star lies somewhere on the low T, low ρ side of its mass track, and it moves toward the hydrogen burning line on a KH timescale.

We would also like to know what it looks like on the H-R diagram, since this is what we can **actually observe**. Therefore, we want to understand the movement of the star in the ($\log T_{eff}$, $\log L$) plane.



Kelvin–Helmholtz Contraction

There are two phases during this Kelvin–Helmholtz contraction:

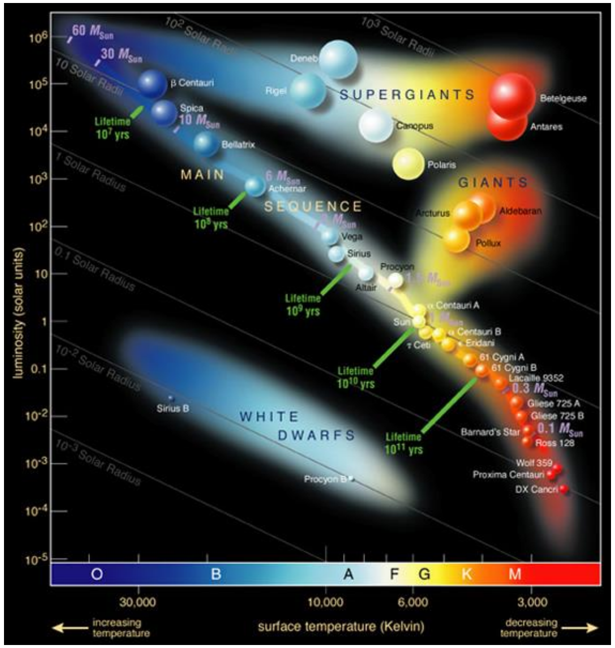
- the **Hayashi** phase, and
- the **Henyey** phase.

The overall timescale for the process is the Kelvin–Helmholtz timescale, as the time for a star to collapse to the main sequence assuming its luminosity is provided solely by energy liberated as a result of gravitational collapse.

Work by Chushiro Hayashi in the 1960s showed that a star **cannot** achieve hydrostatic equilibrium if its outer layers are too cool. Two important concepts in understanding Hayashi's work are opacity and convection.

Hayashi showed that there is a boundary on the right-hand side of the H–R diagram cooler than which hydrostatic equilibrium is impossible, and hence stable stars cannot exist. It lies at an effective surface temperature of around $T_{eff} \approx 3000$ to 5000 K (depending on the star's mass, chemical composition and luminosity).

Objects to the right of that boundary are **out** of hydrostatic equilibrium, and collapse rapidly until the surface temperature reaches the value corresponding to stability. The boundary is almost vertical in the H–R diagram, so collapsing protostars moves vertically along the path which is called a **Hayashi track** and enter the H–R diagram near that boundary.



Hayashi track (1)

$$\left[\frac{GM}{-\xi_1^2(d\theta/d\xi)_{\xi_1}}\right]^{(n-1)}\left(\frac{R}{\xi_1}\right)^{3-n}=\frac{[K(n+1)]^n}{4\pi G}$$

To figure this out, we can approximate the protostellar interior as a polytrope with $P=K_p\rho^{1+\frac{1}{n}}$, or

$$\log P=\log K_p+\left(\frac{n+1}{n}\right)\log\rho$$

Recalling way back to the discussion of polytropes, the polytropic constant K_p is related to the mass and radius of the star by

$$K_p\propto M^{(n-1)/n}R^{(3-n)/n}\Longrightarrow\log K_p=\left(\frac{n-1}{n}\right)\log M+\left(\frac{3-n}{n}\right)\log R+\text{const}$$

so, we have

$$\log P=\left(\frac{n-1}{n}\right)\log M+\left(\frac{3-n}{n}\right)\log R+\left(\frac{n+1}{n}\right)\log\rho+\text{const}$$

Now consider the photosphere of the star, at radius R, where it radiates away its energy into space. If the density at the photosphere is ρ_R, then hydrostatic balance requires that

$$\frac{dP}{dr}=-\rho_R\frac{GM}{R^2}\qquad\Longrightarrow\qquad P_R=\frac{GM}{R^2}\int_R^\infty\rho dr$$

where P_R is the pressure at the photosphere

Hayashi track (2)

$$\frac{dP}{dr} = -\rho_R \frac{GM}{R^2} \quad \Rightarrow \quad P_R = \frac{GM}{R^2}\int_R^\infty \rho dr$$

where P_R is the pressure at the photosphere and we have assumed that GM/R^2 is constant across the photosphere, which is a reasonable approximation since the photosphere is a very thin layer.

The photosphere is the place where the optical depth τ drops to a value below 1 (we will soon learn that the "surface" of a star, which we "see", and which has temperature T_{eff} (by definition) lies at τ=2/3).

Thus, we know that at the photosphere

$$\tau = \kappa \int_R^\infty \rho dr \approx 1$$

where we are also approximating that κ is constant at the photosphere. Putting this together, we have

$$P_R = \frac{GM}{\kappa R^2} \quad \Rightarrow \quad \log P_R = \log M - 2\log R - \log \kappa + \text{constant}$$

For simplicity we will approximate κ as a power-law of the form $\kappa = \kappa_0 \rho T_{\text{eff}}^b$, where T_{eff} is the star's effective temperature, i.e. the temperature at its photosphere:

$$\log P_R = \log M - 2\log R - \log\rho - b\log T_{\text{eff}} + \text{constant}$$

Finally, we know that the ideal gas law applies at the stellar photosphere, so we have

$$\log P_R = \log \rho_R + \log T_{\text{eff}} + \text{constant}$$

$$P = NkT = \frac{\rho}{\mu m_p}kT$$

and we have the standard relationship between luminosity and temperature

$$\log L = 2\log R + 4\log T_{\text{eff}} + \text{constant}$$

$$L \propto R^2\, T^4$$

Hayashi track (3)

We now have four equations:

$$\log P_R = \left(\frac{n-1}{n}\right)\log M + \left(\frac{3-n}{n}\right)\log R + \left(\frac{n+1}{n}\right)\log \rho + \text{constant}$$

$$\log P_R = \log M - 2\log R - \log \rho - b\log T_{\text{eff}} + \text{constant}$$

$$\log P_R = \log \rho_R + \log T_{\text{eff}} + \text{constant}$$

$$\log L = 2\log R + 4\log T_{\text{eff}} + \text{constant}$$

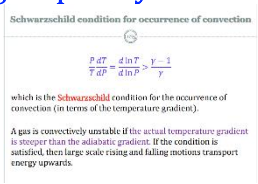
And the four unknowns $\log T_{\text{eff}}$, $\log L$, $\log \rho_{\text{R}}$, and $\log P_{\text{R}}$.
Solving these equations (and skipping over the algebra), we obtain

$$\log L = \left(\frac{9-2n+b}{2-n}\right)\log T_{\text{eff}} + \left(\frac{2n-1}{2-n}\right)\log M + \text{constant}$$

Thus, to figure out the slope of a young star's track in the HR diagram, we need only specify n and b.

Stars at the Hayashi boundary are fully convective. The reason is easy to understand:

1) First, the opacity of protostellar matter decreases with temperature, so cool objects have high opacity.
2) Second, high opacity leads to steep radiative temperature gradients, and
3) Third, steep radiative temperature gradients lead to convective instability → (Slide 186)


Schwarzschild condition for occurrence of convection

$$\frac{P}{T}\frac{dT}{dP} = \frac{d\ln T}{d\ln P} > \frac{\gamma-1}{\gamma}$$

which is the Schwarzschild condition for the occurrence of convection (in terms of the temperature gradient).

A gas is convectively unstable if the actual temperature gradient is steeper than the adiabatic gradient. If the condition is satisfied, then large scale rising and falling motions transport energy upwards.

Putting these three things together, **the coolest stars are more likely to be unstable to convection**.

Hayashi track (4)

Thus, protostars are fully convective, so $n = 1.5$.

Protostars are fully convective due to their high opacities, and they are initially quite cold, ~4000 K.

This makes their opacity very different from that of main-sequence stars.

We will discuss opacities in detail in the following lectures, but now I just note that in main sequence stars like the Sun, the opacity is mostly free-free or, at high temperatures, electron scattering.
At the low temperatures of protostars, however, there are too few free electrons for either of this to be significant, and instead the main opacity source is bound-bound. One species particularly dominates: **H^-, that is hydrogen with two electrons** rather than one (we discussed it a lecture ago).

The H^- opacity is very different than other opacities. It strongly increases with temperature, rather than decreases, because higher temperatures produce more free electrons via the ionization of metal atoms with low ionization potentials, which in turn can combine with hydrogen to make more H^-.

Once the temperature passes several thousand K, H^- ions start falling apart and the opacity decreases again, but in the crucial temperature regime where protostars find themselves (4000 K), opacity increases extremely strongly with temperature: $\kappa_{H^-} \propto \rho T^4$ is a reasonable approximation, giving $b = 4$.

Hayashi track (5)

$$\log L = \left(\frac{9-2n+b}{2-n}\right)\log T_{\text{eff}} + \left(\frac{2n-1}{2-n}\right)\log M + \text{const}$$

Thus, plugging in n=1.5 and b=4, we get

$$\log L = 20\log T_{\text{eff}} + 4\log M + \text{const}$$

Thus, the slope is 20, extremely large.

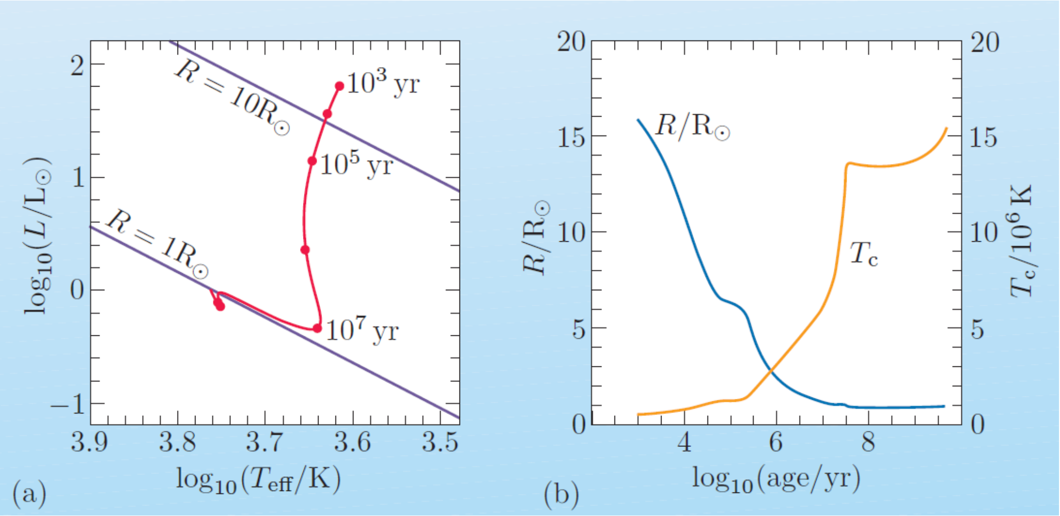
Stars in this phase of contraction make a nearly vertical track in the H–R diagram, **the Hayashi track**.

Remember, T_{eff} and radius of a star R are related by $L = 4\pi R^2\sigma T^4$. Since the star's temperature changes very little during this stage, the luminosity is proportional to the square of the radius.

As R is still decreasing due to contraction, the luminosity **decreases** significantly.

Stars of different masses have Hayashi tracks that are slightly offset from one another due to the $4\log M$ term, but they are all vertical.

(from Ryan & Norton)



(a) H–R diagram with Hayashi track for the Sun. The red dots indicate elapsed times of 10^3, 10^4, 10^5, 10^6, 10^7, 10^8 and 10^9 years.
(b) Evolution of radius and core temperature with time.

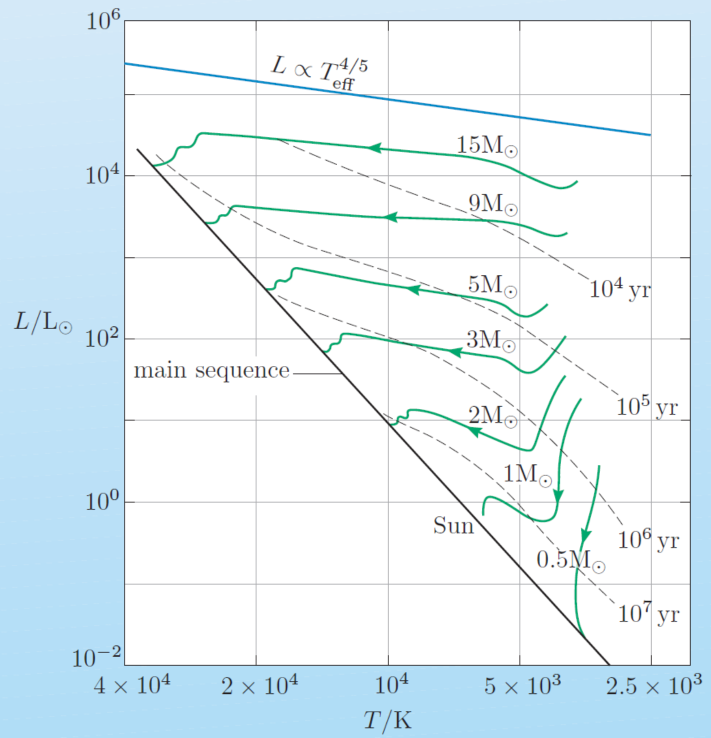
The Henyey contraction

$$\log L = \left(\frac{9 - 2n + b}{2 - n}\right) \log T_{\text{eff}} + \left(\frac{2n - 1}{2 - n}\right) \log M + \text{const}$$

Contraction along the Hayashi track ends once the star contracts and heats up enough for H^- opacity not to dominate, so that b is no longer a large positive number. Once b becomes 0 or smaller, as the opacity changes over to other sources, the track flattens, and the star contracts toward the main sequence at roughly fixed luminosity but increasing temperature.
This is known as **a Heyney track**.

As the protostar continues to contract, its core gets hotter and its opacity decreases, because for a Kramers opacity, $\kappa \propto \rho T^{-3.5}$. As a result of the decreasing opacity, the radiative temperature gradient becomes shallower and the condition for convection eases. Eventually the core becomes non-convective (radiative).

It can be shown that during the Henyey contraction, the slope becomes close to 4/5 (almost flat).



Only stars with masses $\sim M_\odot$ or less have Hayashi phases, and only stars of mass $M \leq 0.5 M_\odot$ reach the main sequence at the bottom of their Hayashi tracks.

More massive stars are "born" hot enough so that they are already too warm to be dominated by H^-.

Stellar evolution

THE ZERO-AGE MAIN SEQUENCE
STELLAR EVOLUTION FROM OBSERVATIONS
EVOLUTION DURING THE MAIN SEQUENCE
POST-MS EVOLUTION OF LOW-MASS STARS
POST-MS EVOLUTION OF HIGH-MASS STARS

The Zero-Age Main Sequence

- Kelvin–Helmholtz contraction continues until the central temperature becomes high enough for nuclear fusion reactions. Once the energy generated by hydrogen fusion compensates for the energy loss at the surface, the star stops contracting and settles on the zero-age main sequence (ZAMS) if its mass is above the hydrogen burning limit of $\sim 0.08\ M_{\odot}$.
- Because contraction is slowest when both R and L are small, the pre-main sequence lifetime is dominated by the final stages of contraction, when the star is already close to the ZAMS.
- Since the nuclear energy source is much more concentrated towards the centre than the gravitational energy released by overall contraction, the transition from contraction to hydrogen burning requires a **rearrangement** of the internal structure.
- When we look at a population of stars that are at many different ages, and thus at many random points in their lives, we expect the number of stars we see in a given population to be proportional to the fraction of its life that a star spends as a member of that population. Since the main sequence is the most heavily populated part of the H–R diagram and the hydrogen nuclear burning phase is the longest evolutionary phase, it seems natural to assume that main sequence stars are burning hydrogen.

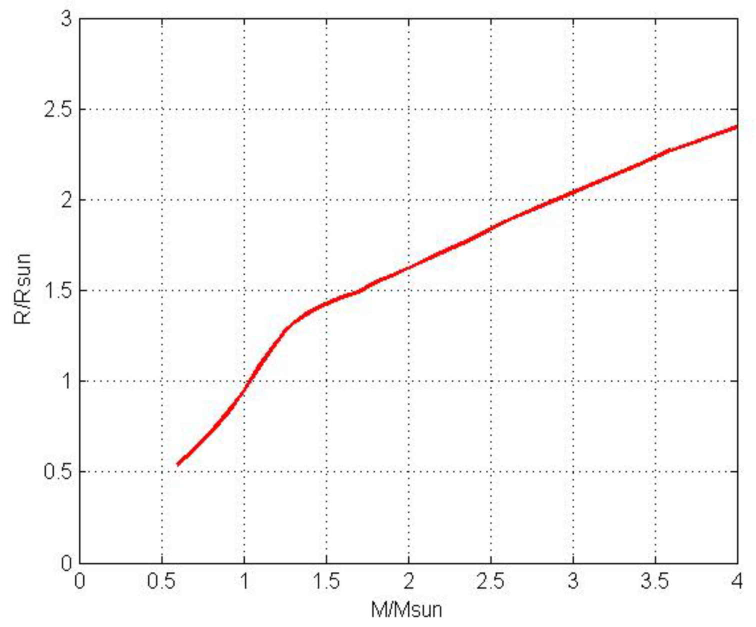
The Main Sequence stars (1)

Main-sequence stars obey several relations:

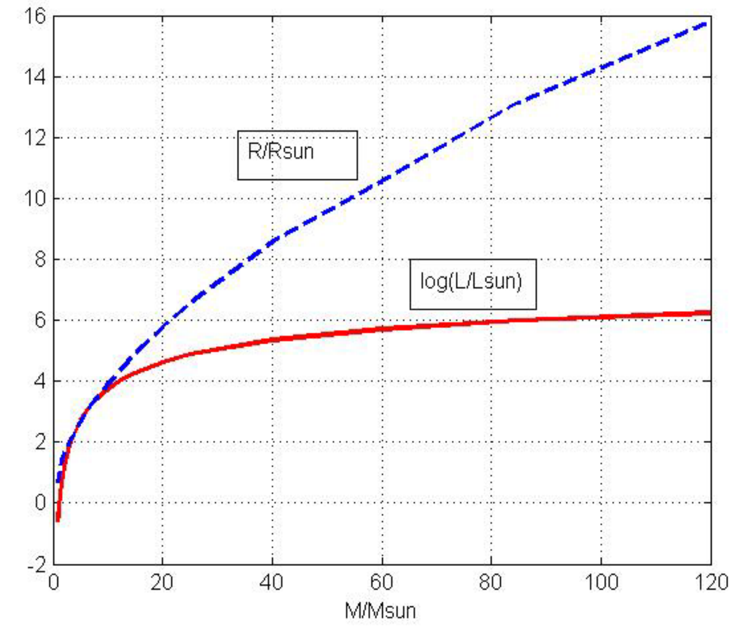
- Main sequence stars obey a mass-luminosity relation, with $L \propto M^{\eta}$. The slope η changes slightly over the range of masses; between 1 and $10M_{\odot}$, $\eta \approx 3.88$. The relation flattens out at higher masses, due to the contribution of radiation pressure in the central core. This helps support the star, and decreases the central temperature slightly. The relation also flattens significantly at the very faint end of the luminosity function. This is due to the increasing important of convection for stellar structure.
- Main sequence stars obey a mass-radius relation. However, the relation displays a significant break around $1M_{\odot}$; $R \propto M^{\beta}$, with $\beta \approx 0.57$ for M > $1M_{\odot}$, and $\beta \approx 0.8$ for M < $1M_{\odot}$. This division marks the onset of a convective envelope. Convection tends to increase the flow of energy out of the star, which causes the star to contract slightly. As a result, stars with convective envelopes lie below the mass-radius relation for non-convective stars. This contraction also increases the central temperature (via the virial theorem) and also moves the star above the nominal mass-luminosity relation.
- The depth of the convective envelope (in terms of M_{env}/M) increases with decreasing mass. Stars with M~$1M_{\odot}$ have extremely thin convective envelopes, while stars with M < $0.3M_{\odot}$ are entirely convective. Nuclear burning ceases around $0.08M_{\odot}$.
- The interiors of stars are extremely hot ($T > 10^6$ K). The fall-off to surface temperatures ($T \sim 10^4$ K) takes place in a very thin region near the surface.

Some Relations for the MS stars

Mass-Radius relation



M-R and *M-L* relations



The Main Sequence stars (2)

Main-sequence stars obey several relations (cont):

- The region of nuclear energy generation is restricted to a very small mass range near the center of the star. The rapid fall-off of ε_n with radius reflects the extreme sensitivity of energy generation to temperature.
- Stars with masses below $\sim 1 M_\odot$ generate most of their energy via the proton-proton chain. Stars with more mass than this create most of their energy via the CNO cycle. This changeover causes a shift in the homology relations for the stellar interior.
- CNO burning exhibits an extreme temperature dependence. Consequently, those stars that are dominated by CNO fusion have very large values of $L/4\pi r^2$ in the core. This results in a large value of $\nabla_{rad} \equiv \frac{d \ln T}{d \ln P}$, and convective instability. In this region, convective energy transport is extremely efficient, and $\nabla \approx \nabla_{ad}$.
- Because of the extreme temperature sensitivity of CNO burning, nuclear reactions in high mass stars are generally confined to a very small region, much smaller than the size of the convective core.
- As the stellar mass increases, so does the size of the convective core (due again to the large increase in ε_n with temperature). Supermassive stars with M $\sim 100 M_\odot$ would be entirely convective.
- A star's position on the ZAMS depends on both its mass and its initial helium abundance. Stars with higher initial helium abundances have higher luminosities and effective temperatures. The higher mean molecular weight translates into lower core pressures. Helium rich stars therefore are more condensed, which (through the virial theorem) mean they have higher core temperatures and larger nuclear reaction rates.

Main-sequence lifetimes

- Stars arrive at the main sequence chemically homogeneous because of convective mixing during the protostellar phase.
- The time of arrival on the main sequence is known as the ZAMS – zero-age main sequence.
- "Where" it ends up depends only on mass and chemical composition.

- Approximate MS lifetimes: $\tau_{MS} \approx 10^{10}(M/M_{\odot})^{-2.5}$ yr

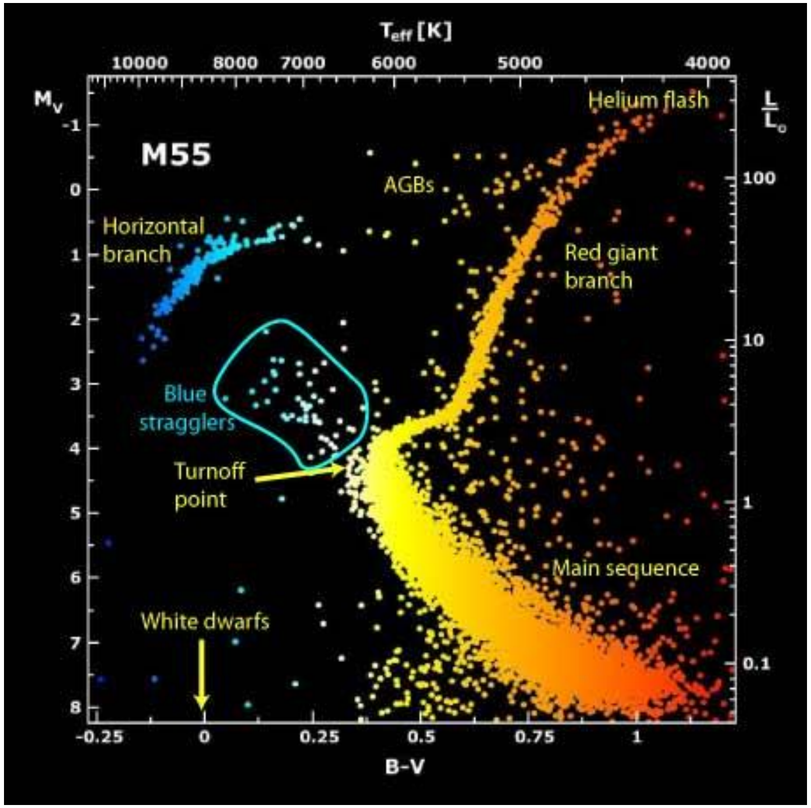
- Stars of all masses live on the main-sequence, but subsequent evolution differs enormously.
- Most of stars form in clusters, open and globular. Because they born at same time, age of cluster will show on the H–R diagram as the upper end, or turn-off of the main-sequence.
- We can use this as a tool (clock) for measuring age of star clusters. Stars with lifetimes less than cluster age, have left main sequence. Stars with main-sequence lifetimes longer than age, still sit on the main-sequence.

Mass, $M_{\odot}$	Time, yr
0.1	6×10^{12}
0.5	7×10^{10}
1.0	1×10^{10}
1.25	4×10^{9}
1.5	2×10^{9}
3.0	2×10^{8}
5.0	7×10^{7}
9.0	2×10^{7}
15	1×10^{7}
25	6×10^{6}

Stellar Evolution from observations

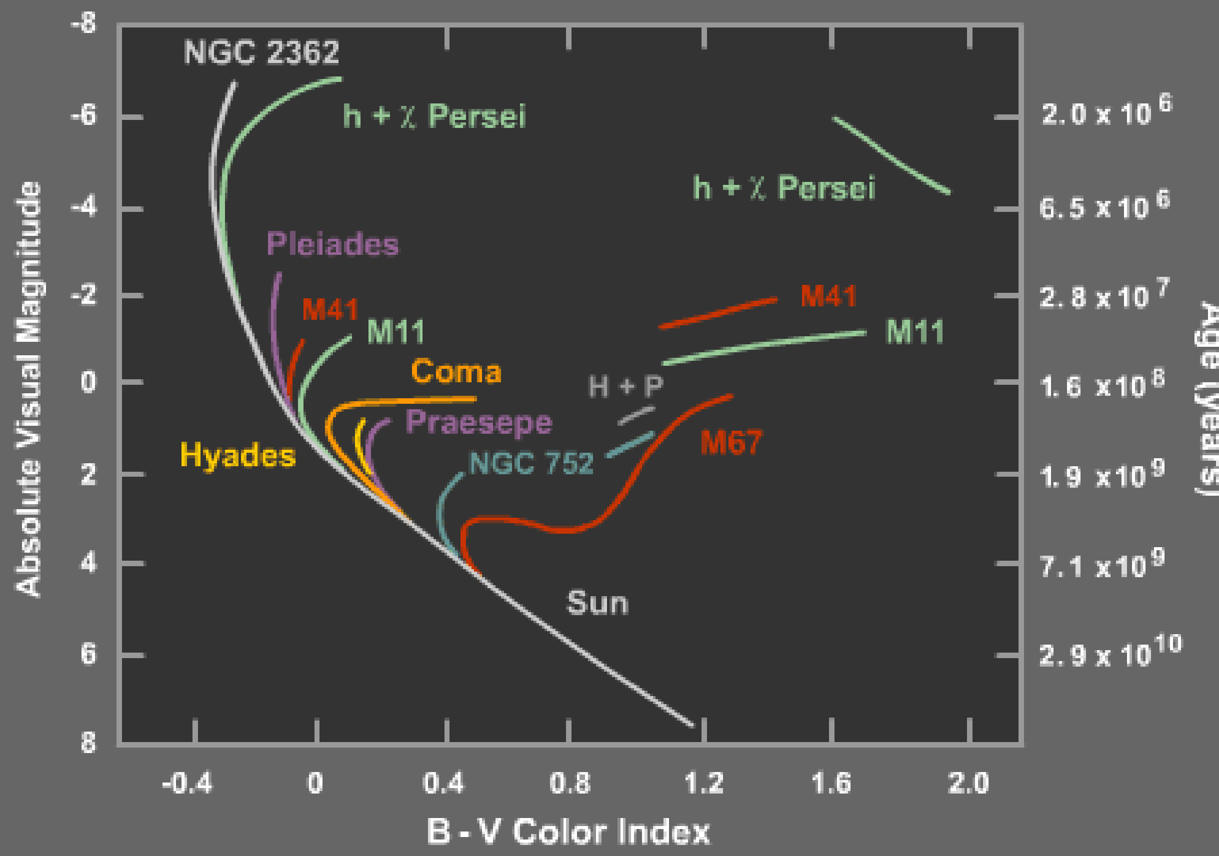
Globular cluster Messier 55

H-R diagram



Stellar Evolution from observations

Different globular clusters have different age, and accordingly different turn-off points.



HR Diagrams for Various Open Clusters

Stellar Evolution from observations

- An interesting effect can be observed in young star clusters.
- Recall, the time needed for a protostar to reach the ZAMS depends on its mass. This time is basically the Kelvin-Helmholtz contraction timescale. Since contraction is slowest when both R and L are small, the pre-main sequence lifetime is dominated by the final stages of contraction, when the star is already close to the ZAMS.
- An estimate of the PMS lifetime is $\tau_{PMS} \approx 10^7 (M/M_\odot)^{-2.5}$ yr
- Thus, massive protostars reach the ZAMS much earlier than lower-mass stars, and they can even leave the MS while low-mass stars still lie above and to the right of it.

Table 8.1 Evolutionary lifetimes (years)

$M/M_\odot$	1–2	2–3	3–4	4–5
15	6.7(2)	2.6(4)	1.3(4)	6.0(3)
9	1.4(3)	7.8(4)	2.3(4)	1.8(4)
5	2.9(4)	2.8(5)	7.4(4)	6.8(4)
3	2.1(5)	1.0(6)	2.2(5)	2.8(5)
2.25	5.9(5)	2.2(6)	5.0(5)	6.7(5)
1.5	2.4(6)	6.3(6)	1.8(6)	3.0(6)
1.25	4.0(6)	1.0(7)	3.5(6)	1.0(7)
1.0	8.9(6)	1.6(7)	8.9(6)	1.6(7)
0.5	1.6(8)			

Note: powers of 10 are given in parentheses.

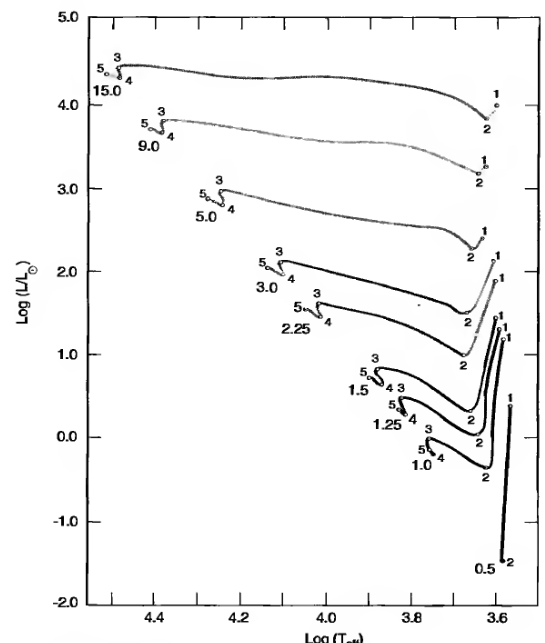
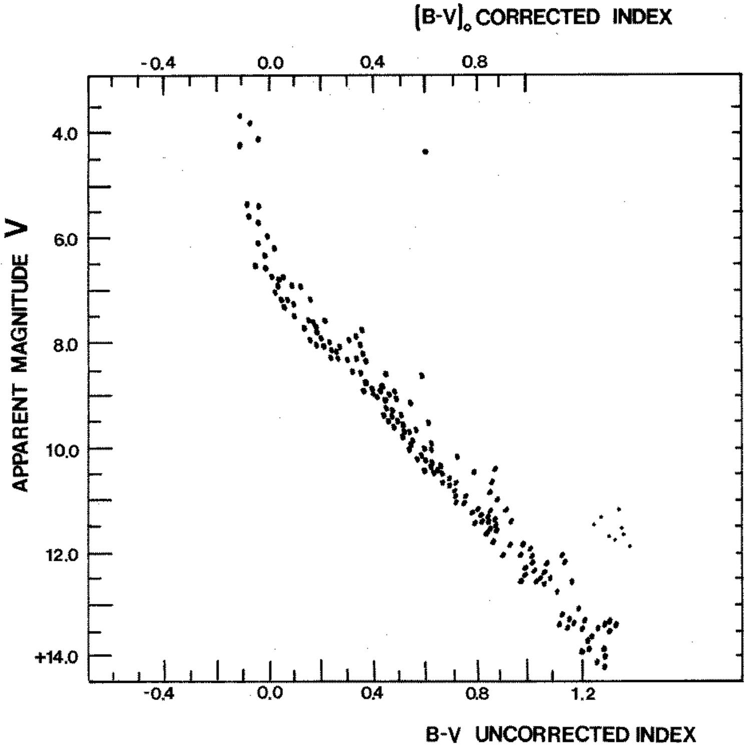


Figure from Prialnik

Pleiades, a young open star cluster

Massive protostars reach the ZAMS much earlier than lower-mass stars, and they can even leave the MS while low-mass stars still lie above and to the right of it.

Pleiades, a young open star cluster, $\sim 10^8$ yr:



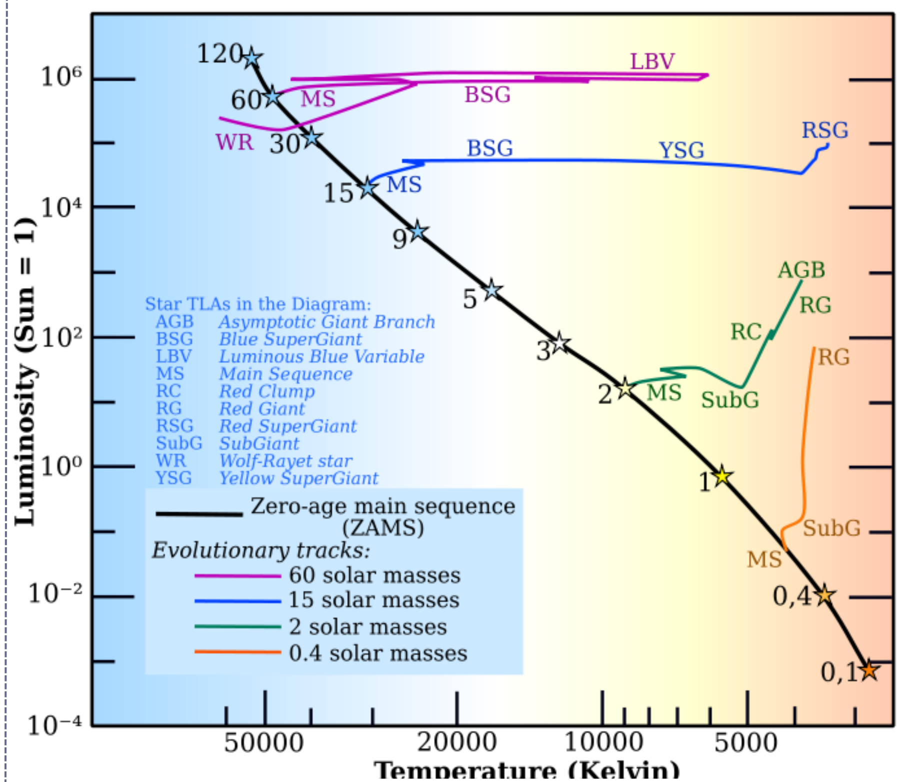
Tracks

Stellar **evolutionary tracks** are trajectories of individual stars in the H-R diagram, which trace the evolution of a **given** mass star as a function of time.

Consider stars of different masses but with the same age. Let's make a plot of $\text{Log}(L/L_{\odot})$ vs. Log T_{eff} for an age of 1Gyr. The result is an **isochrone**.

Describe evolution of a single star with time.

Tables of stellar parameters as function of T: (luminosity, temperature, surface gravity; core temperature, core composition, current mass, etc, etc).



Isochrones

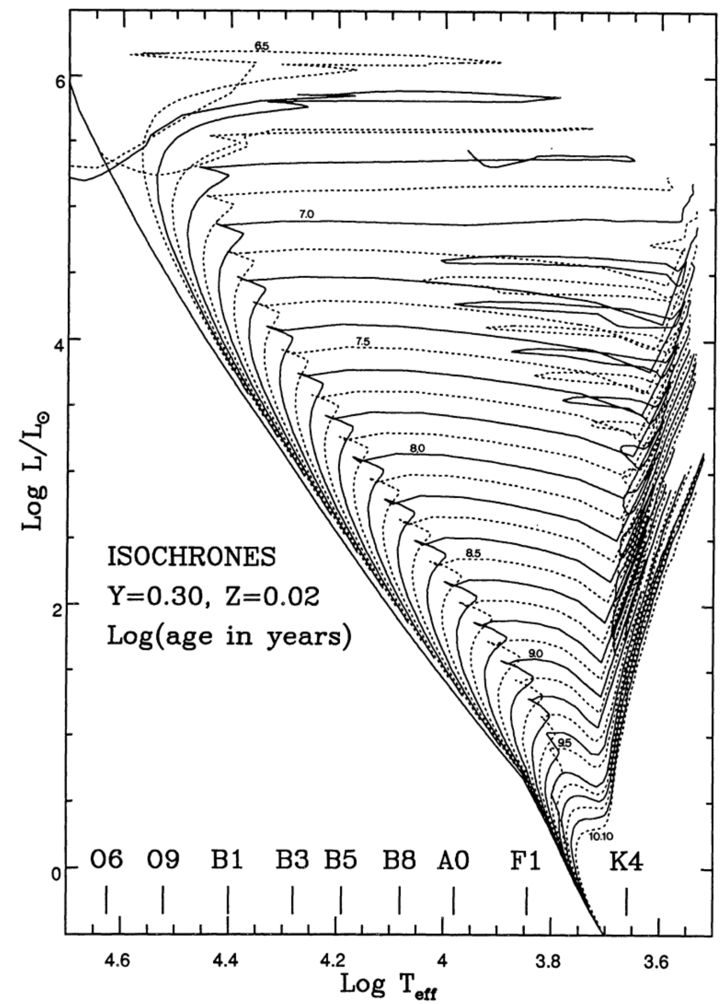
An isochrone: a curve which traces the properties of stars as a function of mass for a **given** age.

Don't be confused with an **evolutionary track** which shows the properties of a star as a function of age for a **fixed** mass.

Isochrones are particularly useful for star clusters - all stars born at the same time with the same composition.

The best way to check stellar evolutionary calculations is to compare calculated isochrones and an observed H-R diagram of a cluster.

Important - think about what we are looking at when we observe a cluster. We see a "freeze-frame" picture at a particular age. We see how stars of different masses have evolved up to that fixed age (this is **not** equivalent to an evolutionary track).



Meynet, Mermilliod, and Maeder, 1993

Convective regions on the ZAMS

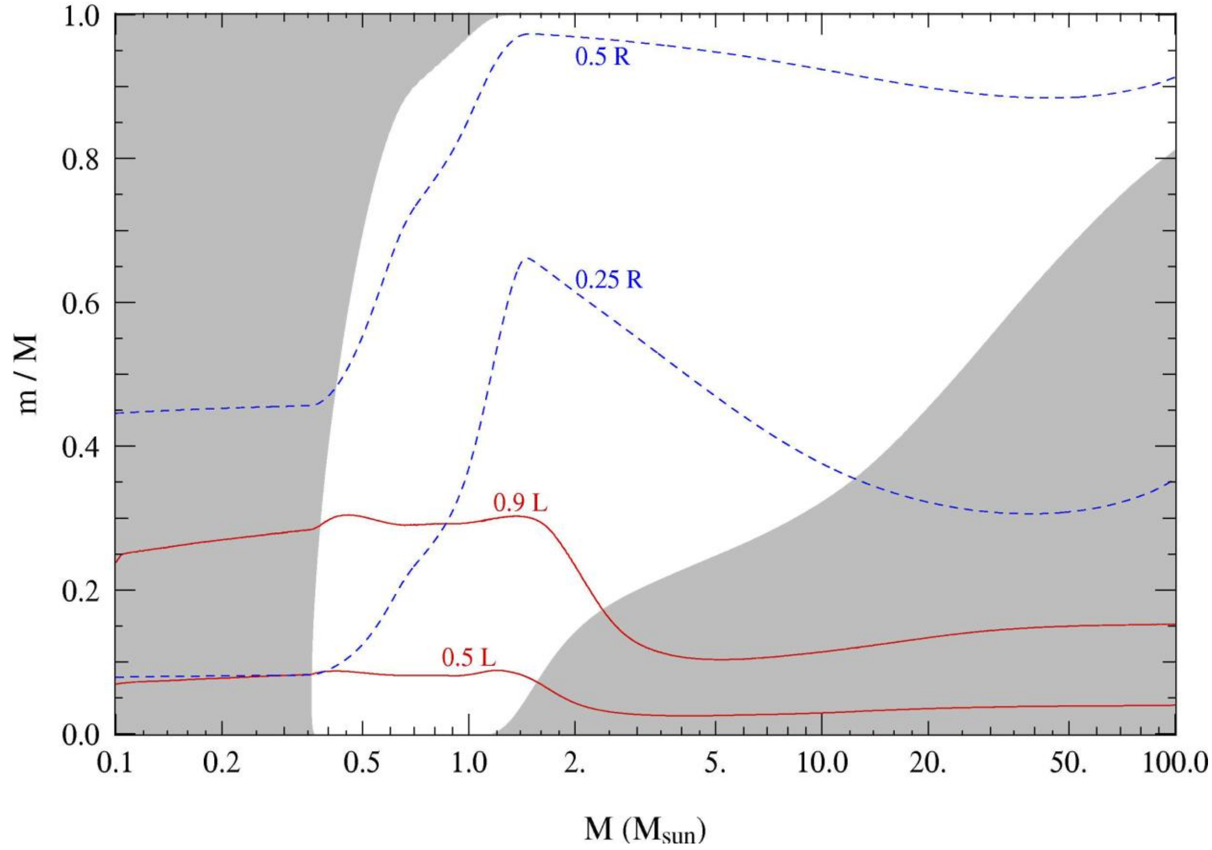
Occurrence of convective regions (gray shading) on the ZAMS in terms of fractional mass coordinate m/M as a function of stellar mass, for detailed stellar models with a composition $X = 0.70, Z = 0.02$.

The solid (red) lines show the mass shells inside which 50% and 90% of the total luminosity are produced.

The dashed (blue) lines show the mass coordinate where the radius r is 25% and 50% of the stellar radius R.

We can distinguish three types of ZAMS stars:

- completely convective, for $M < 0.35 M_\odot$
- radiative core + convective envelope, for $0.35\,M_\odot < M < 1.2\,M_\odot$
- convective core + radiative envelope, for $M > 1.2\,M_\odot$.

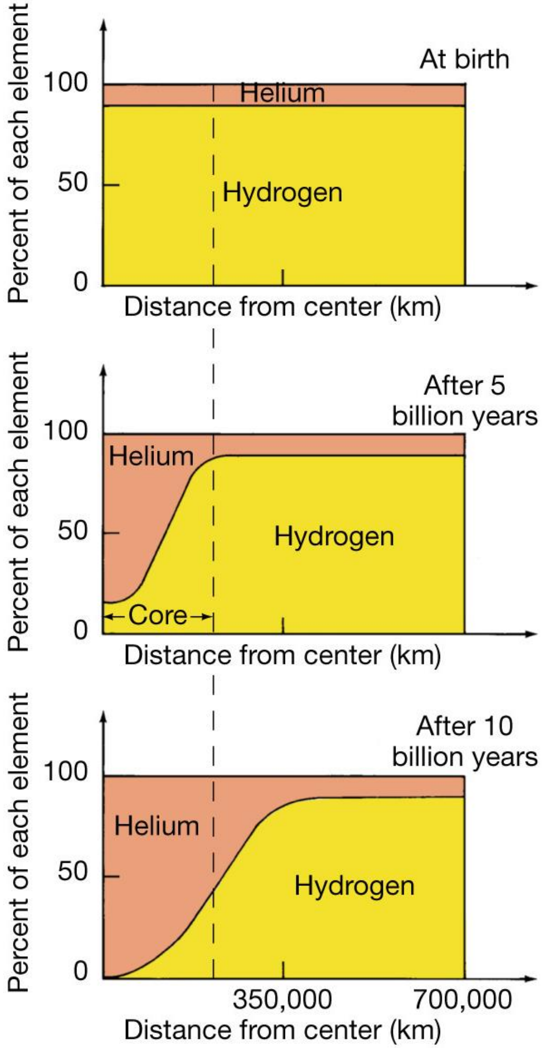


(After Kippenhahn & Weigert)

Evolution During the Main Sequence

During the MS phase H is converted into He in the core. The temperature in the core can only change very little, because fusion is a strong function of T with $\varepsilon \sim T^4$ for P-P chain and $\sim T^{18}$ for the CNO-cycle. Even a small change in T would result in a large change in ε and in L, which is not allowed by the hydrostatic equilibrium requirement. So nuclear fusion acts like a thermostat in the center of the star. The CNO cycle is a better thermostat than the pp chain.

Even while on the main sequence, the composition of a star's core is changing, thus μ_c increases. a Sun-like star →



Recall, when Z is negligible: $\mu = 4/(3+5X)$

For the solar abundance X=0.73, $\mu = 0.6$.

When all H gets converted into He, we then have $\mu = 1.3$. It more than doubles! But ↓

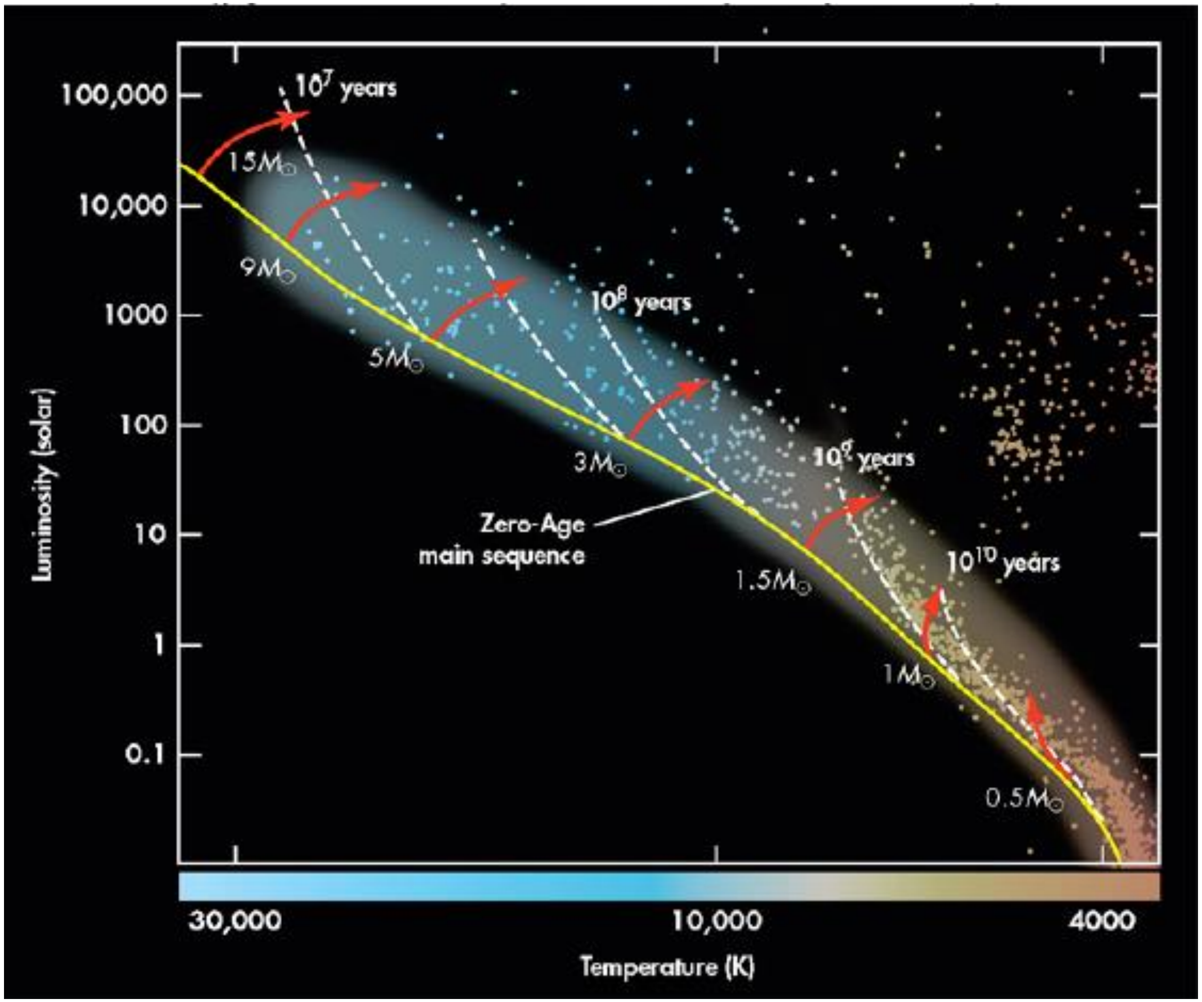
$$P_{gas} = \frac{\rho k T}{\mu m_p} = \frac{\Re \rho T}{\mu}$$

If T_c remains constant during the MS-phase but μ_c increases, then $P_c/\rho_c \sim T_c/\mu_c$ must decrease. So, either P_c decreases or ρ_c increases as more H is converted into He. It turns out that both effects occur. Actually, T_c is also slightly raising.

Stellar Evolution on the Main Sequence

- In Hydrostatic Equilibrium the central pressure is set by the weight of the layers above.
- So, as the central pressure decreases during the MS phase the outer layers of the star must expand.
- So, when μ increases in the center the radius must increase.
- At the same time, the nuclear energy generation increases and then so does the luminosity (μ-effect).
- This causes a slow increase of the star's luminosity over the whole MS phase.

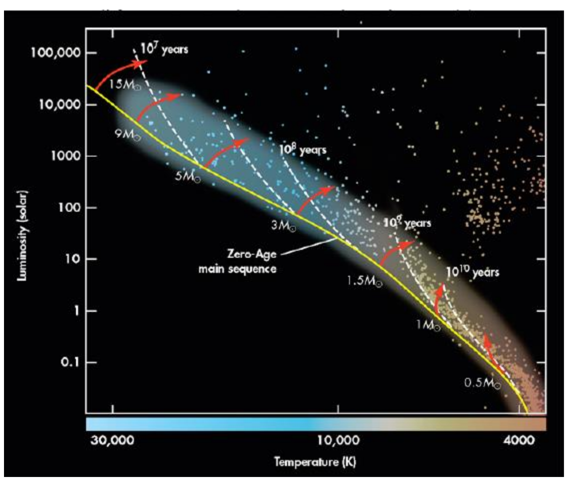
- Because R^2 increases more than L, the effective temperature decreases. This implies that the stars move up and to the right in the HRD during H-fusion in the core.

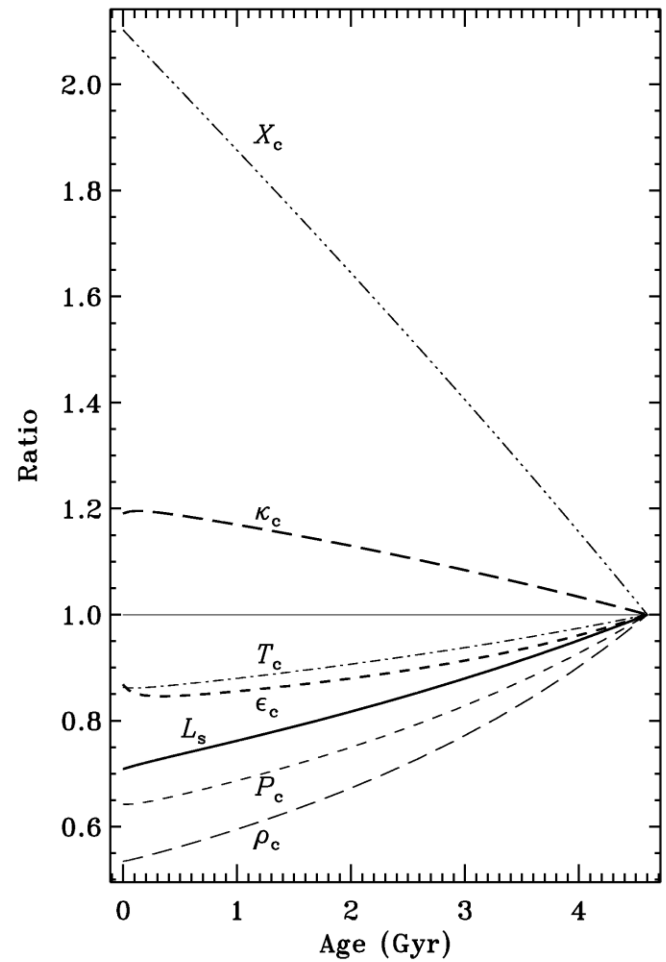


Gradual change of the Sun's parameters

- How much has the solar luminosity changed over time? Calculations show that the Sun's ZAMS luminosity was about 25-30% less than it is today, which has/had implications for the Earth.

- Hence stars of a given mass but different ages populate the main-sequence with a width of ~0.5 dex [for decimal exponent]





From Christensen-Dalsgaard (2008)

Change of the Sun's parameters

Table 13.1. *Distribution of mass, temperature, pressure, density and luminosity for the young sun at the age of 5.4×10^7 years when it had* R = 6.14×10^{10} *cm* = $R_{\odot Z}$, L = 2.66×10^{33} *erg* s^{-1} *and* T_{eff} = *5610* K. *(These data were provided by C. Proffitt.)*

$r/R_{\odot Z}$	$M_r/M_\odot$	T [K]	P_g [dyn cm^{-2}]	ρ [g cm^{-3}]	$L/L_\odot$	$r/R_\odot$
0	0	13.62 (6)	1.49 (17)	8.02 (1)	0	0
0.014	1.00 (−4)	13.62 (6)	1.48 (17)	8.01 (1)	0.001	0.012
0.018	2.22 (−4)	13.60 (6)	1.48 (17)	7.99 (1)	0.003	0.016
0.035	1.64 (−3)	13.49 (6)	1.45 (17)	7.89 (1)	0.020	0.031
0.057	7.23 (−3)	13.23 (6)	1.38 (17)	7.67 (1)	0.076	0.051
0.081	1.99 (−2)	12.84 (6)	1.28 (17)	7.33 (1)	0.164	0.072
0.098	3.42 (−2)	12.49 (6)	1.19 (17)	7.03 (1)	0.233	0.087
0.115	5.32 (−2)	12.09 (6)	1.10 (17)	6.69 (1)	0.309	0.101
0.125	6.71 (−2)	11.84 (6)	1.04 (17)	6.45 (1)	0.358	0.110
0.138	8.75 (−2)	11.50 (6)	9.59 (16)	6.14 (1)	0.418	0.122
0.147	1.05 (−1)	11.24 (6)	9.00 (16)	5.90 (1)	0.461	0.130
0.158	1.26 (−1)	10.94 (6)	8.33 (16)	5.61 (1)	0.506	0.140
0.178	1.69 (−1)	10.40 (6)	7.16 (16)	5.07 (1)	0.575	0.157
0.198	2.18 (−1)	9.85 (6)	6.03 (16)	4.51 (1)	0.625	0.174
0.219	2.75 (−1)	9.28 (6)	4.93 (16)	3.92 (1)	0.655	0.193
0.263	3.99 (−1)	8.18 (6)	3.10 (16)	2.80 (1)	0.682	0.232
0.424	7.63 (−1)	5.26 (6)	4.11 (15)	5.81 (0)	0.692	0.374
0.635	9.45 (−1)	3.13 (6)	2.94 (14)	7.01 (−1)	0.690	0.560
0.731	9.74 (−1)	2.33 (6)	9.15 (13)	2.94 (−1)	0.690	0.645
0.745	9.78 (−1)	2.16 (6)	7.56 (13)	2.62 (−1)	0.690	0.658
0.843	9.93 (−1)	1.18 (6)	1.65 (13)	1.05 (−1)	0.690	0.744
1.00	1.00	5.61 (3)			0.690	0.884

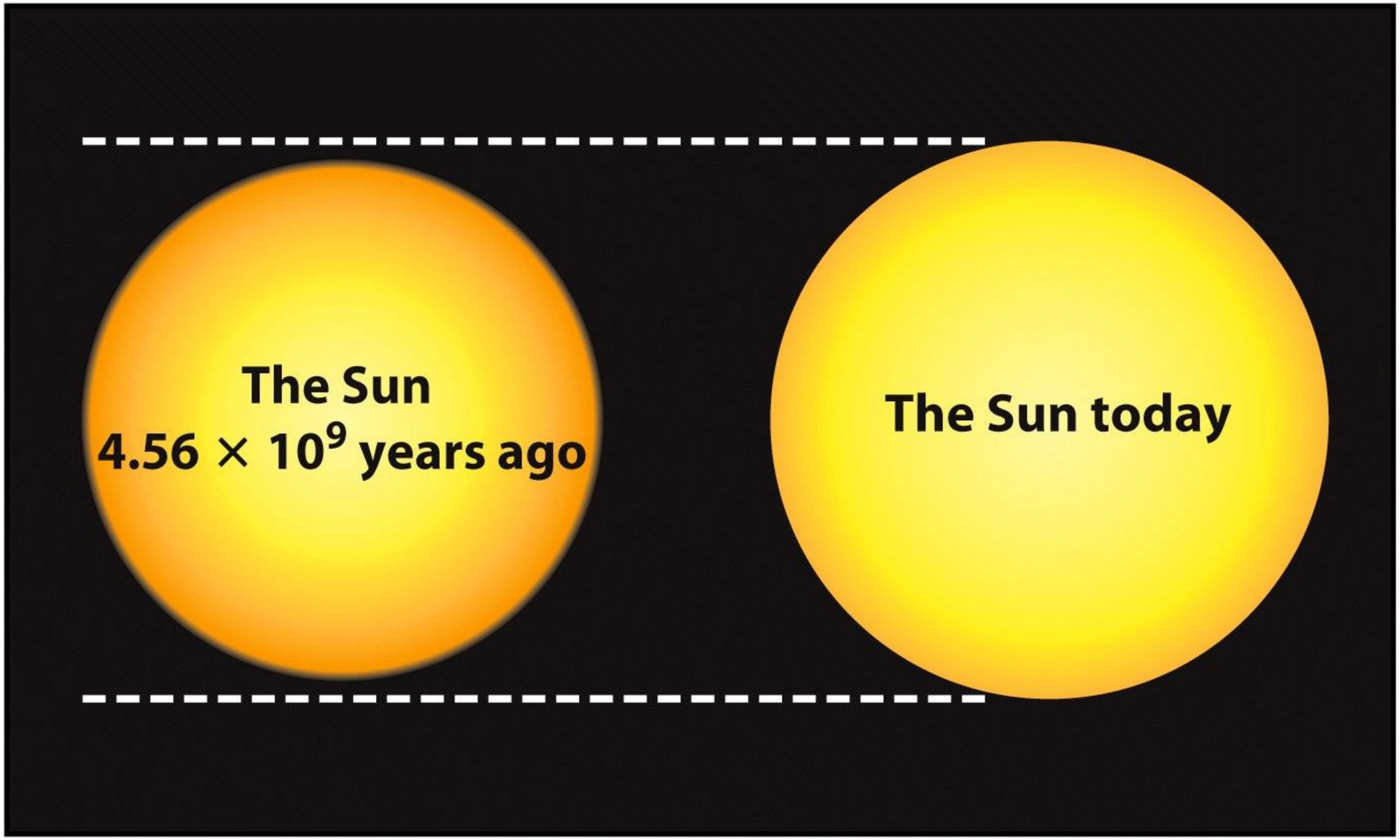
The numbers in brackets give the powers of 10.

Table 13.2. *Distribution of mass, temperature, pressure, density, luminosity, and abundances of H, He, C and N in the present sun according to Bahcall and Ulrich (1988).*

$r/R_\odot$	$M_r/M_\odot$	T [K]	P [dyn cm^{-2}]	ρ [g cm^{-3}]	$L/L_\odot$	H	He	C	N
0.0	0.0	1.56 (7)	2.29 (17)	1.48 (2)	0.0	0.341	0.639	2.61 (−5)	6.34 (−3)
0.024	0.0014	1.55 (7)	2.21 (17)	1.42 (2)	0.012	0.359	0.621	2.50 (−5)	6.22 (−3)
0.048	0.0108	1.49 (7)	1.99 (17)	1.26 (2)	0.085	0.408	0.571	2.24 (−5)	5.98 (−3)
0.071	0.0307	1.42 (7)	1.72 (17)	1.08 (2)	0.217	0.467	0.513	1.98 (−5)	5.84 (−3)
0.095	0.0654	1.33 (7)	1.41 (17)	8.99 (1)	0.400	0.530	0.450	1.71 (−5)	5.78 (−3)
0.115	0.1039	1.25 (7)	1.18 (17)	7.64 (1)	0.553	0.577	0.403	1.50 (−5)	5.77 (−3)
0.135	0.1500	1.17 (7)	9.60 (16)	6.45 (1)	0.688	0.615	0.364	1.68 (−5)	5.77 (−3)
0.149	0.186	1.12 (7)	8.25 (16)	5.72 (1)	0.766	0.637	0.342	1.84 (−4)	5.57 (−3)
0.162	0.222	1.07 (7)	7.11 (16)	5.10 (1)	0.826	0.654	0.325	1.09 (−3)	4.52 (−3)
0.174	0.258	1.02 (7)	6.14 (16)	4.55 (1)	0.872	0.667	0.312	2.39 (−3)	3.00 (−3)
0.188	0.300	9.74 (6)	5.16 (16)	3.99 (1)	0.912	0.679	0.301	3.42 (−3)	1.80 (−3)
0.211	0.370	9.00 (6)	3.84 (16)	3.18 (1)	0.954	0.692	0.288	4.01 (−3)	1.11 (−3)
0.235	0.440	8.32 (6)	2.81 (16)	2.51 (1)	0.978	0.699	0.280	4.12 (−3)	9.86 (−4)
0.259	0.510	7.67 (6)	2.00 (16)	1.94 (1)	0.992	0.704	0.274	4.13 (−3)	9.66 (−4)
0.318	0.655	6.39 (6)	8.69 (15)	1.01 (1)	1.000	0.708	0.271	4.14 (−3)	9.63 (−4)
0.504	0.900	3.88 (6)	6.59 (14)	1.27 (0)	1.000	0.710	0.271	4.14 (−3)	9.63 (−4)
0.752	0.985	1.82 (6)	2.98 (13)	1.22 (−1)	1.00	0.710	0.271	4.14 (−3)	9.63 (−4)
0.886	0.998	6.92 (5)	2.60 (12)	2.84 (−2)	1.00	0.710	0.271	4.14 (−3)	9.63 (−4)
0.920	0.999	4.54 (5)	8.95 (11)	1.50 (−2)	1.00	0.710	0.271	4.14 (−3)	9.63 (−4)
1.000	1.000	5.77 (3)			1.00	0.710	0.271	4.14 (−3)	9.63 (−4)

The numbers in brackets give the powers of 10.

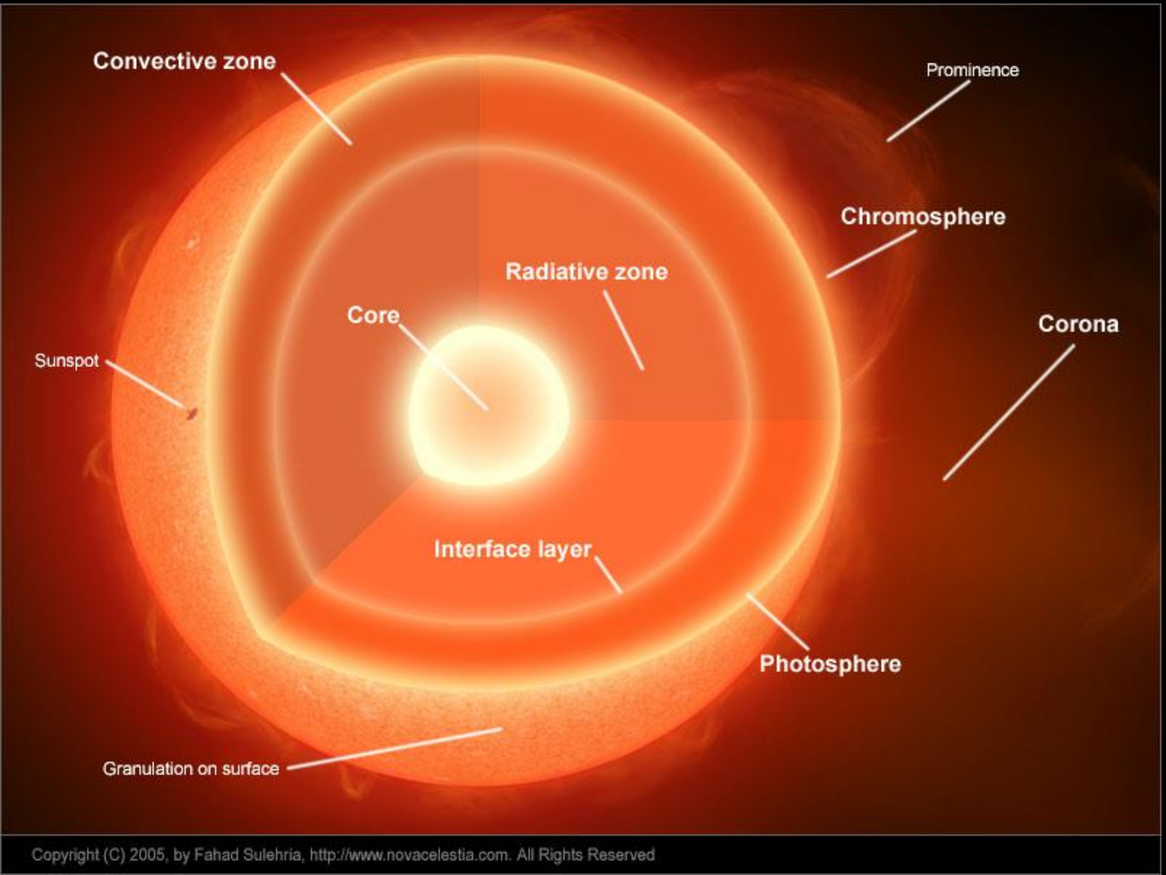
Gradual change in size of the Sun

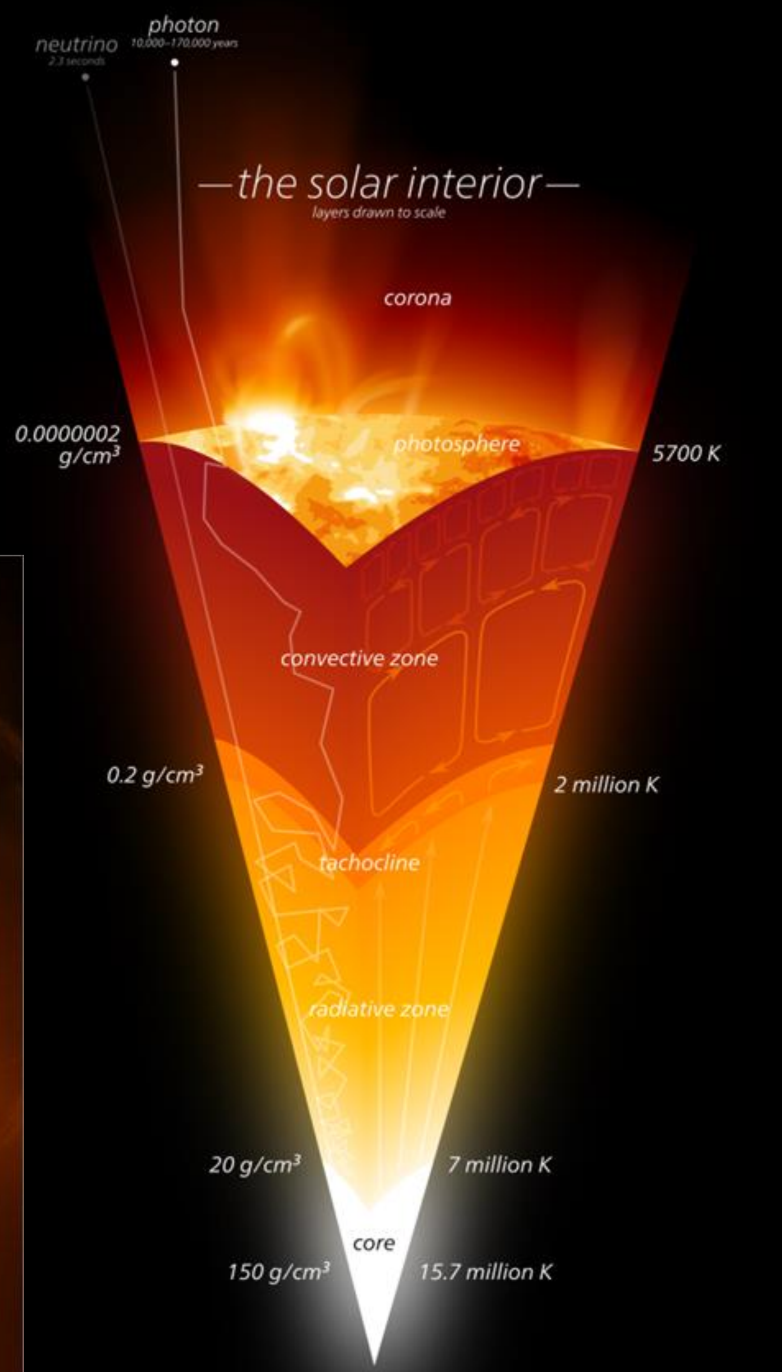


Now 30% brighter, 12% larger, 3% hotter

The main-sequence phase of the Sun

- 50% of mass is within radius $0.25R_{\odot}$
- Only 1% of total mass is in the convection zone and above
- Pressure increases steeply in the centre





Post-MS evolution through helium burning

After the main-sequence phase, stars are left with a hydrogen-exhausted core surrounded by a still hydrogen-rich envelope. To describe the evolution after the main sequence, it is useful to make a division based on the mass. Note that all masses are approximate, boundaries overlap depending on definition.

- ***Red dwarfs:*** stars whose main-sequence lifetime exceeds the present age of the Universe (estimated as 1-2$\times 10^{10}$ yr). Models yield an upper mass limit of $0.7 M_\odot$ of stars that must still be on main-sequence, even if they are as old as the Universe.
- ***Low-mass stars:*** stars in the region $0.7 \leq M \leq 2\ M_\odot$. After shedding considerable amount of mass, they will end their lives as white dwarfs and possibly planetary nebulae. We will follow the evolution of a $1 M_\odot$ star in more detail.
- ***Intermediate mass stars:*** stars of mass $2 \leq M \leq 8\text{-}10\ M_\odot$. Similar evolutionary paths to low-mass stars, but always at higher luminosity.
  Give planetary nebula and higher mass white dwarfs.
  Complex behaviour on the AGB (asymptotic giant branch).
- ***High mass (or massive) stars:*** $M > 8\text{-}10\ M_\odot$. Distinctly different lifetimes and evolutionary paths, huge variation.

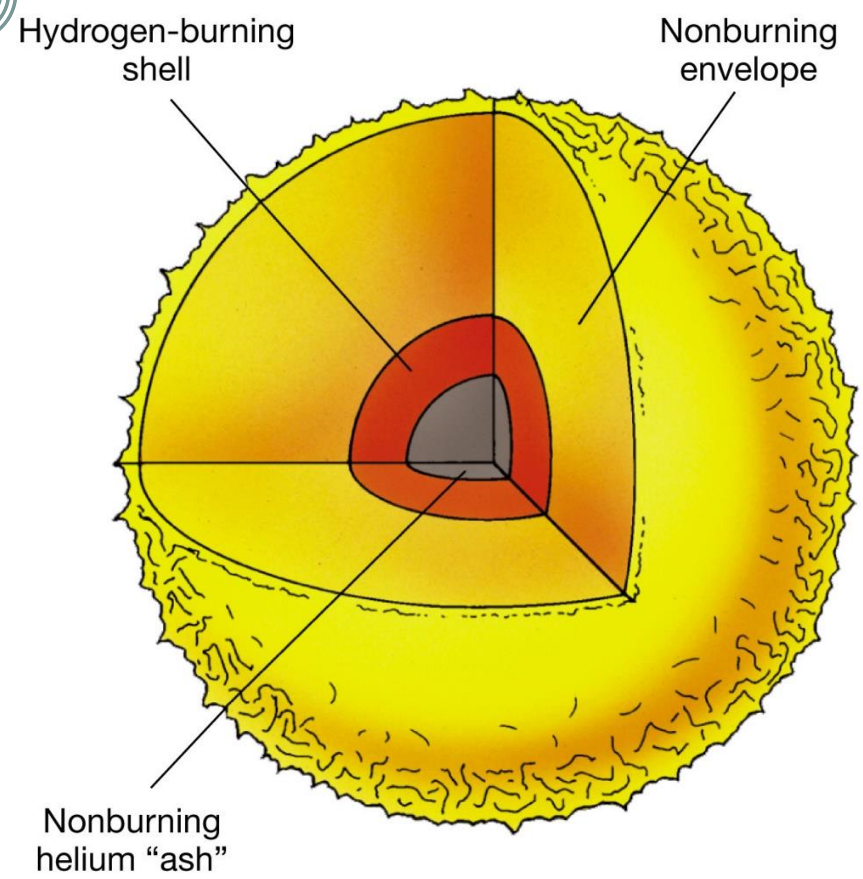
The evolution of low-mass stars

The Terminal-Age MS and Subgiant Branch

The cores of $1M_\odot$ stars become He rich. Fusion is most efficient in the centre, where T is highest.

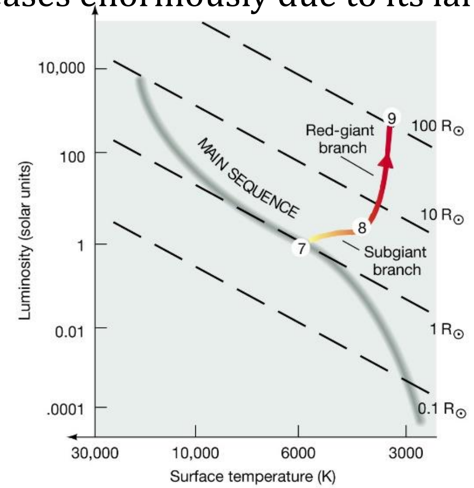
- As He content increases, core shrinks and heats up → He rich core grows
- The T is not high enough for the triple-α process
- H-burning continues in a shell around the core, and as T increases, the CNO process can occur in the shell
- As $\varepsilon_{CNO} \sim T^{18}$ energy generation is concentrated in the regions of highest T and highest H content (in shell $T \sim 20 \times 10^6$ K)

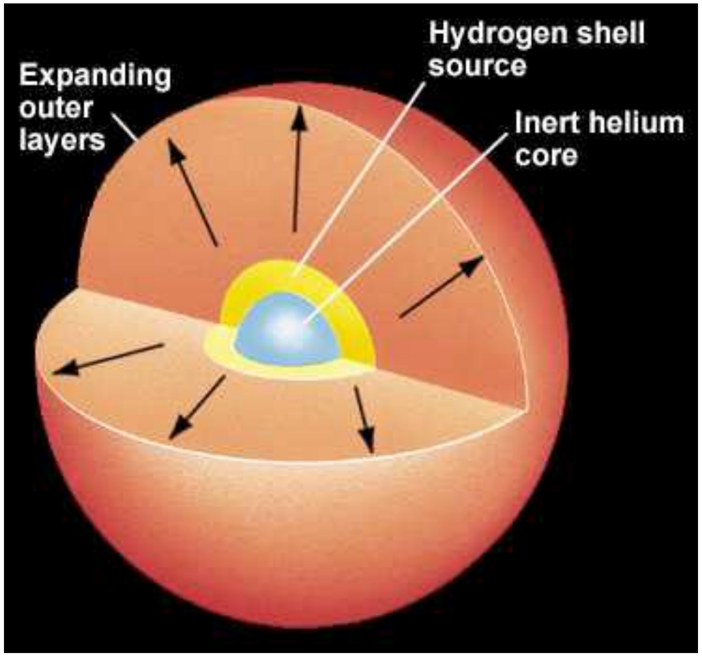
- The envelope expands using the gravitational energy released during core contraction and thermal energy generated by efficient H shell burning.
- This expansion **terminates** the life at the main-sequence
- Luminosity remains approximately constant, hence $T_{\rm eff}$ must decrease, star moves along the red subgiant branch.



The red-giant phase of a Sun-like Star

- The shell source slowly burns, moving through the star, as the He core grows in mass and contracts. But the star cannot expand and cool indefinitely.
- When the temperature of the outer layers reach <5000 K the envelopes become fully convective.
- This enables greater luminosity to be carried by the outer layers and hence quickly forces the star almost vertically in the HR diagram. Despite its cooler temperature, its luminosity increases enormously due to its large size.

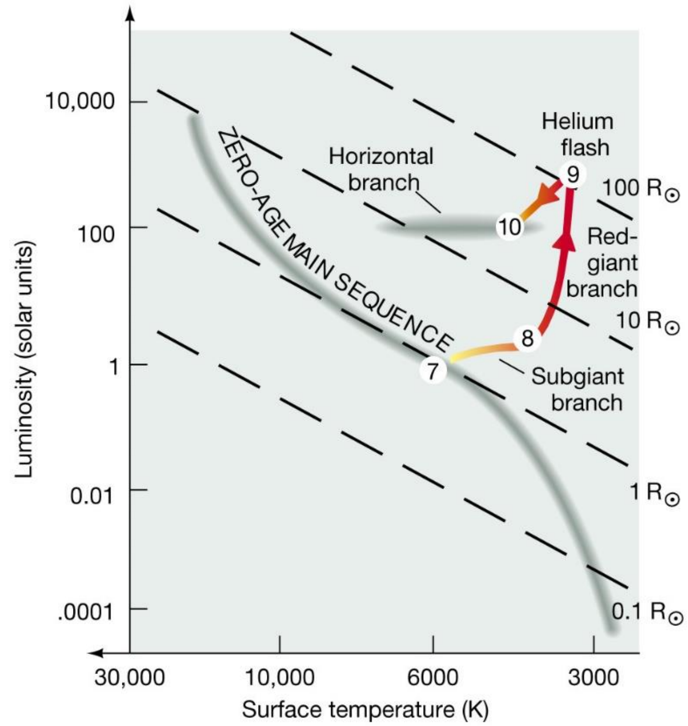




- The star approaches the Hayashi line, and a small increase in the He core mass causes a relatively large expansion of the envelope.
- It is now a **red giant**, extending out as far as the orbit of Mercury.

The He-flash and core He-burning

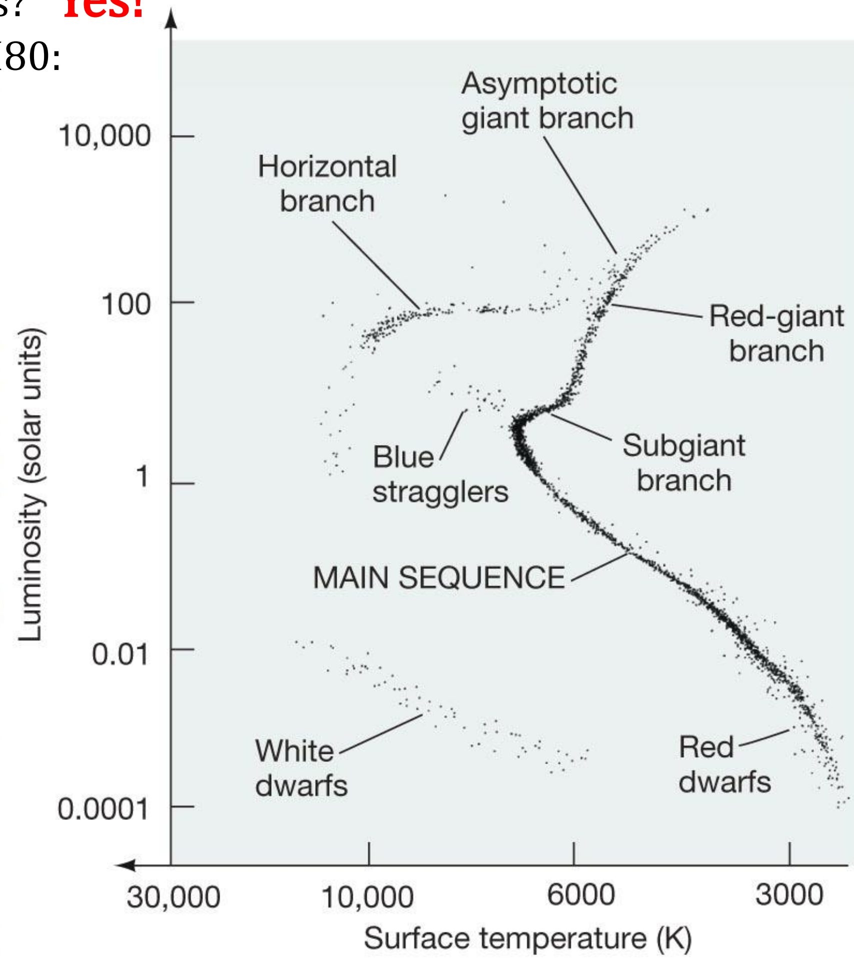
- The helium core does not reach threshold *T* for further burning as it ascends the RGB, and as it is not producing energy it continues to contract until it becomes **degenerate**.
- At tip of the RGB the e^- in core are completely **degenerate**. *P* is due to **degenerate** electron pressure, which is independent of *T*.
- *T* is defined mainly by the energy distribution of the heavy particles (He nuclei). Gravitational collapse is resisted by e^- degeneracy pressure.
- At $T \sim 10^8$ K, triple-α reactions start in the very dense core. They generate energy, heating core, and kinetic energy of He nuclei increases, increasing the energy production. Energy generation and heating under degenerate conditions leads to runway - the **He flash**.
- During the He-flash, the core temperature changes within seconds. The rapid increase in *T* leads the e^- again following Maxwellian velocity distribution and degeneracy is **removed**. The pressure increases and core expands. The luminosity actually drops as most of the released energy goes into expansion.



The star finds a new equilibrium configuration with an expanded **non**-degenerate core which is hot enough to burn He. The H-burning shell source is also expanded and has lower *T* and density and generates less energy than before. The star sits in the Red Clump (metal rich stars) or the Horizontal Branch (metal poor stars).

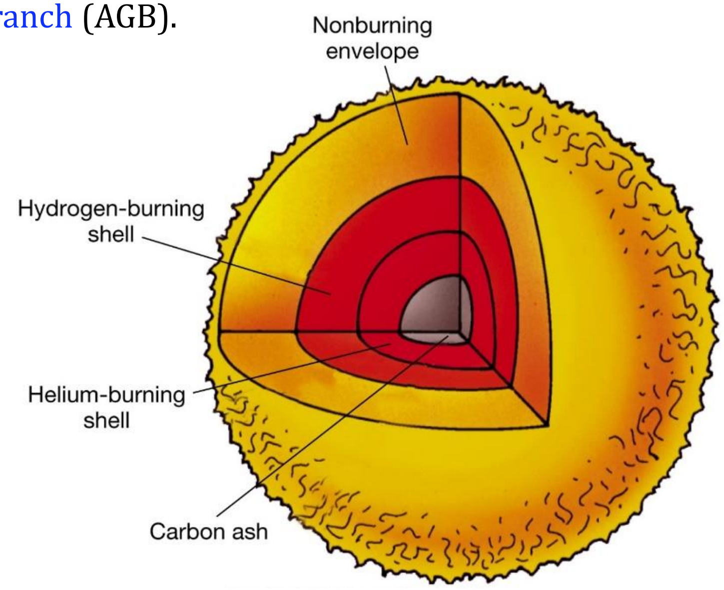
Observations of these evolution stages

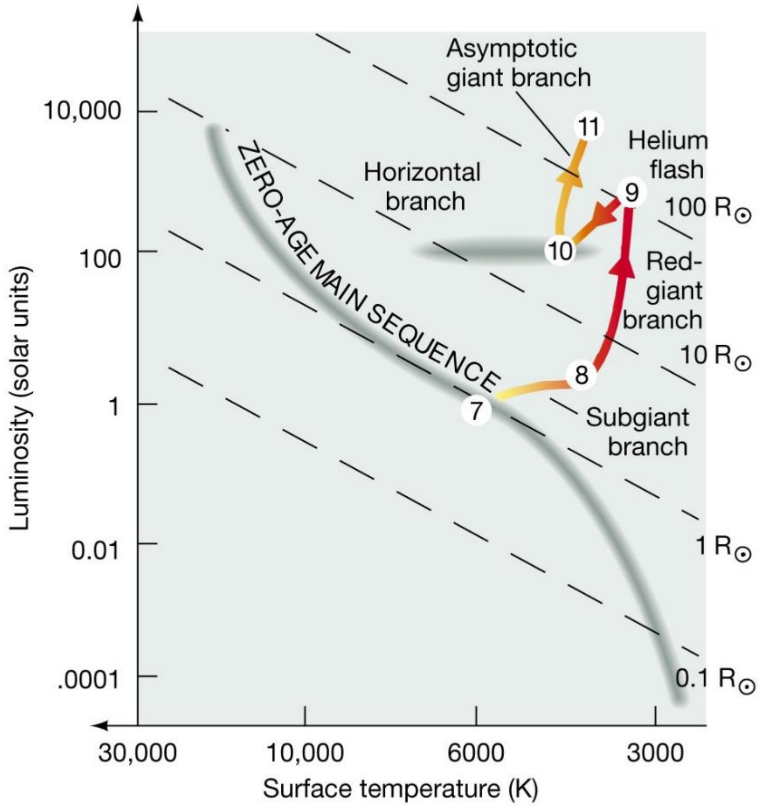
Can we observe these evolution stages in real stars? **Yes!**
Look at the H-R diagram for the globular cluster M80:



The AGB and thermal pulses

- The triple-α reaction is even more T-dependent ($\varepsilon \sim T^{40}$), hence energy generation is even more centrally condensed. Note, the H-burning shell is still generating energy.
- The core will soon consist only of C+O, and in a similar way to before, the CO-core grows in mass and contracts, while a He-burning shell source develops.
- These two shell sources force expansion of the envelope and the star evolves up the red giant branch a second time - this is called the asymptotic giant branch (AGB).





- For high metallicity stars, the AGB coincides closely with the first RGB.
- For globular clusters (typical heavy element composition 100 times lower than solar) they appear separated.

The stellar wind

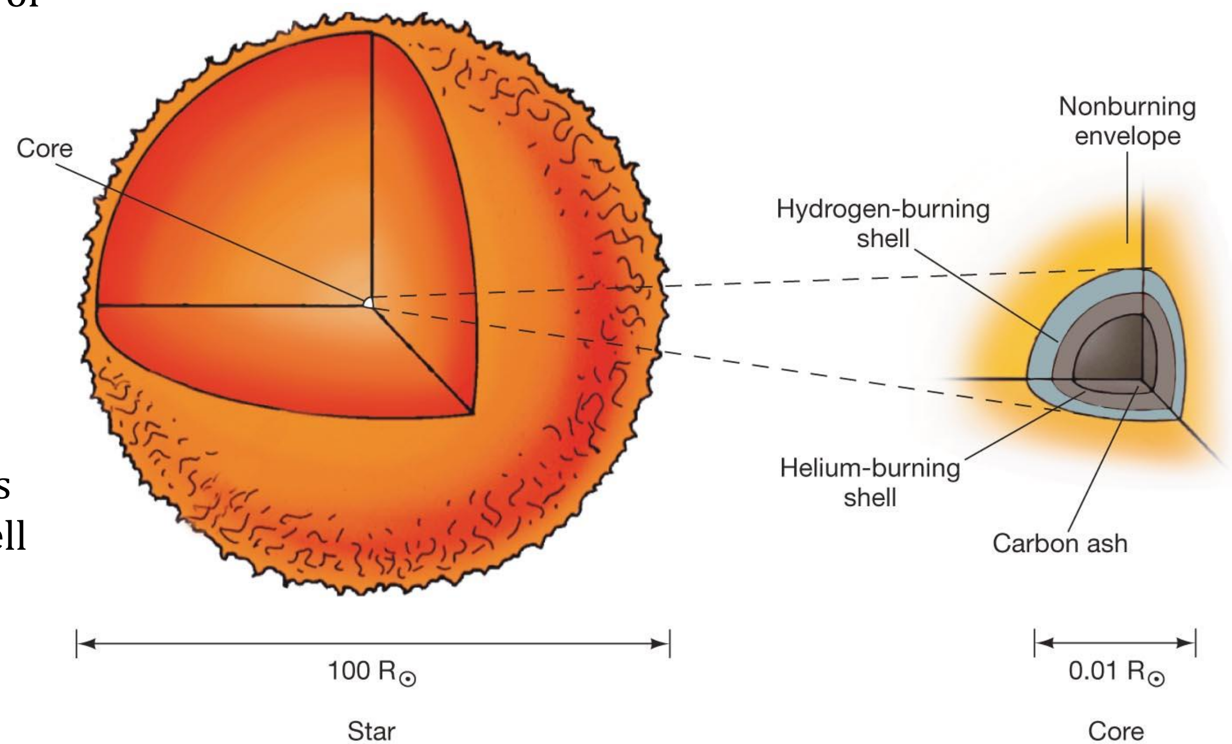
- Large radiation pressure (and dust formation) at tip of AGB probably drives mass-loss. Particles may absorb photons from radiation field and be accelerated out of the gravitational potential well. Observations of red giants and supergiants (more massive evolved stars) give the mass loss rate in the range 10^{-9} to 10^{-4} $M_\odot$ yr^{-1}.
- Mass-loss is generally classified into two types of wind.
  1. Stellar wind: described by empirical formula (Dieter Reimers), linking mass, radius, luminosity with simple relation and a constant from observations. Typical wind rates are of order 10^{-6} $M_\odot$ yr^{-1}

$$\dot{M} = 10^{-13} \frac{L}{L_\odot} \frac{R}{R_\odot} \frac{M_\odot}{M} \ \mathrm{M_\odot\ yr^{-1}}$$

  2. A superwind: a stronger wind, leading to stellar ejecta observable as a shell surrounding central star
- The existence of a superwind is suggested by two independent variables. The high density observed within the observed shells of stellar ejecta, and relative paucity of very bright stars on the AGB.
- The latter comes from the number of AGB stars expected compared to observed is >10. Hence a process prevents them completing their movement up the AGB, while losing mass at the Reimer's rate.

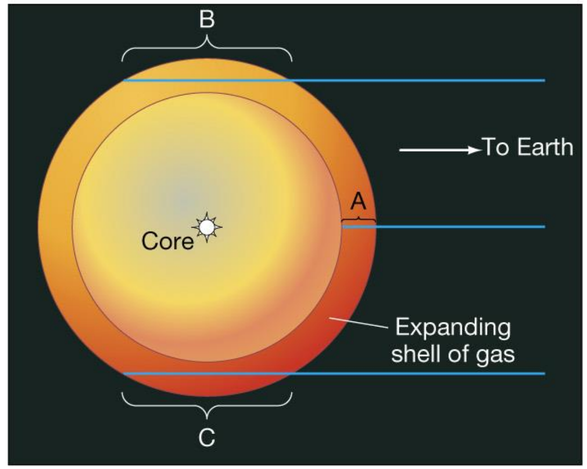
The planetary nebula phase

- This is a superwind which removes the envelope mass before the core has grown to it's maximal possible size. Direct observations of some stars indicate mass-loss rates of order $10^{-6}\ M_{\odot}\ yr^{-1}$. Probably this is due to pulsational instability and thermal pulses (unstable He shell burning) in envelope e.g. Mira-type variables.
- Superwind causes ejection of outer layers of gas which form **planetary nebula**.
- The cores collapse into C-O white dwarfs.
- Core mass at a tip of AGB $\sim 0.6\text{-}1\ M_{\odot}$ and most white dwarfs have masses close to this.
- Planetary nebula nucleus is still burning H in a thin shell above C-O core.



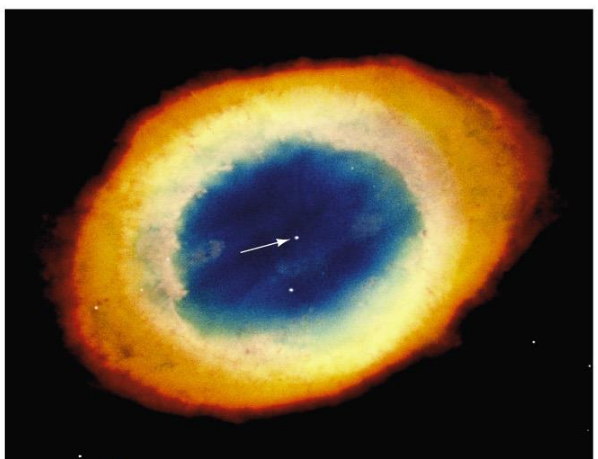
The Death of a Low-Mass Star


(a)



(b)

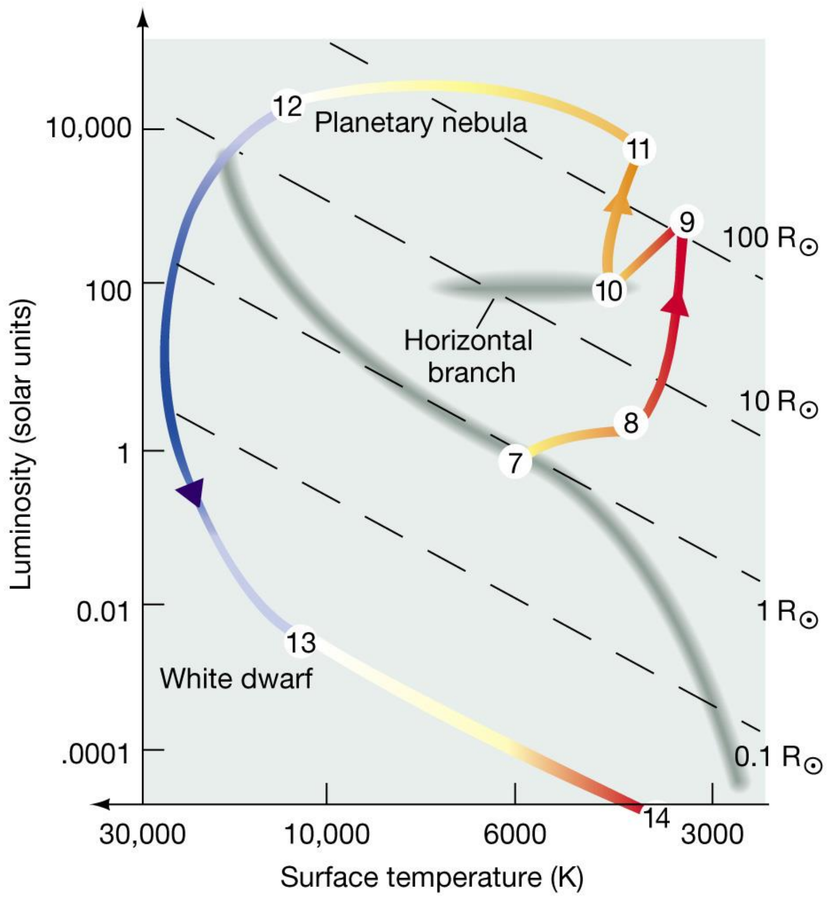
- Planetary Nebula is ejected (25 - 60% of mass ejected) and expands for about 10^5 yrs before dissipating into the surroundings.
- An envelope is about the size of our solar system.
- The ejected envelope is ionized by the hot white dwarf and glows.

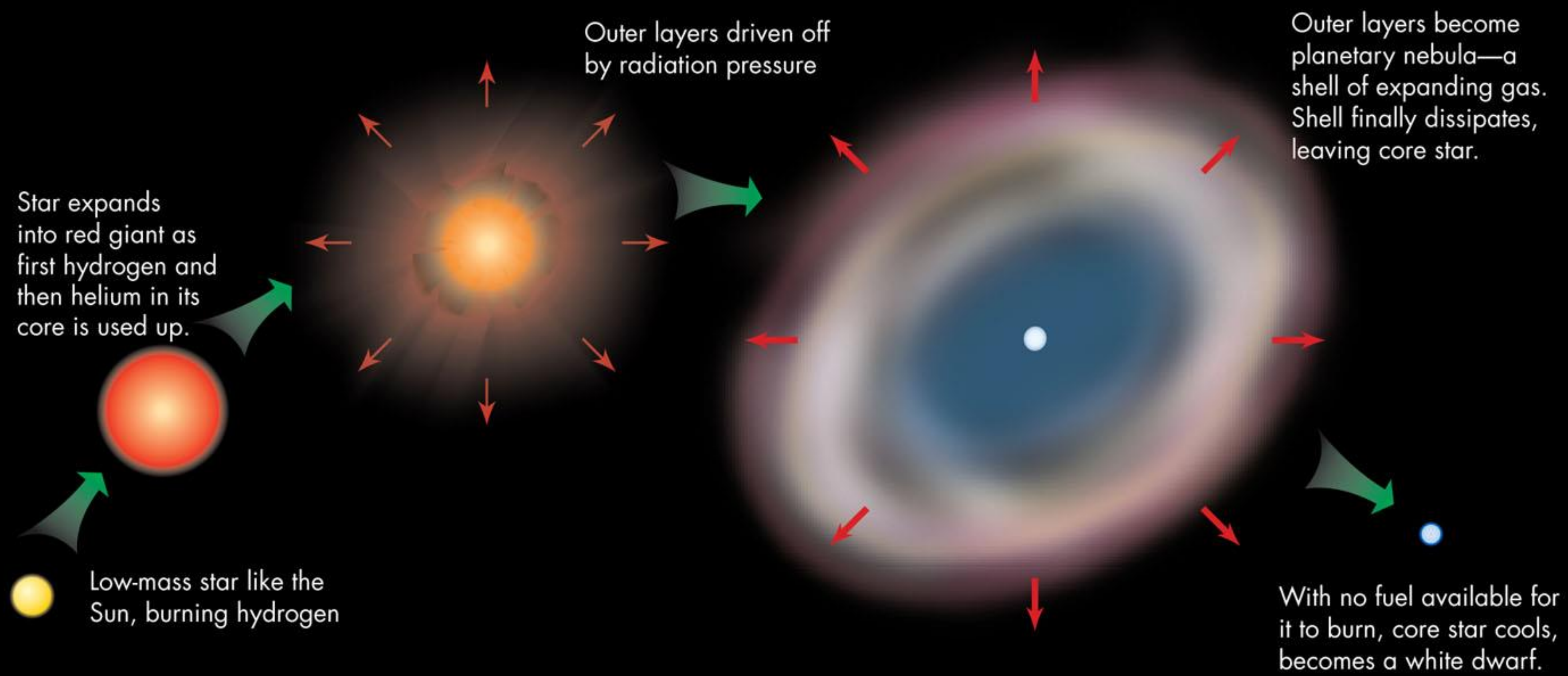


- Planetary Nebulae are very common (20,000 to 50,000 in the Milky Way).
- Planetary nebulae can have many shapes.
- As the dead core of the star cools, the nebula continues to expand, and dissipates into the surroundings.

The end point: white and black dwarfs

- Once the nebula has gone, the remaining core is extremely dense and extremely hot, but quite small.
- It is luminous only due to its high temperature.
- Core is all carbon, very dense – degenerate.
- Carbon ignition never occurs
- As the white dwarf cools, its size does not change significantly; it simply gets dimmer and dimmer, and finally ceases to glow.
- Estimated cooling time for the white dwarf is longer than the current age of the Universe => no black dwarfs yet.

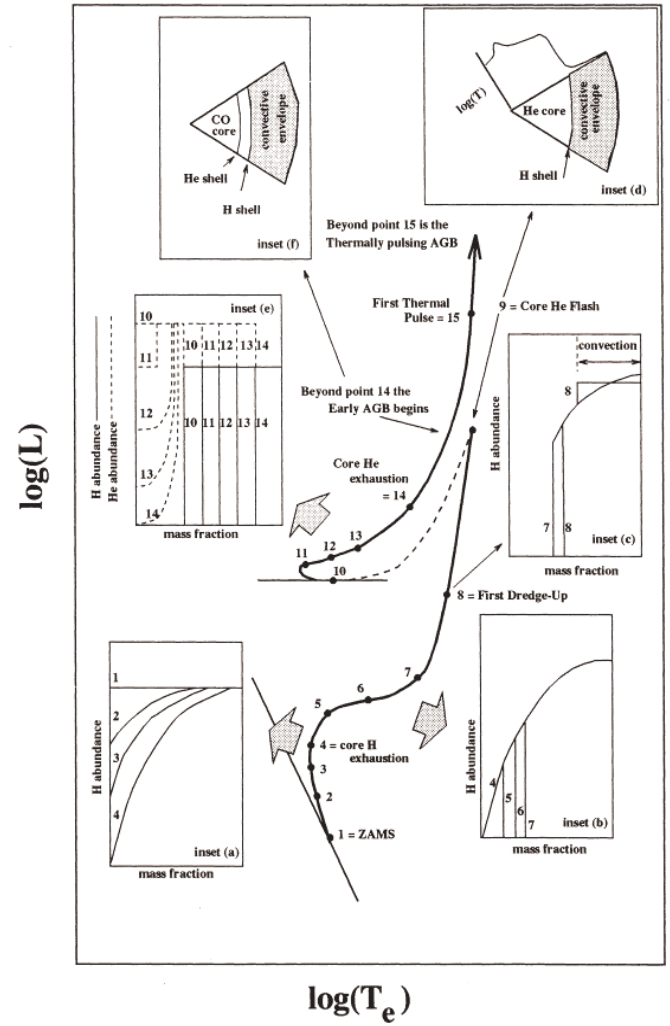



Outer layers driven off
by radiation pressure
Outer layers become
planetary nebula—a
shell of expanding gas.
Shell finally dissipates,
leaving core star.
Star expands
into red giant as
first hydrogen and
then helium in its
core is used up.
Low-mass star like the
Sun, burning hydrogen
With no fuel available for
it to burn, core star cools,
becomes a white dwarf.

Summary of 1 $M_\odot$ evolution

Approximate typical timescales:

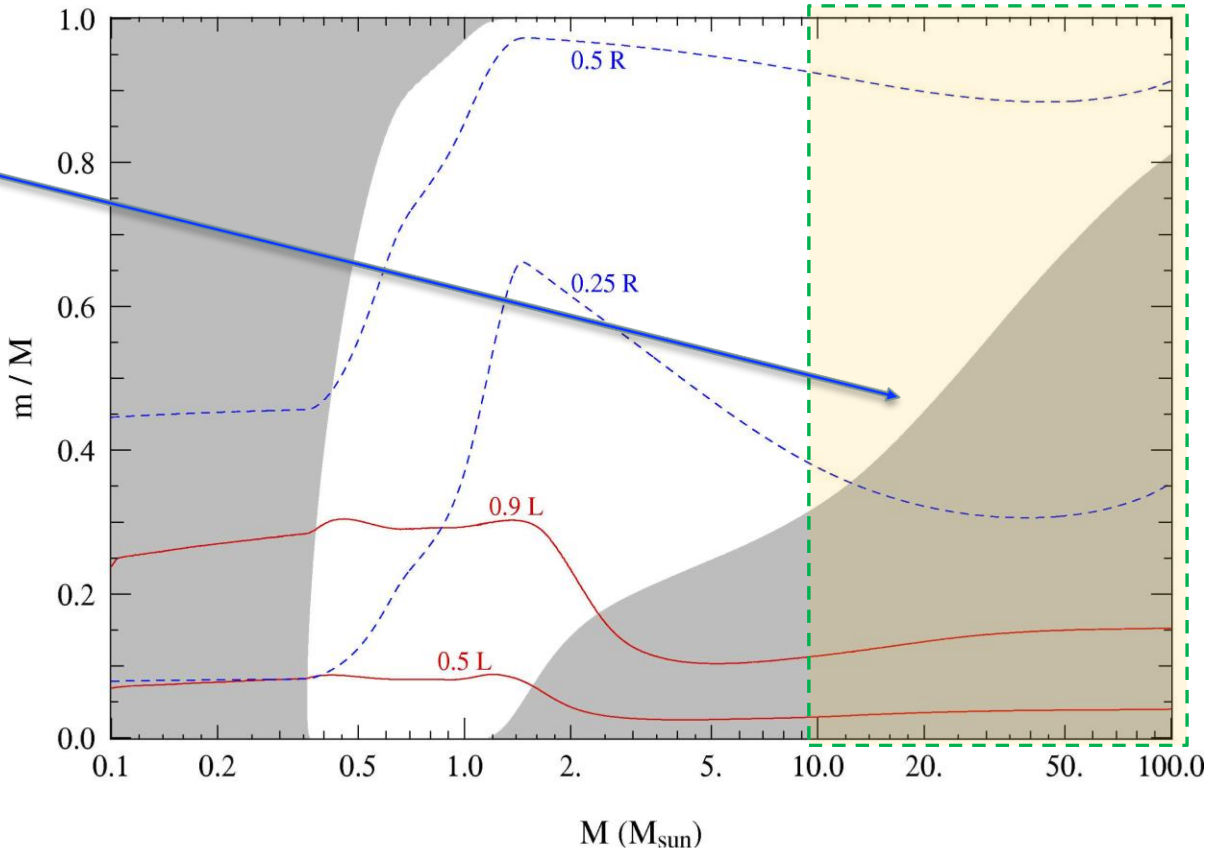
Phase	T (yrs)	Steps →
Main-sequence	9×10^{9}	1-4
Subgiant	3×10^{9}	5-7
Red Giant Branch	1×10^{9}	8-9
Red clump	1×10^{8}	10
AGB evolution	$\sim5\times10^{6}$	11-15
PNe	$\sim1\times10^{5}$	
WD cooling	$>8\times10^{9}$	



The evolution of massive stars

Convective regions on the ZAMS

The cores of **massive** stars on the ZAMS are convective.



Occurrence of convective regions (gray shading) on the ZAMS in terms of fractional mass coordinate m/M as a function of stellar mass, for detailed stellar models with a composition $X = 0.70$, $Z = 0.02$.

The solid (red) lines show the mass shells inside which 50% and 90% of the total luminosity are produced.

The dashed (blue) lines show the mass coordinate where the radius r is 25% and 50% of the stellar radius R.

(After Kippenhahn & Weigert)

From the main-sequence to He burning

1. The cores of massive stars are convective, hence newly formed He is evenly mixed in the core.
2. As the hydrogen is consumed, the core contracts and also shrinks in mass.
3. The convective core becomes exhausted of hydrogen homogeneously, while it contracts to a smaller volume and becomes hotter.
4. The star also develops a H-burning shell around the He dominated core.
5. The temperature at the bottom of the hydrogen envelope is too high to sustain hydrostatic equilibrium. The envelope expands and the star becomes cooler, moving to the red region of the HRD. It becomes a red supergiant star.
6. Due to the rapid drop in temperature throughout the outer atmosphere, the criterion for convection is reached in this region and a convection zone develops, reaching deep into the star.
7. It dredges up some of the material from the original convective core.
   This core material can appear at the stellar surface in the atmosphere of the red supergiants.

High-mass stars create heavy elements in their cores

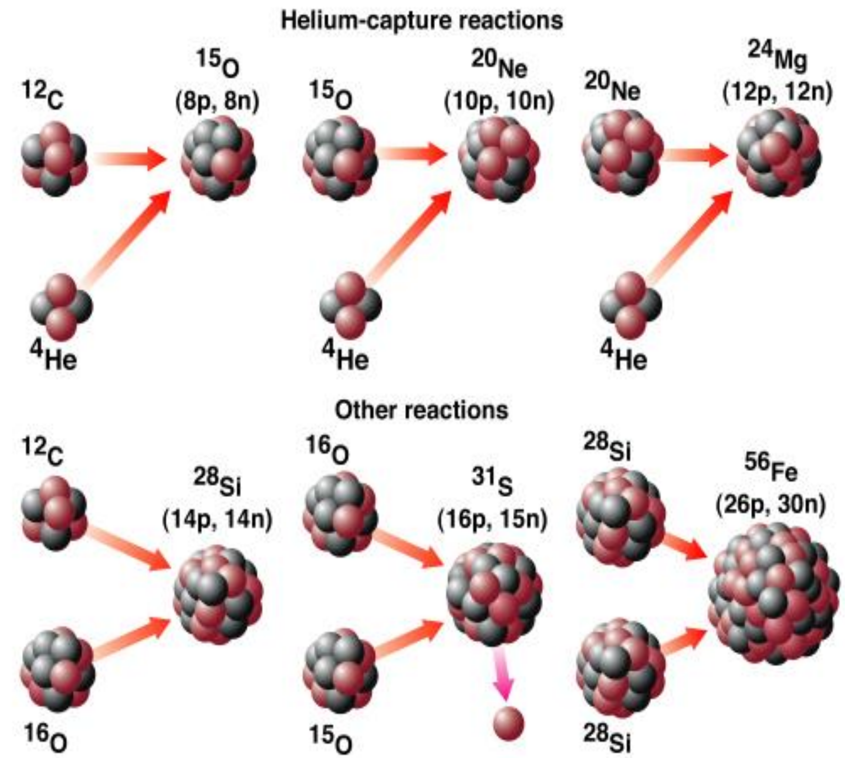
- Unlike low-mass stars, **massive** stars ($M > 8\ M_{\odot}$) can go through all phases of nuclear fusion, up to the Fe-core.
- The internal evolution goes by a series of successive fusion-phases, ending with a shell-like structure, consisting of shells of different compositions.
- Massive stars of course spend most of their lives on the main-sequence, and illustrative timescales for $25\ M_{\odot}$ - stars given below.

Evolutionary Stages of a 25-$M_{\odot}$ Star

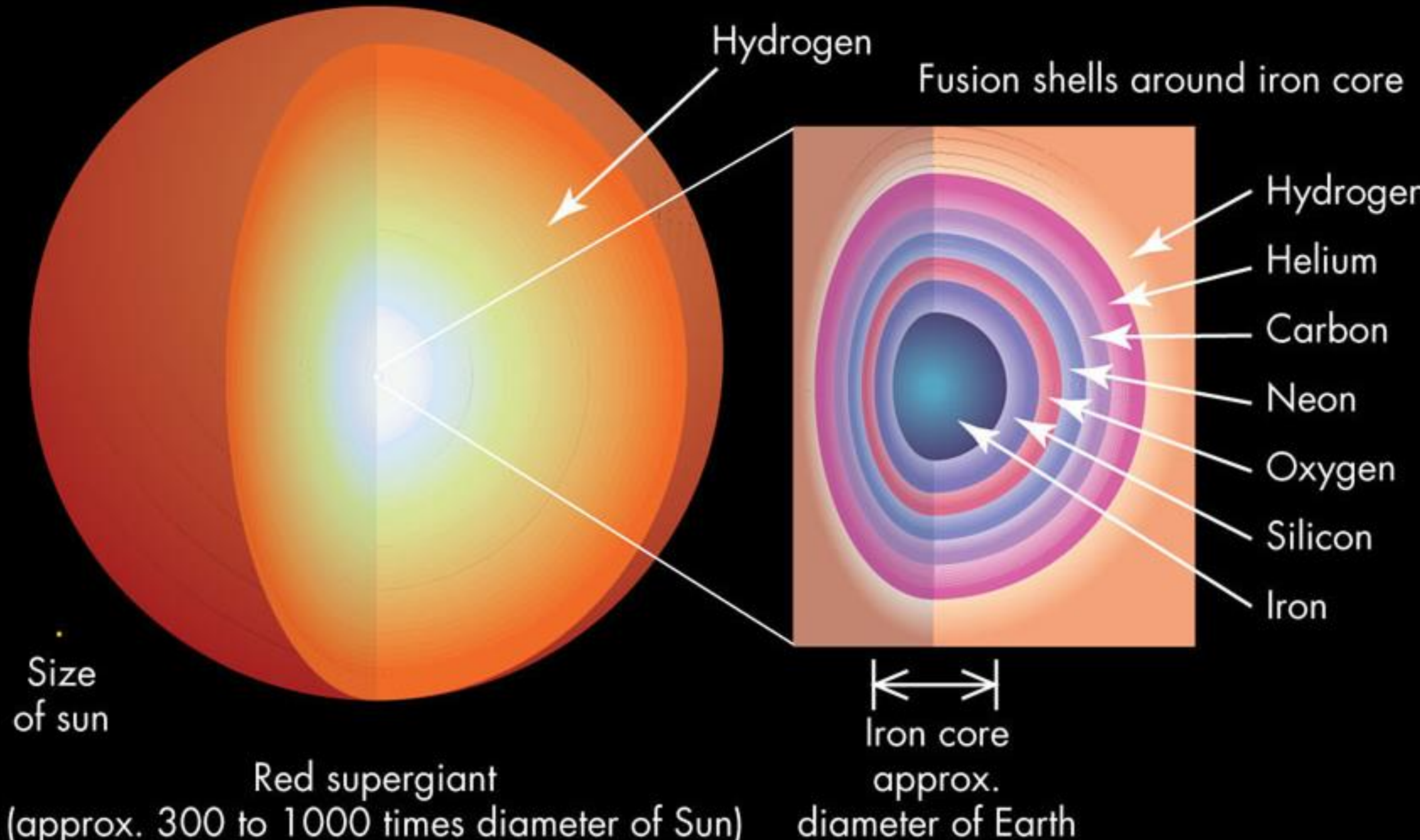
Stage	Core temperature (K)	Core density (kg/m^3)	Duration of stage
Hydrogen fusion	4×10^7	5×10^3	7×10^6 years
Helium fusion	2×10^8	7×10^5	7×10^5 years
Carbon fusion	6×10^8	2×10^8	600 years
Neon fusion	1.2×10^9	4×10^9	1 year
Oxygen fusion	1.5×10^9	10^{10}	6 months
Silicon fusion	2.7×10^9	3×10^{10}	1 day
Core collapse	5.4×10^9	3×10^{12}	¼ second
Core bounce	2.3×10^{10}	4×10^{15}	milliseconds
Explosive (supernova)	about 10^9	varies	10 seconds

From He burning to core-collapse

1. The He burning core is surrounded by a H-burning shell.
2. The triple-α process liberates less energy per unit mass than for H-burning (~10%). Hence the lifetime is shorter, again around 10%.
3. There is **no** He-flash as densities in the He-core are not high enough for electron degeneracy.
4. We have now core of ^{12}C and ^{16}O, surrounded by He and H burning shells.
5. The core will again contract and the temperature will rise (~0.5-1.5 billion K), allowing C and O burning to Mg, Ne, Si.
6. Neutrinos from pair annihilation contribute a substantial amount to the luminosity (energy loss), even more than the nuclear burning.
7. So, the star contracts to make up the loss with gravitational energy.
8. The photons released in above reactions have such high energies that they can photo-disassociate surrounding nuclei.
9. Silicon burning sets in through interactions with alpha particles.
10. This process continues, with increasing Z, building up heavier and heavier elements until the iron group of elements of Ni, Fe and Co are formed.



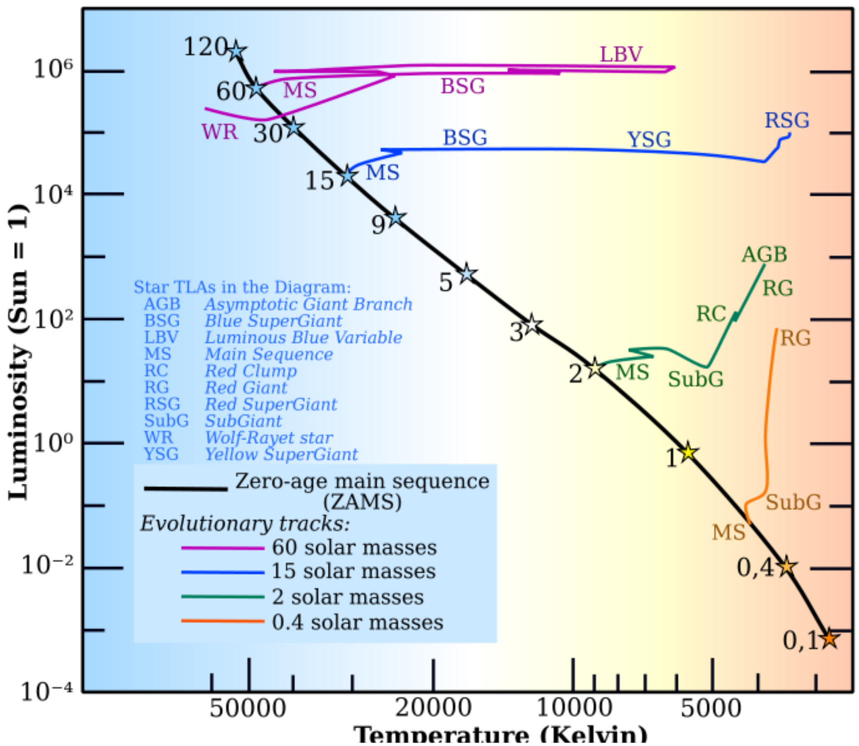
The core is surrounded by a series of shells at lower T, and lower ρ.



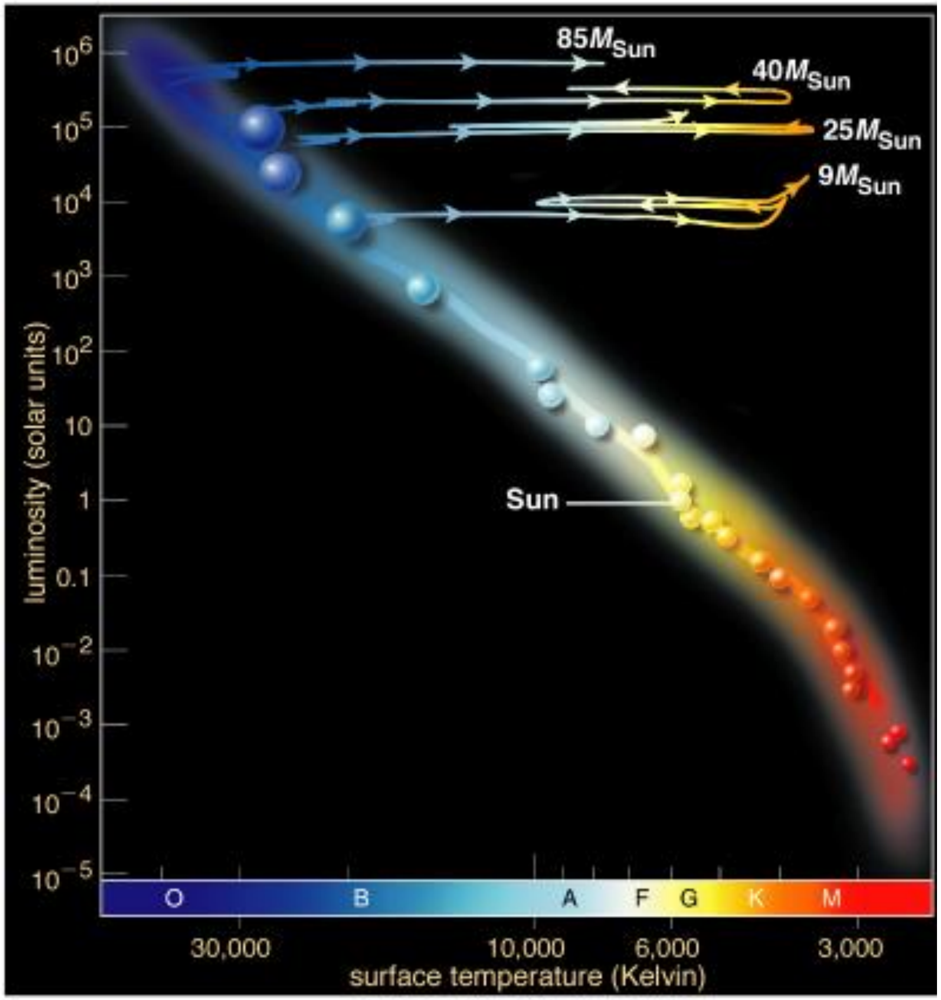
Schematic description of the evolution of massive stars

The evolution of massive stars have the following general characteristics and differences to lower mass evolution:

1. The electrons in their cores do **not** become degenerate until the final burning stages, when iron core is reached.
2. The external evolution of massive stars proceeds mostly along horizontal lines in the HRD, i.e. about constant luminosity, because the star does not develop a degenerate core and most of the mass is in radiative equilibrium.
3. Mass-loss plays an important role in the entire evolution. Stars above 30 $M_\odot$ typically lose about 15 percent of their mass during the main sequence phase so the products of nuclear fusion appear at the surface right after the MS phase. The mass-loss time-scale can be shorter than the main sequence timescale.
4. The luminosity remains approximately constant in spite of internal changes. The track on the HRD is therefore horizontal. For a 15-25 $M_\odot$ star there is a gradual red-wards movement. But for higher mass (or stars with different initial compositions) the star moves back and forth between low and high effective temperatures.



Supergiants on the H-R Diagram



- As the shells of fusion around the core increase in number:
  - thermal pressure overbalances the lower gravity in the outer layers
  - the surface of the star expands
  - the surface of the star cools
- The star moves toward the upper right of H-R Diagram
  - it becomes a red supergiant
- For the most massive stars:
  - the core evolves too quickly for the outer layers to respond
  - they explode before even becoming a red supergiant

The evolution proceeds differently in three distinct mass ranges

1) The stars with $8 < M < 25\ M_{\odot}$ become **red supergiants** (RSG) in their H-shell fusion phase. (This is similar to the RGB (Red Giant Brunch) of lower mass stars, but their He-core is not degenerate). During their He-core fusion phase they make a leftward loop in the HRD and temporarily become yellow supergiants. (This is similar to the HB – Horizontal Brunch – of low mass stars). The later fusion phases are all spend as a RSG (Hayashi limit). In the end these RSGs explode as supernova (SN).
   So their evolution track in the HRD is quite similar to those of stars of M>4 $M_{\odot}$ but they never develop a degenerate core.
2) The stars with $25 < M < 50\ M_{\odot}$ also become RSG, but their mass loss rate is so high that after a short time at the Hayashi limit they have lost most of their envelope and move to the left of the HRD. (This is similar to the post AGB evolution of lower mass stars, but they do not become WDs). When they are in the left of the HRD they have a very high mass loss rate and their atmospheres are dominated by He, N and C. These are Wolf-Rayet stars (WR-stars). They explode as SN in the WR phase.
3) Stars with $M > 50\ M_{\odot}$ become unstable due to radiation pressure in their envelope immediately after the MS, because they are very close to the photospheric Eddington limit. They become Luminous Blue Variables with high mass loss rates and occasional eruptions. They stay on the blue side in the HRD where they become Wolf-Rayet stars and explode as SN.

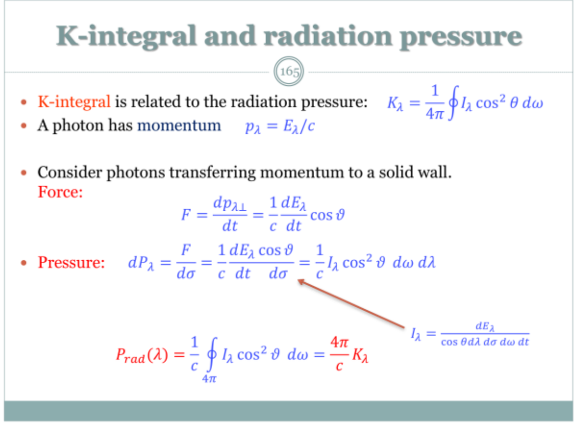
The importance of Radiation Pressure

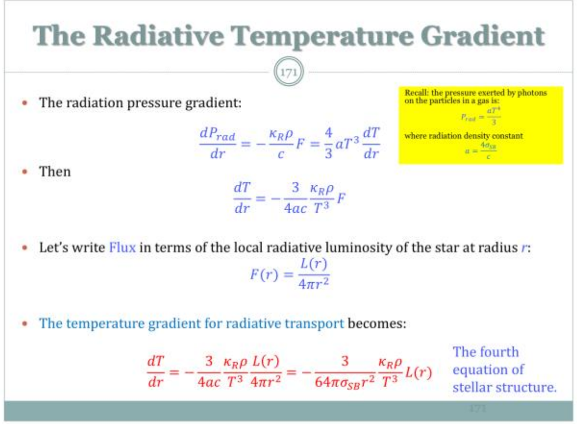
Radiation is important:

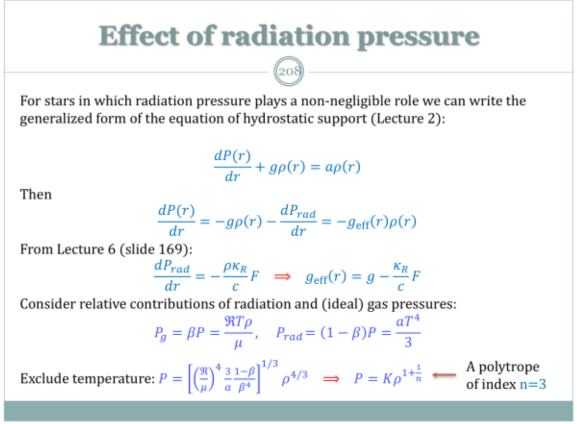
- inside stars, as a source of energy transport
- outside stars and other sources, from its effect on surrounding gas.

Radiation is an inefficient carrier of momentum (velocities have the highest possible value), but when a photon is absorbed or scattered by matter, it imparts not only its energy to that matter, but also its momentum $h\nu/c$.

We already discussed the importance of radiation pressure several times (e.g., slides 165, 169-171, 208):







Let's now shortly repeat it again and introduce the Eddington limit on Luminosity.

Radiation vs Gravity

We already discussed the importance of radiation pressure several times (e.g., slides 165, 169-171, 208). The radiation pressure gradient near the surface may be written as

$$\frac{dP_{rad}}{dr} = -\frac{\bar{\kappa}\rho}{c}F_{\lambda} = \frac{\bar{\kappa}\rho}{4\pi c r^2}L$$

$$F(r) = \frac{L(r)}{4\pi r^2}$$

But hydrostatic equilibrium demands that the pressure gradient near the star's surface must also be given by

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2}$$

Combining, and solving for the luminosity, we have :

$$L_{edd} = L = \frac{4\pi c G M}{\bar{\kappa}}$$

The Eddington limit

Radiation pressure balances gravity when:

$$L_{edd} = L = \frac{4\pi cGM}{\bar{\kappa}}$$

L_{edd} is the maximum radiative luminosity that a star can have and **still remain** in hydrostatic equilibrium. If L is larger than this value, then the pressure due to radiation exceeds the gravitational force at all radii, and gas will be blown away.

Critical luminosity is called **the Eddington limit**.

Depends upon:
- the mass of the star
- the opacity of the gas surrounding the star / source

How large is the opacity?

$$L_{edd} = \frac{4\pi cGM}{\bar{\kappa}}$$

Opacity depends upon temperature, density, and composition.

In the outer regions of massive stars, it is generally ionized hydrogen. Scattering is due to the electrons with Thomson cross-section σ_T, while the mass is set by protons, so $\kappa = \sigma_T / m_p$. In that case,

$\kappa_\lambda \rho = n\sigma_\lambda$

$$L_{edd} = 6.3 \times 10^4 (M/\text{gram}) \text{ erg/s}$$

$$L_{edd} = 1.25 \times 10^{38} \left(M/M_\odot\right) \text{ erg/s}$$

$$L_{edd} = 3.2 \times 10^4 \left(M/M_\odot\right) L_\odot$$

Only important for sources that are much more luminous that the Sun.

Remember?
Main sequence stars obey a mass-luminosity relation, with $L \propto M^\eta$. The slope η changes slightly over the range of masses; between 1 and $10M_\odot$, $\eta \approx 3.88$.

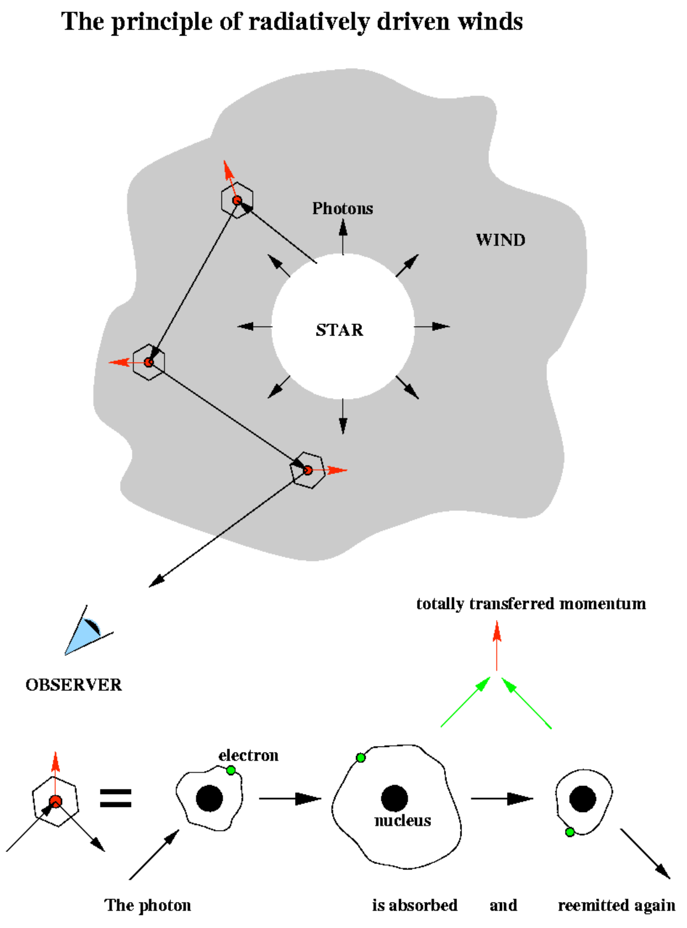
So… why is there a Solar wind?

Mass-loss from high mass stars

Large amount of evidence now that high mass stars loose mass through a strong stellar wind.

- The winds are driven by radiation pressure - UV photons from a hot, very luminous star absorbed by the optically thick outer atmosphere layers.
- atmosphere is optically thick at the wavelength of many strong UV transitions (resonance transitions) of lines of Fe, O, Si, C (and others).
- photons absorbed, imparting momentum to the gas and driving an outward wind.
- It leads to wind velocities of up to 4000 km/s and mass-loss rates of up to $5 \times 10^{-5}\ M_\odot\ \mathrm{yr}^{-1}$.
- Huge effect on massive stellar evolution - the outer layers are effectively stripped off the star.

$$L_{edd} = \frac{4\pi c G M}{\kappa_\lambda}$$



Stellar Winds

Hot Stars possess dense outflows, or stellar winds, which are produced via radiation pressure which influence their subsequent evolution – e.g. Wolf-Rayet Star 124.

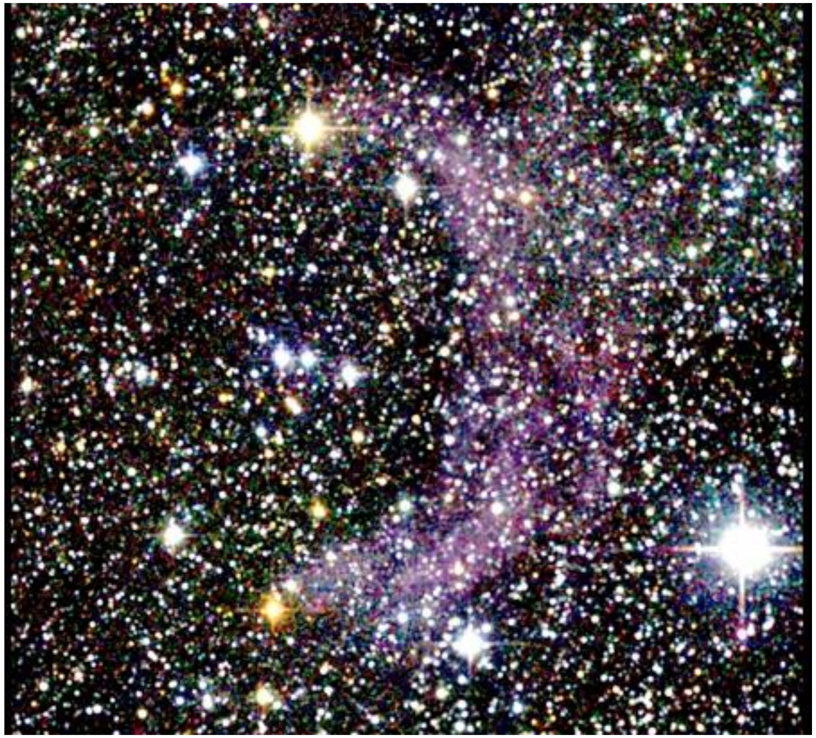
Wolf-Rayet Stars

H-deficient massive stars. Spectra show either high abundances of He+N or C+O.

These are products of H-burning (CNO cycle) and He burning. Likely there is a relation between initial mass and final WR star produced.

Possible evolution scenario:

main-sequence star →
blue supergiant →
red supergiant →
lost of the envelope and WR star.



On the right: NGC 3199, in the constellation Carina, which is the wind-blown partial "ring“ around the WR star WR 18 (= HD 89358), the easternmost (leftmost) of the three bright blue stars near the center of the 2MASS image. NGC 3199 and WR 18 are at a distance of about 3.6 kpc from us.

The death of a high-mass star: supernova explosions

- Nuclear reactions in a high-mass star stop at the iron group, and the last reaction is ^{52}Fe capturing an alpha particle to make nickel.

- Further reactions would require energy to proceed.

- The star dies in a violent cataclysm in which its core collapses and most of its matter is ejected into space at high speeds.

- The luminosity of the star increases suddenly by a factor of around 10^8 during this explosion, producing a supernova.

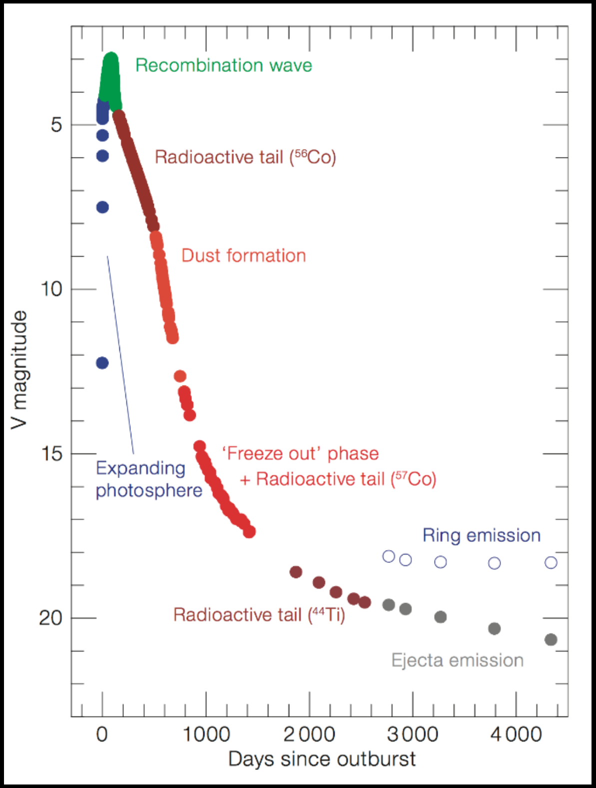
- The matter ejected from the supernova, moving at supersonic speeds through interstellar gases and dust, glows as a nebula called a supernova remnant.

- Leaves either a neutron star or black hole behind.

In 1987 a nearby supernova gave us a close-up look at the death of a massive star

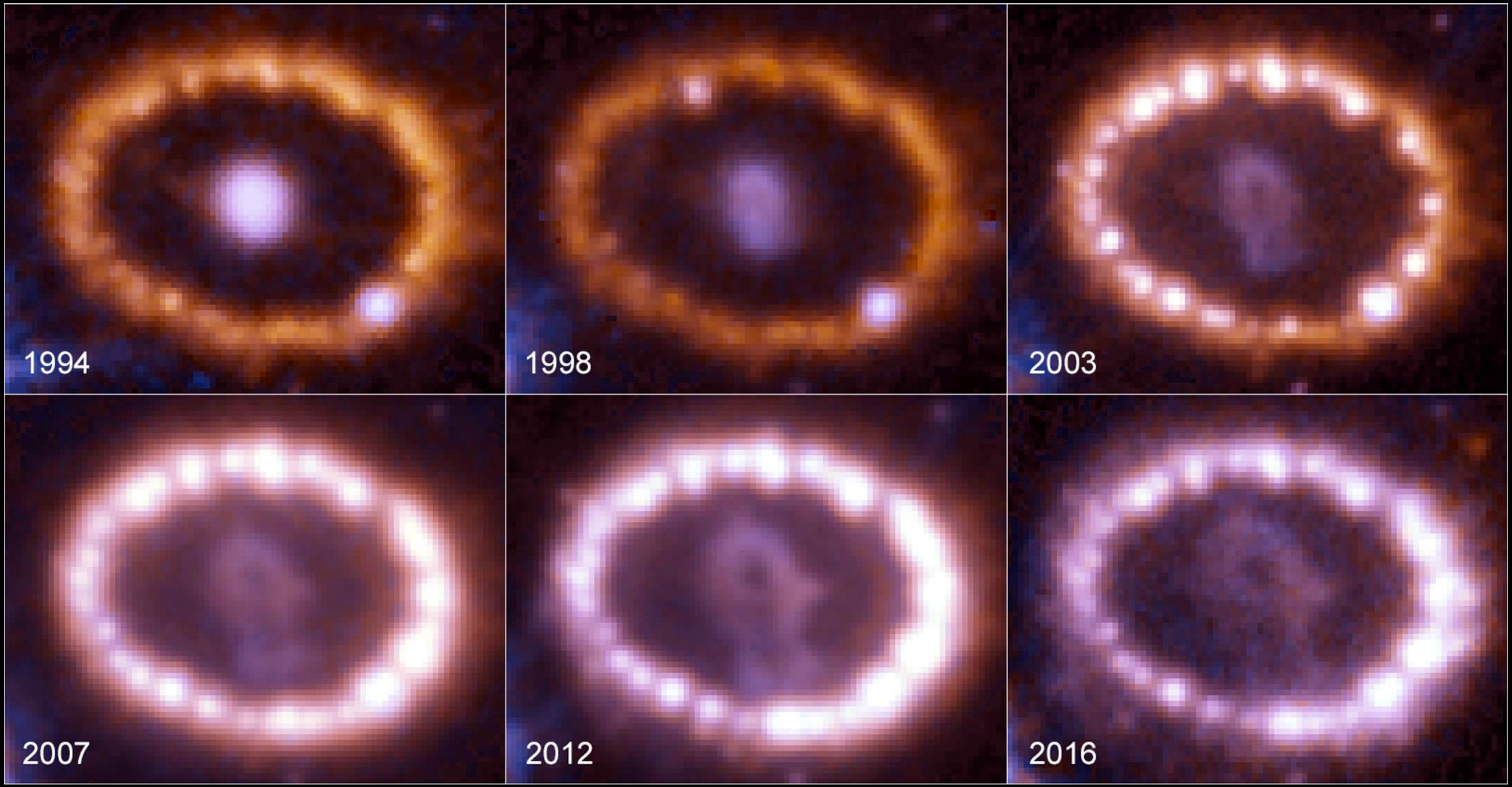
After the star exploded

Before the star exploded



Light curve

The evolution of SN 1987A



Summary of the differences between high- and low-mass stars

Compared to low-mass stars, **high-mass stars**:

- live much shorter lives.
- fuse Hydrogen via the CNO cycle instead of the p-p chain.
- die as a supernova.
  - low-mass stars die as a planetary nebula.
- can fuse elements heavier than Carbon.
- may leave either a neutron star or black hole behind.
  - low-mass stars leave a white dwarf behind.
- are far less numerous.

The initial mass function

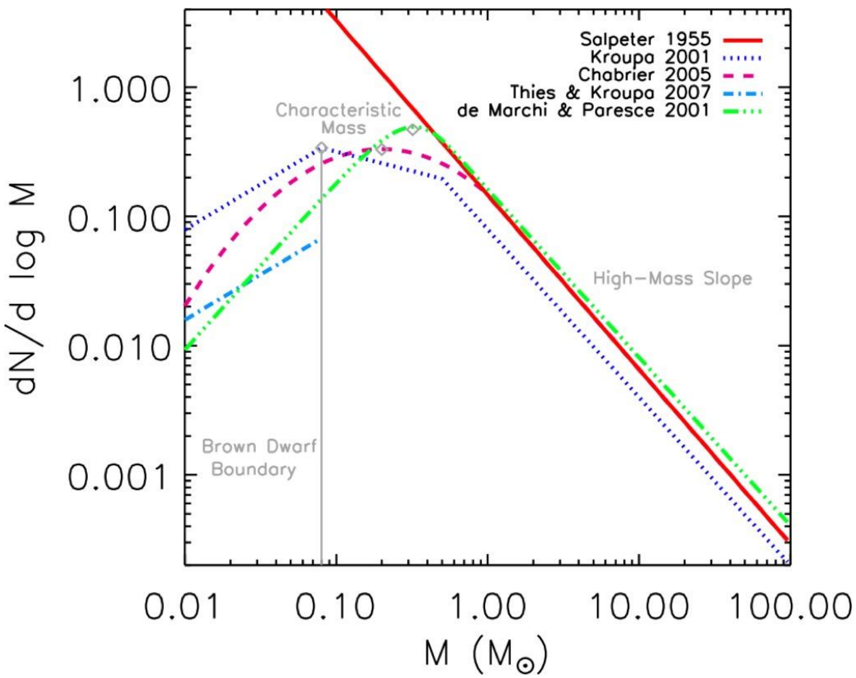
The initial mass function (IMF)

How many stars are formed at each mass in a star cluster, or a star forming region?
Is it always the same distribution ?
Is it constant across environments and galaxies?

Define the number of stars formed at a given time within a given volume, with masses in the range (M, M+dM) as a function solely of M:

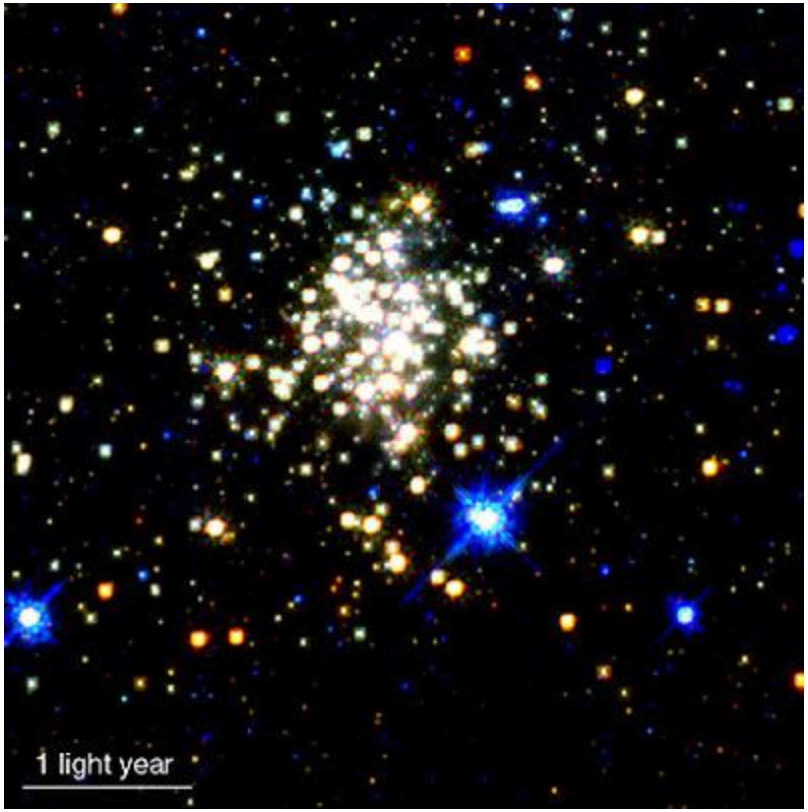
$$dN \propto M^{-\alpha} dM$$

This was the form originally adopted by Salpeter (1955), who determined α = 2.35 by fitting observational data. Over 65 years later, this value is still considered the standard for stars above 1 $M_\odot$.
However, this function diverges as it approaches zero.



Upper mass limit for stars

- How big do stars get?
  Is there an upper mass cut-off?
  What are the masses of the most massive stars?
- The most luminous stars in the Galaxy have inferred masses of ~150-300 $M_\odot$ but this value depends on estimate of $\log L/L_\odot$ and T_{eff}.
  The former requires distance, reddening, bolometric correction and the later requires reliable model atmosphere.
  Probably uncertain within a factor 2.



- The IMF was determined for the very massive Arches cluster (Figer 2005, Nature, 434, 152), which is massive enough that there is a reasonable probability that stars of masses >500 $M_\odot$ could exist in cluster, if they form.
- Most of the bright white stars near the centre of the image have M~120 $M_\odot$.

IMF is constant at different Z

- Solar neighbourhood composition: X=0.70, Y=0.28, Z=0.02.
- The Large Magellanic Cloud (LMC): $Z=0.5Z_{\odot}$
- The Small Magellanic Cloud (SMC): $Z=0.2Z_{\odot}$.
- Star formation of massive stars proceeds independently of metallicity.
- Local Group galaxies SMC and LMC are excellent laboratories to study massive star populations.

No evidence for environment influence

- Whatever the star-formation rate, the IMF seems constant,
- Starburst regions, “normal” young clusters, low mass clusters in Milky Way, LMC, SMC, all similar.
- IMF not measured well beyond the Magellanic Clouds.

Summary

- We have covered qualitative description of the evolution of star from modern calculations.
- The theoretical HRD in general, and $1M_{\odot}$ and $25M_{\odot}$ stars in detail.
- Time-scales for evolutionary stages: 90% of massive star's life is on main-sequence. Final stages of C-burning and beyond last few hundred years.
- Massive stars loose mass - most massive become WR stars, with final masses significantly less than birth mass.
- The main-sequence stars should have upper mass limit.
- The initial mass function (IMF) implies significantly less massive stars than low mass stars born - implications for galactic evolution.

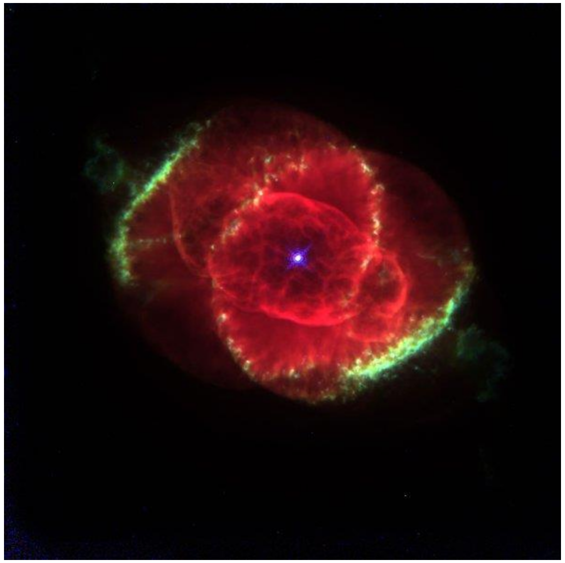
The end point: Stellar remnants

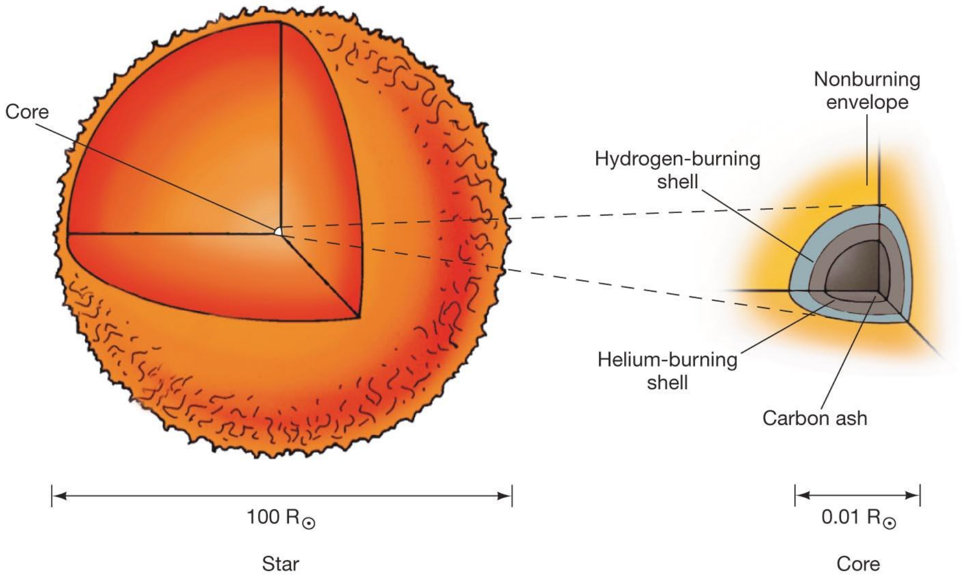
WHITE DWARFS
NEUTRON STARS
BLACK HOLES
SUPERNOVAE

White dwarfs

As mentioned in previous lectures, the end state of stars with masses $M < 8\text{–}10\ M_{\odot}$ is a white dwarf: a stellar remnant in which the pressure is provided by degenerate electrons and there is no significant nuclear burning. No fusion, no contraction, just cooling. For eternity.

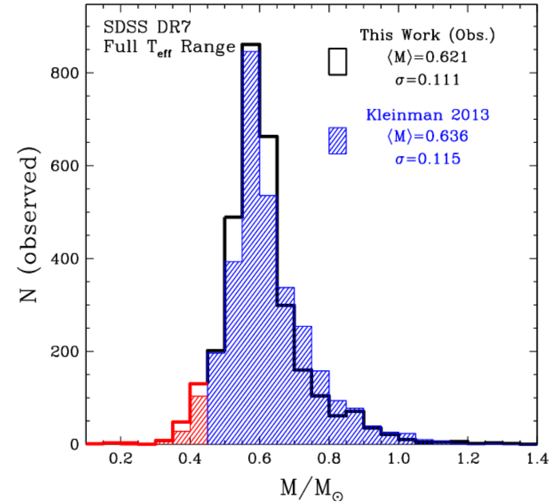
Simply speaking, white dwarfs are the remnants of former stellar cores supported by electron degeneracy pressure and left behind by stellar envelopes dissipated as planetary nebulae.





White dwarfs

Observed WD masses are mostly in a narrow range around 0.6 $M_\odot$, which corresponds to the mass of the C-O cores of low-mass ($\lesssim$2 $M_\odot$) AGB progenitors.



The great majority of white dwarfs are indeed composed of C and O.

White dwarfs with $M > 1.2\ M_\odot$, on the other hand, are mostly O-Ne or O-Ne-Mg white dwarfs. They form from stars with $M \approx 8–10\ M_\odot$ the core temperature of which is sufficient to fuse carbon but not neon.

Stars of very low mass will be unable to fuse helium; hence, a helium white dwarf should form, and they are indeed discovered.

However, the formation of known He WDs is not expected to happen in single stars, but can result by mass loss from binary interaction and indeed most low-mass WDs are found in binary systems.

Why?

White dwarfs

White dwarfs are very well described as polytropes.

In lecture 11 we obtained the Chandrasekhar mass limit – the **maximum mass possible** for a white dwarf.

$$M = M_{Ch} = \frac{5.826}{\mu_e^2} M_\odot$$

For a highly relativistic electron gas, there is only a **single** possible mass which can be in hydrostatic equilibrium. White dwarfs are typically formed of helium, carbon or oxygen, for which $\mu_e = 2$ and therefore $M_{Ch} = 1.456\, M_\odot$.

Thus, stars with masses $M < 8–10\ M_\odot$ never develop a degenerate core more massive than the Chandrasekhar limit (and even for such massive stars, this requires a lot of mass loss).

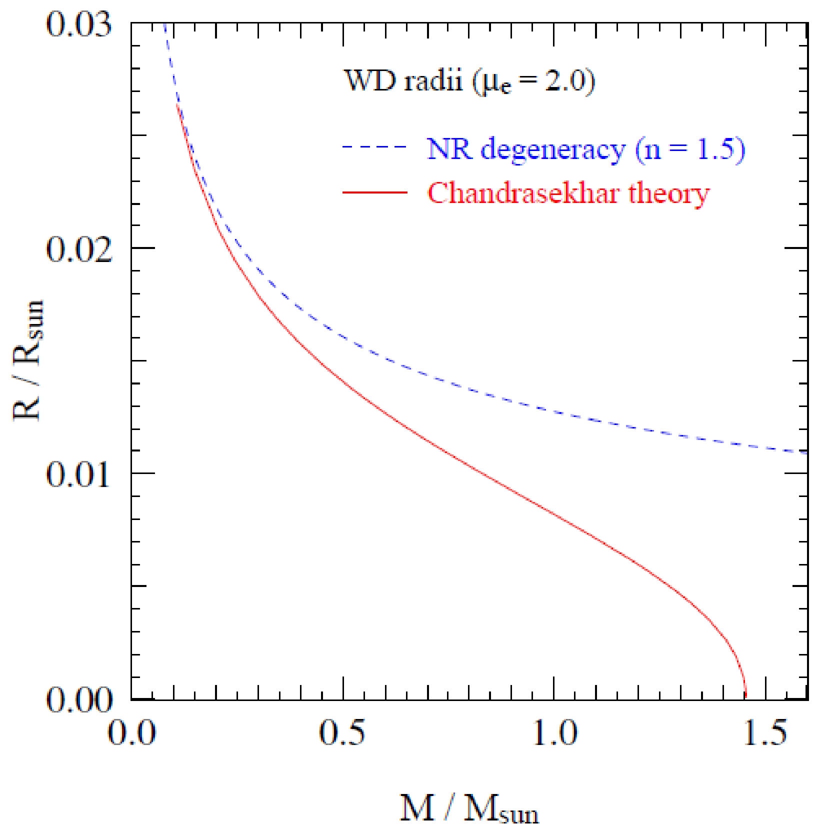
A further increase of the mass (e.g., due to accretion from a companion star) leads to the loss of stability and collapse. This is the cause of supernovae type Ia explosions (will discuss later).

Radius—Mass relation for WDs

White dwarfs are very well described as polytropes with n=1.5, and also in Lecture 11 we obtained that for n=1.5, $R \sim M^{-1/3}$, i.e. the stellar radius is inversely proportional to the mass.

In reality, a proper theory for WDs should take into account that the most energetic electrons in the Fermi sea can move with relativistic speeds, even in fairly low-mass white dwarfs. This means that the EOS is generally not of polytropic form but has a gradually changing n between 1.5 and 3. The pressure in the central region is therefore somewhat smaller than that of a purely non-relativistic electron gas.

Thus, WD radii are smaller than given by the polytropic relation, the difference growing with increasing mass (and increasing central density).

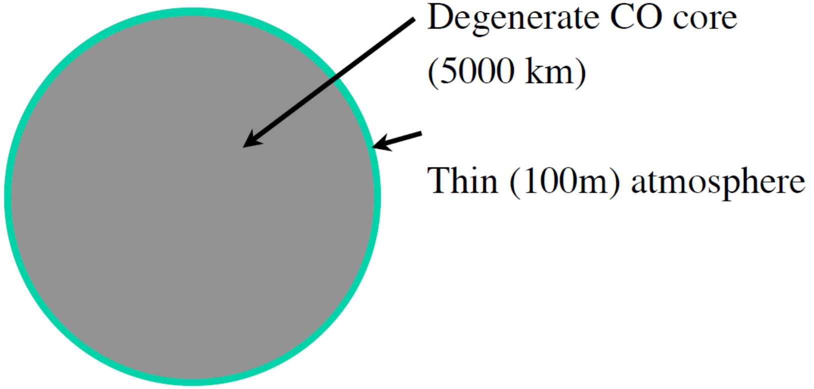


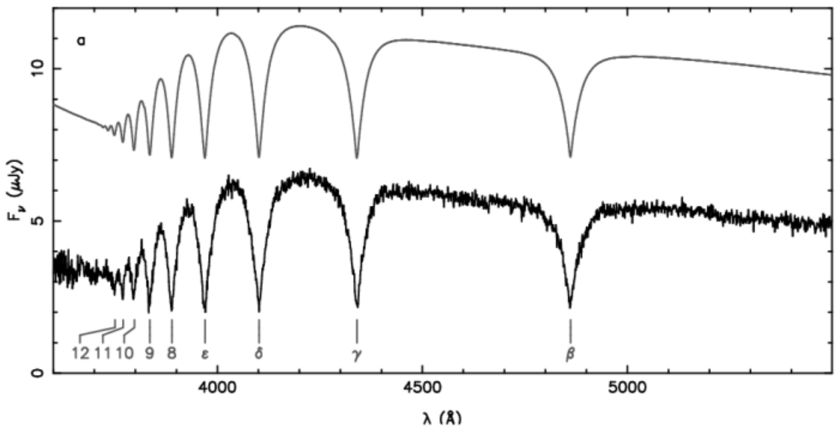
Structure of white dwarfs

The surface composition of white dwarfs is usually completely different than their interior composition.

The strong surface gravity has resulted in separation of the elements, such that any hydrogen left is found as the surface layer while all heavier elements have settled at deeper layers.

Most white dwarfs, regardless of their interior composition, show spectra completely dominated by H lines and are classified as DA white dwarfs.
A minority of white dwarfs show only helium lines and have spectroscopic classification DB. These have lost all hydrogen from the outer layers during their formation process, probably as a result of a late or very late thermal pulse.

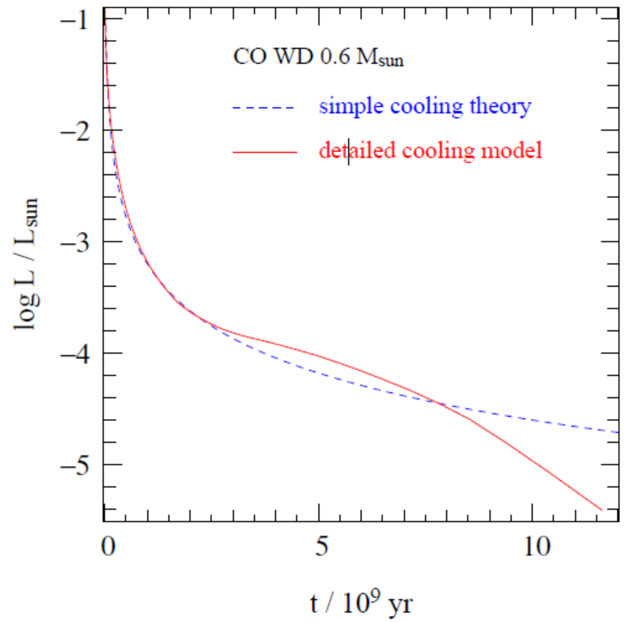




He and CO WDs also usually appear to be **pure** hydrogen. Only distinguishable by mass.

Cooling of white dwarfs

- In the interior of a white dwarf, the degenerate electrons provide a high thermal conductivity. This leads to a very small temperature gradient, especially because L is also very low. The degenerate interior can thus be considered to have a constant temperature.
- However, the outermost layers have much lower density and are non-degenerate, and here energy transport is provided by radiation. Due to the high opacity in these layers, radiation transport is much less effective than electron conduction in the interior. The non-degenerate outer layers thus act to insulate the interior from outer space, and here a substantial temperature gradient is present.
- It can be shown that more massive WDs evolve more slowly, because more ionic thermal energy is stored in their interior.
- Simple cooling law, shown in Figure for a $0.6\ M_\odot$ CO WD, predicts cooling times greater than 1Gyr when $L < 10^{-3}\ L_\odot$, and greater than the age of the Universe when $L < 10^{-5}\ L_\odot$.
- A more detailed WD cooling model show that white dwarfs that have cooled for most of the age of the Universe cannot have reached luminosities much less than $10^{-5}\ L_\odot$ and should still be detectable.
- Observed WD luminosities thus provide a way to derive the age of a stellar population.



Exceeding the Chandrasekhar mass

Suppose we add mass to a white dwarf, for example in a mass transfer binary system, to bring it up to the Chandrasekhar limit. What happens?

Possibility 1:

- A further increase of the mass leads to the loss of stability – the pressure of degenerate electrons can no longer hold the star up – and collapse. If this accretion-induced collapse occurs, the end state would be a neutron star (see below).
- The collapse would produce very little in the way of observable phenomenon.

Possibility 2:

- As M approaches M_{Ch}, the temperature and density in the core ignite fresh nuclear reactions. Unlike in the case of ordinary stellar nuclear reactions, this is devastating to the star.

Recall: $$P_e = K_{NR}\left(\frac{\rho}{\mu_e}\right)^{5/3}$$ ...no temperature dependence

- Hence, large energy release from nuclear reactions heats the material up without changing the pressure or density. This is the cause of supernovae type Ia explosions, production of $\sim 1 M_{\odot}$ of radioactive nickel.

Neutron stars (1)

- When a star's total mass is above around $8 - 10\ M_\odot$, the mass around the core will be larger than the Chandrasekhar limit. In this case after helium fusion ends and the core temperature is not hot enough for carbon fusion to occur, the gravitational pressure will overwhelm the electron degeneracy pressure and the core will collapse past the white dwarf stage.
- The further compression heats up the core, allowing the remaining stages of fusion to occur, making elements up to iron. After these stages finish, the star collapses again.
- As the core collapses its temperature becomes so high, $T > 10^{10}$ K, that the photons are energetic enough to break up heavy nuclei into lighter ones their constituent protons and neutrons (photodisintegration):

$$^{56}\mathrm{Fe} \rightarrow 13\ ^{4}\mathrm{He} + 4\ \mathrm{n}$$

Since this is an endothermic reaction that costs energy, rather than produces it, the core quickly cools and the collapse accelerates.

- the protons smash into the electrons producing neutrons and neutrinos:

$$p + e^{-} \rightarrow n + \nu$$

This reaction is energetically favorable because the neutrinos stream out of the star relieving some of the pressure. Ultimately, only neutrons are left.

Neutron stars (2)

Neutrons are fermions, and the core of neutrons in a neutron star is a Fermi gas, like a WD.

- Once the core reaches nuclear densities ($\rho \sim 10^{15}$ g cm^{-3}), degeneracy pressure of neutrons provides the pressure support against gravity.
- Formation of a proto-neutron star stops the collapse and produces a bounce which sends a shock wave back out into the star.

Shock wave can explode the star (a supernova explosion), if it can propagate out through the infalling matter.

- Core may leave a neutron star, or if it is too massive, collapse further to form a black hole.

We can again use our polytropic equations to estimate parameters of a WD, but matters are complicated because nuclear forces are important. Exact EOS is still unknown.

- Masses of NSs are between 1.2 and 2.5 $M_\odot$ and radius is ~10 km.
- The maximum mass of a neutron star depends on the existence of a general-relativistic instability (interactions between nucleons), quite commonly accepted to be ~3 $M_\odot$.
- Most neutron star observations are in a very narrow range of masses $M_{NS} = 1.35 \pm 0.04\ M_\odot$.

Neutron stars (3)

Newly formed NSs are very hot and embedded within young supernova remnants. Rapidly cool due to neutrino emission (in the very early stages), then photon emission.

Neutron stars cool down faster than WDs.

Galactic population of NSs must be enormous - a very small number have been detected directly from this cooling radiation.

Idea that neutron stars might be formed in supernova explosions was suggested by Walter Baade & Fritz Zwicky in 1934.

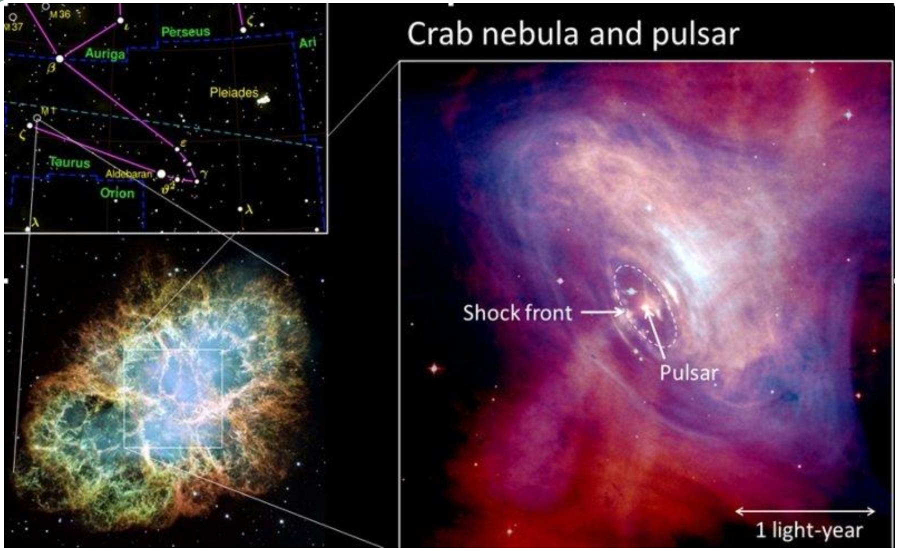
Detailed models for neutron stars were derived by Robert Oppenheimer and George Volkoff in 1939.

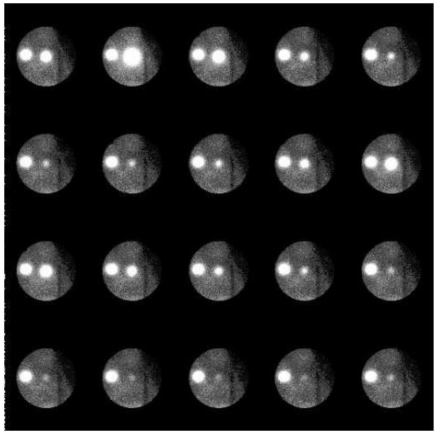
Obvious that isolated neutron stars would be very faint and hard to detect.

Neutron stars in binaries are extremely luminous, but only in X-rays which at the time were undetectable.

Surprisingly, first discovered by accident via pulsed radio emission - pulsars, by Jocelyn Bell in 1967.

Large number (~ thousand) of radio pulsars are now known, some of which also pulse in X-rays and even in optical





Golden et al., 2000, A&A, 363, 617

Stellar mass black holes

Maximum mass of a neutron star is unknown but is probably $\lesssim 3\ M_\odot$.

No known source of pressure can support a stellar remnant with a higher mass - collapse to a black hole appears to be inevitable.

Strong observational evidence for black holes in two mass ranges:

Stellar mass black holes: $M_{BH} = 5 - 100\ M_\odot$.

- produced from the collapse of very massive stars ($\gtrsim 25\ M_\odot$).
- lower mass examples could be produced from the merger of two neutron stars.

Supermassive black holes: $M_{BH} = 10^6 - 10^9\ M_\odot$.

- present in the nuclei of most galaxies
- formation mechanism unknown

Other types of black hole could exist too (intermediate mass black holes: $M_{BH} \sim 10^3\ M_\odot$).

Basic properties of black holes (1)

Black holes are solutions to Einstein's equations of General Relativity.
Numerous theorems have been proved about them, including, most importantly:

The 'No-hair' theorem:

A stationary black hole is uniquely characterized by its:

- Mass M
- Angular momentum J
- Charge Q

} Conserved quantities

Remarkable result: Black holes completely 'forget' how they were made – from stellar collapse, merger of two existing black holes, etc…

Astrophysical black holes are highly unlikely to have any significant charge.

Basic properties of black holes (2)

Astrophysical black holes are highly unlikely to have any significant charge, so $Q=0$.

Then there are 2 interesting cases:

$J=0$: Schwarzschild black hole

Spherically symmetric. Most important property – existence of an event horizon at a radius:

$$R_s = \frac{2GM}{c^2} \qquad \text{(Schwarzschild radius)}$$

No matter, radiation, or information can propagate outwards through this radius.

For $M_{BH} = 7\ M_\odot$: $R_s = 20$ km - very similar to the size of a neutron star. Unless we can measure the mass, hard to observationally distinguish between stellar mass black holes and neutron stars…

Basic properties of black holes (3)

Astrophysical black holes are highly unlikely to have any significant charge, so $Q=0$.
Then there are 2 interesting cases:

J = arbitrary: Kerr black hole

Axisymmetric solution – black hole has a preferred rotation axis.

Let's define the amount of angular momentum via a dimensionless spin parameter:

$$a = \frac{cJ}{GM^2}$$

Maximum angular momentum of a Kerr black hole corresponds to a spin parameter a = 1. A Kerr black hole cannot spin up beyond this limit.

For comparison, $J_\odot = 1.6 \times 10^{48}$ g cm^2 s^{-1}, $M = 1.99 \times 10^{33}$ g, so for the Sun a = 0.185.

Event horizon → $\frac{GM}{c^2}$ as a tends to maximal value.

Black holes

Stellar mass black holes in mass transfer binaries are the best studied systems. Show many very complex phenomena.

Best hope for studying fundamental physics of black holes is probably to observe gravitational radiation as they either form, or as pre-existing black holes in a binary merge.

Axisymmetric systems do not emit gravitational radiation (e.g. pulsars, or Kerr black holes) at all, so need to look for:

- inspiral or merging of black hole binaries
- merger of neutron stars (binary pulsar will merge in a few hundred million years, so we know for certain such events occur)
- signals from black hole formation in core collapse supernovae - **if** the collapse is not axisymmetric…

Supernovae and Supernova remnants

Explosion ejects outer layers of the star with $v \sim 10^4$ km s^{-1}.

This gas expands and collides with the circumstellar medium producing a supernova remnant.

Supernovae are classified according to the spectral lines seen in their spectra:

- Type I: no lines of hydrogen in the spectrum
- Type II: lines of hydrogen seen in spectrum

Overwhelming observational evidence that Type II supernovae are associated with the endpoints of massive stars. Thought to represent core collapse of massive stars with
$M > 8\text{–}10\ M_\odot$.

Type I is further divided into subclasses (Ia, Ib and Ic) again based on their spectral properties:

- Type Ia supernovae are believed to result from the explosion of Chandrasekar mass white dwarfs.
- Type Ib and Type Ic are thought to result from the collapse of massive stars that have lost their outer hydrogen envelopes prior to the explosion.

Note: classification predates any physical understanding, and so is potentially confusing!

Summary of Star Lifecycles

- The formation, evolution and death of stars is a cyclical process.
- Starts off with big cloud of gas.
- Cloud collapses under gravity until it becomes hot enough to burn and shine.
- When the fuel runs out the star dies.
- Massive stars end in supernova explosions which returns material to the interstellar medium.
- This is recycled into new stars!

Stellar Mass