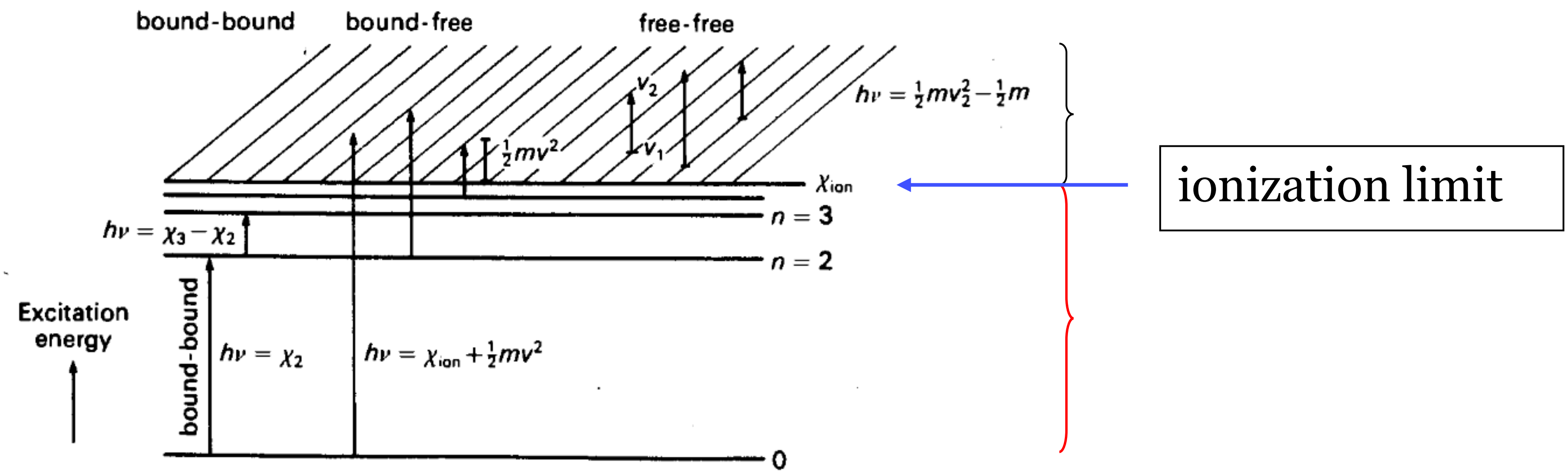
The degree of ionization of an atom



The degree of ionization of any atom or ion can be obtained from the Saha equation, which can be derived from the Boltzmann formula if we extend it to states with positive energies, i.e., to free electrons with the appropriate statistical weights (the upper state is now an ion plus free electron, with energy $\chi_{ion}+1/2m_e v^2$).

The statistical weight of the ion in the ground state plus electron is the product of the statistical weight of the ion g_1^+ and the statistical weight of the electron g_e: $g_{ion+e} = g_1^+ g_e$

The Saha Equation

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The (differential) statistical weight of the electron, g_e, i.e. the number of available states in interval ($v, v+dv$) is (from quantum mechanics)

$$g_e = \frac{1}{N_e} \frac{8\pi m_e^3 v^2 dv}{h^3}$$

The $1/N_e$ factor comes from the space volume element. It is the volume per electron.

Inserting this into Boltzmann's equation, we arrive at **the Saha equation:**

$$\frac{N_1^+}{N_1} = \frac{2g_1^+}{N_e g_1} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{ion}/kT}$$

This relates the **ground** state populations of the atom and ion.

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To derive the ratio of the total number of ions (N^+) to the total number of atoms (N^0) we can use the conventional Boltzmann formula for each level n of the atom and ion, N_n/N_1 and $N_n{}^+/N_1{}^+$ i.e.:

$$\frac{N_n}{N_1} = \frac{g_n}{g_1} e^{-\chi_n/kT} \qquad \frac{N_n^+}{N_1^+} = \frac{g_n^+}{g_1^+} e^{-\chi_n^+/kT}$$

Partition function (1)

If N^0 is the sum of *all neutral* particles in their different quantum states:

$$N^0 = N_1^0 + \sum_{n=2}^{\infty} N_n^0 = N_1^0 + \frac{N_1^0}{g_1} \sum_{n=2}^{\infty} g_n\, e^{-\chi_n/kT}$$

We find:

$$N^0 = \frac{N_1^0}{g_1}\left(g_1 + \sum_{n=2}^{\infty} g_n\, e^{-\chi_n/kT}\right) = \frac{N_1^0}{g_1} u^0(T)$$

where we have introduced $\boldsymbol{u^0}$, the partition function of the atom.
This is the weighted sum of the number of ways it can arrange its electrons with the same energy, and can be used to calculate the probability that at the given temperature, the atom is on the given energy level.

Similarly for the ion,

$$N^+ = N_1^0 + \frac{N_1^+}{g_1^+} u^+(T) \qquad u^+(T) = g_1^+ + \sum_{n=2}^{\infty} g_n^+\, e^{-\chi_n^+/kT}$$

For H^+, $u^+=1$, since no electrons left.

Partition function (2)

If we multiply N_1^+/N_1^0 from earlier by N^+/N_1^+ and N_1^0/N^0 we again obtain the **Saha equation**:

$$\frac{N^+ N_e}{N^0} = \frac{2u^+}{u^0}\frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{ion}/kT} = 4.83 \times 10^{15}\frac{u^+}{u^0} T^{3/2} e^{-\chi_{ion}/kT}$$

In logarithmic form Saha equation can be written as:

$$\log\frac{N^+}{N^0} = \log\frac{u^+}{u^0} + \log 2 + \frac{5}{2}\log T - \chi_{ion}\Theta - \log P_e - 0.48$$

where χ_{ion} is measured in eV, $\Theta=5040/T$ and the electron pressure P_e is related to the electron density via the ideal gas law ($P_e=N_e kT$). In stellar atmospheres, P_e lies in the range 1 dyn/cm^2 (cool stars) to 1000 dyn/cm^2 (hot stars).

High temperature favours ionization, high pressure favours recombination.

Note that 1dyn/cm^2=0.1N/m^2 (SI units), so for SI calculations the final constant is -1.48 instead of -0.48

Partition functions (Gray App D2)

Table D.2. *Partition functions, log u(T).*

$\Theta = 5040/T$

	θ										
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	log g_0
H	0.368	0.303	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301
He	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
He^+	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301
Li	–	0.987	0.488	0.359	0.320	0.308	0.304	0.302	0.302	0.302	0.301
Be	–	0.328	0.087	0.025	0.007	0.002	0.001	0.000	0.000	0.000	0.000
Be^+	0.541	0.334	0.307	0.302	0.301	0.301	0.301	0.301	0.301	0.301	0.301
B	1.191	0.831	0.786	0.778	0.777	0.777	0.777	0.777	0.777	0.776	0.778
B^+	0.435	0.051	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
C	1.163	1.037	0.994	0.975	0.964	0.958	0.954	0.951	0.950	0.948	0.954
C^+	0.853	0.782	0.775	0.774	0.773	0.772	0.771	0.770	0.769	0.767	0.778
C^{++}	0.143	0.010	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
N	1.060	0.729	0.645	0.616	0.606	0.603	0.602	0.602	0.602	0.602	0.602
N^+	1.073	0.993	0.965	0.953	0.946	0.942	0.939	0.937	0.934	0.932	0.954
O	1.095	0.991	0.964	0.953	0.947	0.944	0.941	0.939	0.937	0.935	0.954
O^+	0.895	0.655	0.614	0.604	0.602	0.602	0.602	0.602	0.602	0.602	0.602
F	0.788	0.772	0.768	0.765	0.762	0.759	0.756	0.753	0.750	0.747	0.778
F^+	1.034	0.968	0.949	0.940	0.935	0.930	0.926	0.923	0.919	0.915	0.954
Ne	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Ne^+	0.771	0.766	0.760	0.754	0.748	0.743	0.737	0.732	0.727	0.723	0.778
Na	4.316	1.043	0.493	0.357	0.320	0.309	0.307	0.306	0.306	0.306	0.301
Na^+	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Mg	2.839	0.478	0.110	0.027	0.007	0.002	0.001	0.001	0.001	0.000	0.000

Partition functions (Gray, old edition)

$$\log u\,(T) = c_0 + c_1 \log \Theta + c_2 \log^2 \Theta + c_3 \log^3 \Theta + c_4 \log^4 \Theta$$

$$\Theta = 5040/T$$

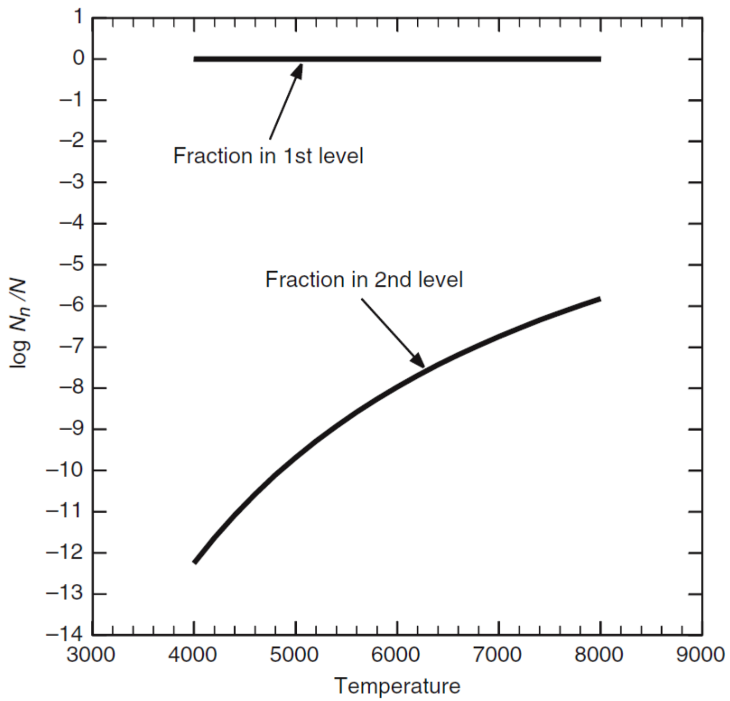
Element		c_0	c_1	c_2	c_3	c_4
H	1	0.30103	−0.00001			
He	2	0.00000	0.00000			
He⁺		0.30103	0.00000			
Li	3	0.31804	−0.20616	0.91456	−1.66121	1.04195
Be	4	0.00801	−0.17135	0.62921	−0.58945	
Be⁺		0.30389	−0.00819			
B	5	0.78028	−0.01622			
B⁺		0.00349	−0.01035			
C	6	0.96752	−0.09452	0.08055		
C⁺		0.77239	−0.02540			
N	7	0.60683	−0.08674	0.30565	−0.28114	
N⁺		0.94968	−0.06463	−0.01291		
O	8	0.05033	−0.05703			
O⁺		0.60405	−0.03025	0.04525		
F	9	0.76284	−0.03582	−0.05619		
Ne	10	0.00000	0.00000			
Ne⁺		0.74847	−0.06562	−0.07088		
Na	11	0.30955	−0.17778	1.10594	−2.42847	1.70721
Mg	12	0.00556	−0.12840	0.81506	−1.79635	1.26292
Mg⁺		0.30257	−0.00451			
Al	13	0.76786	−0.05207	0.14713	−0.21376	
Al⁺		0.00334	−0.00995			
Si	14	0.97896	−0.19208	0.04753		
Si⁺		0.75647	−0.05490	−0.10126		
P	15	0.64618	−0.31132	0.68633	−0.47505	
P⁺		0.93588	−0.18848	0.08921	−0.22447	
S	16	0.95254	−0.15166	0.02340		
S⁺		0.61971	−0.17465	0.48283	−0.39157	
Cl	17	0.74465	−0.07389	−0.06965		
Cl⁺		0.92728	−0.15913	−0.01983		
K	19	0.34419	−0.48157	1.92563	−3.17826	1.83211
Ca	20	0.07460	−0.75759	2.58494	−3.53170	1.65240
Ca⁺		0.34383	−0.41472	1.01550	0.31930	
Sc	21	1.08209	−0.77814	1.78504	−1.39179	
Sc⁺		1.35894	−0.51812	0.15634		
Ti	22	1.47343	−0.97220	1.47986	−0.93275	
Ti⁺		1.74561	−0.51230	0.27621		
V	23	1.68359	−0.82055	0.92361	−0.78342	
V⁺		1.64112	−0.74045	0.49148		
Cr	24	1.02332	−1.02540	2.02181	−1.32723	
Cr⁺		0.85381	−0.71166	2.18621	−0.97590	−2.72893
Mn	25	0.80810	−0.39108	1.74756	−3.13517	1.93514
Mn⁺		0.88861	−0.36398	1.39674	−1.86424	−2.32389
Fe	26	1.44701	−0.67040	1.01267	−0.81428	
Fe⁺		1.63506	−0.47118	0.57918	−0.12293	
Co	27	1.52929	−0.71430	0.37210	−0.23278	
Ni	28	1.49063	−0.33662	0.08553	−0.19277	
Ni⁺		1.03800	−0.69572	0.53893	0.28861	

Ionization Potentials

Atom		Stage of ionization I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
1	H	13.598 44													
2	He	24.587 41	54.417 78												
3	Li	5.391 72	75.640 18	122.454											
4	Be	9.322 63	18.211 16	153.897	217.713										
5	B	8.298 03	25.154 84	37.931	259.366	340.22									
6	C	11.260 30	24.383 32	47.888	64.492	392.08	489.98								
7	N	14.534 14	29.601 3	47.449	77.472	97.89	552.06	667.03							
8	O	13.618 06	35.117 30	54.936	77.413	113.90	138.12	739.29	871.41						
9	F	17.422 82	34.970 82	62.708	87.140	114.24	157.17	185.19	953.91	1 103.1					
10	Ne	21.564 54	40.963 28	63.45	97.12	126.21	157.93	207.28	239.10	1 195.8	1 362.2				
11	Na	5.139 08	47.286 4	71.620	98.91	138.40	172.18	208.50	264.25	299.9	1 465.1	1 648.7			
12	Mg	7.646 24	15.035 28	80.144	109.265	141.27	186.76	225.02	265.96	328.1	367.5	1 761.8	1 963		
13	Al	5.985 77	18.828 56	28.448	119.99	153.83	190.49	241.76	284.66	330.1	398.8	442.0	2 086	2 304	
14	Si	8.151 69	16.345 85	33.493	45.142	166.77	205.27	246.49	303.54	351.1	401.4	476.4	523	2 438	2 673
15	P	10.486 69	19.769 4	30.203	51.444	65.03	220.42	263.57	309.60	372.1	424.4	479.5	561	612	2 817
16	S	10.360 01	23.337 9	34.79	47.222	72.59	88.05	280.95	328.75	379.6	447.5	504.8	564	652	707
17	Cl	12.967 64	23.814	39.61	53.465	67.8	97.03	114.20	348.28	400.1	455.6	529.3	592	657	750
18	Ar	15.759 62	27.629 67	40.74	59.81	75.02	91.01	124.32	143.46	422.5	478.7	539.0	618	686	756
19	K	4.340 66	31.63	45.806	60.91	82.66	99.4	117.56	154.88	175.8	503.8	564.7	629	715	787
20	Ca	6.113 16	11.871 72	50.913	67.27	84.50	108.78	127.2	147.24	188.5	211.3	591.9	657	727	818
21	Sc	6.561 44	12.799 67	24.757	73.489	91.65	111.68	138.0	158.1	180.0	225.2	249.8	688	757	831
22	Ti	6.828 2	13.575 5	27.492	43.267	99.30	119.53	140.8	170.4	192.1	215.9	265.1	292	788	863
23	V	6.746 3	14.66	29.311	46.71	65.28	128.1	150.6	173.4	205.8	230.5	255.1	308	336	896
24	Cr	6.766 64	16.485 7	30.96	49.16	69.46	90.64	161.18	184.7	209.3	244.4	270.7	298	355	384
25	Mn	7.434 02	15.639 99	33.668	51.2	72.4	95.6	119.20	194.5	221.8	248.3	286.0	314	344	404
26	Fe	7.902 4	16.187 8	30.652	54.8	75.0	99.1	124.98	151.06	233.6	262.1	290.2	331	361	392
27	Co	7.881 0	17.083	33.50	51.3	79.5	103	131	160	186.2	276.2	305	336	379	411
28	Ni	7.639 8	18.168 84	35.19	54.9	75.5	108	134	164	193	224.6	321	352	384	430
29	Cu	7.726 38	20.292 40	36.841	55.2	79.9	103	139	167	199	232	266	369	401	435
30	Zn	9.394 05	17.964 40	39.723	59.4	82.6	108	136	175	203	238	274	311	412	454

Degree of ionization of H in stars

We can use the Saha equation to study the degree of ionization of H in general in stellar photospheres. The fraction of ionized hydrogen to the total is defined below. We find that H switches from mostly neutral below 7000K to mostly ionized above 11000K for typical N_e. This allows us to understand why hydrogen lines are strongest in A-type stars, with temperatures of 7500-10000K.



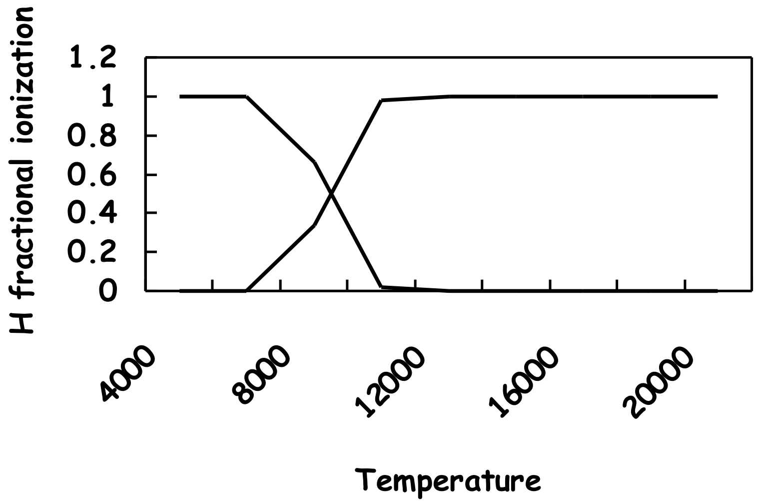
$$\frac{H^+}{H} = \frac{H^+}{H^0 + H^+} = \frac{H^+/H^0}{1 + H^+/H^0}$$

$$\frac{N^+ N_e}{N^0} = 2.4 \times 10^{15}\, T^{3/2} e^{-158000/T}$$

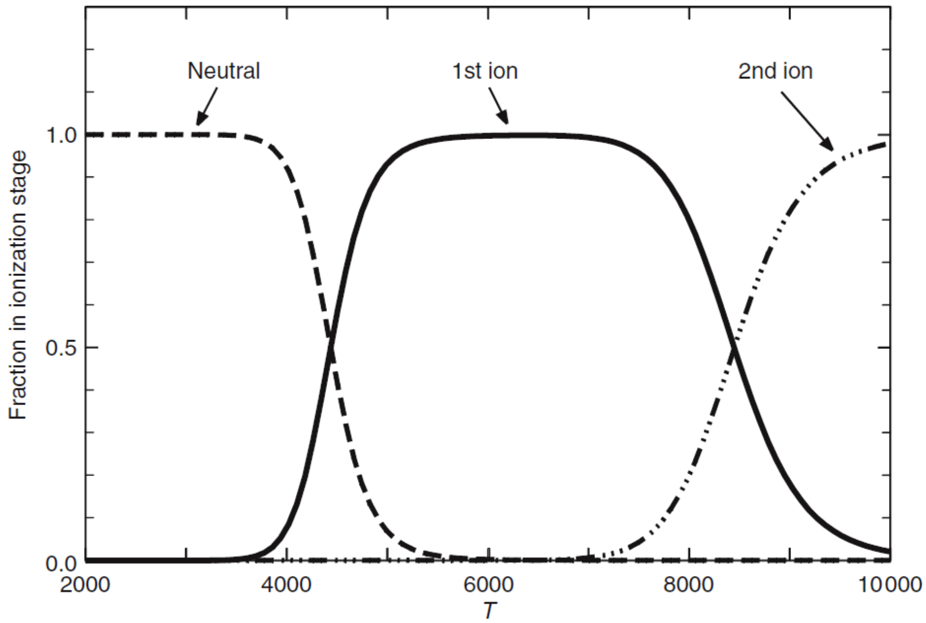
Using 1eV per particle, the hydrogen is heated from 0 to 10^4 K. Supplying 13.6 eV more, the temperature increases only up to 2×10^4 K. Ionization is an extremely energy consuming process. Ionization happens within a very small temperature interval.

Degree of ionization in stars

Hydrogen:



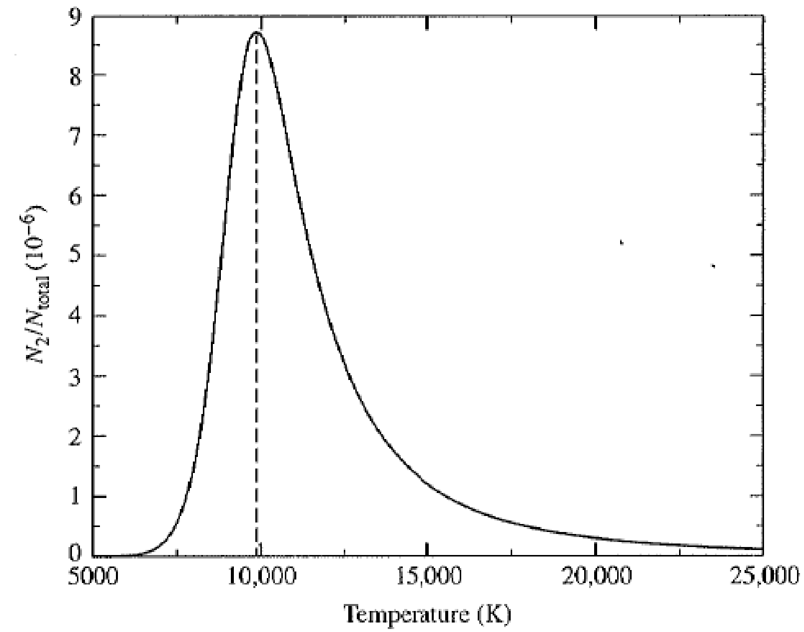
Iron:



As temperature increases, ionization occurs rather abruptly. In a stellar photosphere, elements exist mainly in just two ionization stages.

Strong Balmer lines in A stars – why?

From recent example, a very high *T* was required to populate level *n*=2 of H relative to the ground state. We can now use the Boltzmann & Saha equations to measure H(*n*=2)/H(total) as a function of *T*. For increasing *T*, the *n*=2 population increases due to the Boltzmann equation, reaching a maximum value around 10,000K (equivalent to A spectral type) and then reduces as H becomes mostly ionized. This is why A stars have strong Balmer lines.



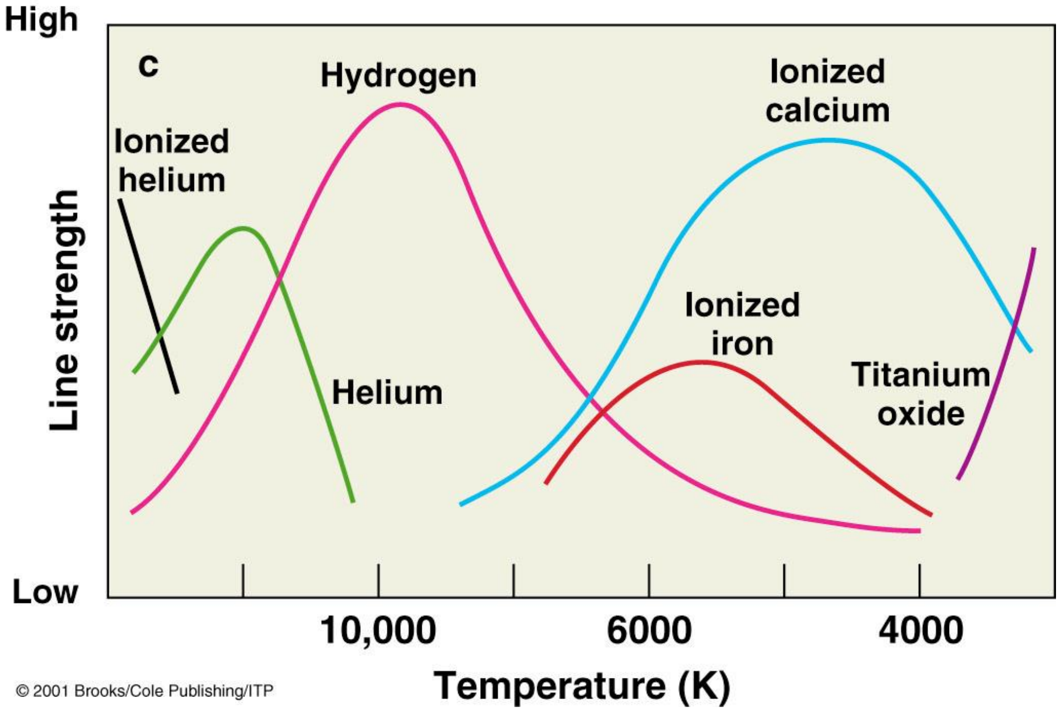
Note: He in stellar atmospheres complicates this calculation since ionized He provides excess electrons with which H ions can recombine, so it takes higher temperatures to achieve the same degree of ionization.

Strong lines in Solar photosphere

λ	Element	W(Å)	Name	λ	Element	W(Å)	Name
3581.21	Fe I	2.14	N	4920.51	Fe I	0.43	
3719.95	Fe I	1.66		4957.61	Fe I	0.45	
3734.87	Fe I	3.03	M	5167.33	Mg I	0.65	b_4
3749.50	Fe I	1.91		5172.70	Mg I	1.26	b_2
3758.24	Fe I	1.65		5183.62	Mg I	1.58	b_1
3770.63	H_{11}	1.86		5232.95	Fe I	0.35	
3797.90	H_{10}	3.46		5269.55	Fe I	0.41	
3820.44	Fe I	1.71	L	5324.19	Fe I	0.32	
3825.89	Fe I	1.52		5238.05	Fe I	0.38	
3832.31	Mg I	1.68		5528.42	Mg I	0.29	
3835.39	H_9	2.36		5889.97	Na I	0.63	D_2
3838.30	Mg I	1.92		5895.94	Na I	0.56	D_1
3859.92	Fe I	1.55		6122.23	Ca I	0.22	
3889.05	H_8	2.35		6162.18	Ca I	0.22	
3933.68	Ca II	20.25	K	6562.81	H_α	4.02	C
3968.49	Ca II	15.47	H	6867.19	O_2	tell	B
4045.82	Fe I	1.17		7593.70	O_2	tell	A
4101.75	H_δ	3.13	h	8194.84	Na I	0.30	
4226.74	Ca I	1.48	g	8498.06	Ca II	1.46	
4310±10	—	—	G	8542.14	Ca II	3.67	
4340.48	H_γ	2.86		8662.17	Ca II	2.60	
4383.56	Fe I	1.01		8688.64	Fe I	0.27	
4861.34	H_β	3.68		8736.04	Mg I	0.29	
4891.50	Fe I	0.31					

Ca II in the Sun

The photosphere of the Sun has only two calcium atoms for every million H atoms, yet the Ca II **H and K lines** (produced by the ground state of singly ionized calcium, Ca^+) are ***stronger*** than the Balmer lines of H (produced by the 1st excited state of neutral H). Why?



Saha-Boltzmann applied to Ca

From the Saha equation we can find that H is essentially neutral in the Solar photosphere:

P_e=200 dyn/cm^2, χ_{ion}=13.6 eV, Θ=5040/($T_{=5777}$)=0.872, the partition function u^0=2, u^+=1 (i.e. log u^+=0)

$$\log \frac{N^+}{N^0} = \log u^+ - \log u^0 + \log 2 + \frac{5}{2}\log T - \chi_{ion}\Theta - \log P_e - 0.48 = -5.235 \rightarrow \ N^+/N^0 \approx 0.0006\%$$

yet from the Boltzmann formula $\log \frac{N_u}{N_l} = \log \frac{g_u}{g_l} - \frac{5040}{T}\chi_{ul}(eV)$: H(n=2)/H(n=1)=5x10^{-9}

i.e. **very little** H is available to produce Balmer absorption lines.

For Ca, χ_{ion}=6.1 eV, and partition functions may be determined from tables (Slide 103) via

$$\log u\,(T) = c_0 + c_1 \log \Theta + c_2 \log^2 \Theta + c_3 \log^3 \Theta + c_4 \log^4 \Theta$$

For Θ=5040/T=0.872, the partition function of neutral Ca

$$\log u^0\,(T) = 0.075 - 0.757 \log \Theta + 2.58 \log^2 \Theta + 3.53 \log^3 \Theta - 1.65 \log^4 \Theta$$

i.e. u^0=1.3. Similarly, u^+=2.3.

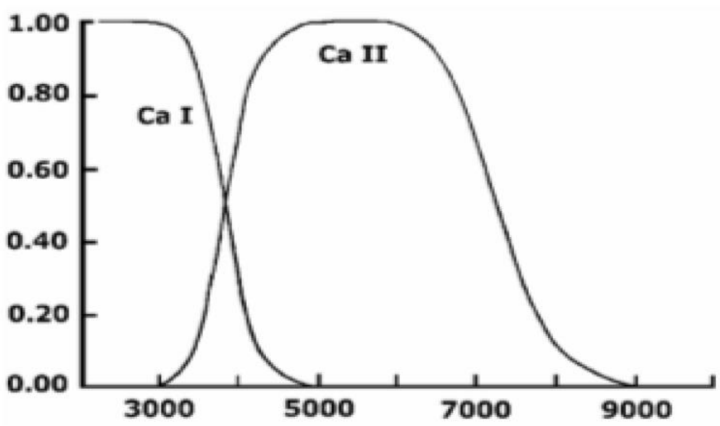
$$\log \frac{\text{Ca}^+}{\text{Ca}^0} = \log \frac{2.3}{1.3} + \log 2 + 9.40 - 5.34 - 1.18 - 0.48 = +2.95 \rightarrow \ \text{Ca}^+/\text{Ca}^0 \approx 900$$

Essentially ***all*** Calcium is singly ionized.

Saha-Boltzmann applied to Ca

Essentially ***all*** Calcium is singly ionized.

$N(Ca^{+})$ in the first excited state relative to the ground state (g_1=2, g_2=4, χ=3.12eV) is 1/265 from Boltzmann eqn, so nearly all Calcium in the Sun's photosphere is in the ground state of Ca^{+}.



Combining these results:

$N(Ca^{+}_{g.s.})/N(H_{n=2}) = N(Ca^{+}_{g.s.})/N(Ca) \times N(Ca)/N(H) \times N(H)/N(H_{n=2}) = 400$

There are 400 times more Ca^{+} ions with electrons in the ground-state (which produce the Ca II H&K lines) than there are neutral H atoms in the first excited state (which produce Balmer lines).

The Ca II lines in the Sun are so strong due to *T* dependence of excitation and ionization (**not** high Ca/H abundance).

More from Saha

- Another observational effect that can be understood using the Saha equation is that supergiants and giants have *lower temperatures* than dwarfs of the same spectral type.
- Spectral classes are defined by line ratios of different ions, e.g. He II 4542A / He I 4471 for O stars. At higher temperatures the fraction of He II will increase relative to He I, so the above ratio will increase.
- However, supergiants have lower surface gravities (or pressure) than main-sequence stars, so from Saha equation a lower P_e at the same temperature will give a higher ion fraction, N^+/N^0
- Assuming a given spectral class corresponds to a fixed ratio N^+/N^0, a star with a lower pressure can have a lower $T_{\rm eff}$ for the same ratio and spectral class

Summary

- LTE = Maxwell + Boltzmann + Saha.
- Boltzmann equation describes degree of excitation of an atom or ion, e.g. $N\,(\mathrm{H}_{n=2})/N\,(\mathrm{H}_{n=1})$.
- Saha equation describes degree of ionization of successive ions, e.g. $N\,(\mathrm{He}^{+})/N(\mathrm{He}^{0})$ or $N\,(\mathrm{He}^{2+})/N\,(\mathrm{He}^{+})$.
- The Partition function is the weighted sum of the number of ways an atom or ion can arrange its electrons with the same energy.
- Ionization is an extremely energy consuming process. Ionization happens within a very small temperature interval.
- Saha-Boltzmann explains the spectral type (or temperature) dependence of lines in stellar atmospheres, e.g. Strongest Balmer series at spectral type A and strong CaII lines in Solar-type stars.

Boltzmann equation & Saha Equation

- Bolzmann equation:

$$\frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-(E_u - E_l)/kT}$$

Boltzmann constant
k=8.6174x10^{-5} **eV/K**

$$\log \frac{N_u}{N_l} = \log \frac{g_u}{g_l} - \frac{5040}{T} \chi_{ul}(eV)$$

- Saha Equation

$$\frac{N_1^+}{N_1} = \frac{2g_1^+}{N_e g_1} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{ion}/kT}$$

Θ=5040/T

$$\log \frac{N^+}{N^0} = \log \frac{u^+}{u^0} + \log 2 + \frac{5}{2} \log T - \chi_{ion}\Theta - \log P_e - 0.48$$

Stellar Opacity

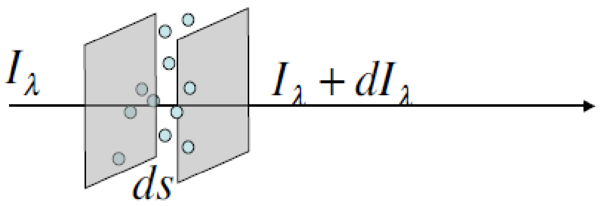
ROSSELAND MEAN OPACITY
ROSSELAND DEPTH
BOUND-BOUND (LINE) ABSORPTION
BOUND-FREE AND FREE-FREE (CONTINUOUS) ABSORPTION

Opacity

- We first introduced the concept of opacity when deriving the equation of radiative transport.
- Opacity is the resistance of material to the flow of heat, which in most stellar interiors is determined by all the processes which scatter and absorb photons.
- The removal of energy from a beam of photons as it passes through matter is governed by
  - line absorption (**bound-bound**),
  - photoelectric absorption (**bound-free**),
  - inverse bremsstrahlung (**free-free**), and
  - photon scattering.
- Stimulated emission acts as negative opacity by creating photons that add to the beam.
- Stellar atmospheres are predominantly hydrogen (90% by number), whilst helium makes up almost all the rest. These two elements provide most of the opacity over most wavelengths for most (hot) stars.

Absorption coefficient

- The monochromatic absorption coefficient specifies the energy fraction taken from a light beam. It may be defined per particle, per gram, or in terms of a geometrical cross-section in cm^2:



- Per gram: $\boldsymbol{dI_\lambda \equiv -\kappa_\lambda \rho I_\lambda ds}$, where κ_λ is the mass absorption coefficient [$cm^2\ g^{-1}$], ρ is the density [$g\ cm^{-3}$].

- Per cm path length: $\boldsymbol{dI_\lambda \equiv -\alpha_\lambda I_\lambda ds}$, where $\boldsymbol{\alpha_\lambda}$ is the absorption coefficient [cm^{-1}]
$$\alpha_\lambda = \kappa_\lambda \rho$$

- Per particle: $\boldsymbol{dI_\lambda \equiv -\sigma_\lambda n\, I_\lambda ds}$, where $\boldsymbol{\sigma_\lambda}$ is the absorption cross-section per particle for individual transitions and $\boldsymbol{n}$ is the number density [particles cm^{-3}]

$$\boldsymbol{\alpha_\lambda = \sigma_\lambda n = \kappa_\lambda \rho}$$

$$\boldsymbol{d\tau_\lambda = \alpha_\lambda ds = \sigma_\lambda n\, ds = \kappa_\lambda \rho\, ds}$$

The mean absorption coefficient

- The grey approximation (α, $\boldsymbol{\kappa}$ =const) is very coarse but can still be useful. Is there a sensible mean value $\bar{\alpha}$ to use? What choice to make for a mean value?
- We demand flux conservation and hope to keep the temperature structure.
- **From the third radiative equilibrium condition:** $\int_0^\infty \frac{dK_\lambda}{d\tau_\lambda} d\lambda = \frac{F(\tau)}{4\pi}$

$$F = \int F_\lambda \, d\lambda = 4\pi \int \frac{dK_\lambda}{d\tau_\lambda} d\lambda = 4\pi \int \frac{dK_\lambda}{\alpha_\lambda ds} d\lambda = \frac{4\pi}{3} \int \frac{dB_\lambda}{\alpha_\lambda ds} d\lambda$$

$$K_\lambda(\tau_\lambda) = \frac{1}{3} J_\lambda(\tau_\lambda) = \frac{1}{3} B_\lambda$$

the Eddington approximation

$$F = \frac{4\pi}{3} \frac{1}{\alpha_R} \int \frac{dB_\lambda}{ds} d\lambda = \frac{4\pi}{3} \frac{1}{\alpha_R} \frac{dB}{ds}$$

$$\frac{1}{\alpha_R} = \frac{\int \frac{dB_\lambda}{\alpha_\lambda ds} d\lambda}{\frac{dB}{ds}}$$

Rosseland mean opacity

$F = \pi B$

$$\frac{1}{\alpha_R} = \frac{\int \frac{1}{\alpha_\lambda}\frac{dB_\lambda}{ds}d\lambda}{\frac{dB}{ds}}$$

$$\frac{dB}{ds} = \frac{dB}{dT}\frac{dT}{ds} \quad \text{and} \quad \frac{dB}{dT} = \frac{d}{dT}\left(\frac{\sigma}{\pi}T^4\right) = \frac{4\sigma}{\pi}T^3$$

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty \frac{1}{\alpha_\lambda}\frac{dB_\lambda}{dT}d\lambda}{\frac{4\sigma}{\pi}T^3}$$

Definition of Rosseland mean opacity

The Rosseland mean $1/\alpha_R$ is a weighted (harmonic) mean of opacity, for which there is a corresponding optical depth (Rosseland depth):

$$\tau_{Ross}(s) = \int_0^s \alpha_R(z)dz$$

We hoped for the temperature structure:

$$T^4(\tau) = \frac{3}{4}(\tau + \frac{2}{3})T_{eff}^4 = \frac{3}{4}(\tau_{Ross} + \frac{2}{3})T_{eff}^4$$

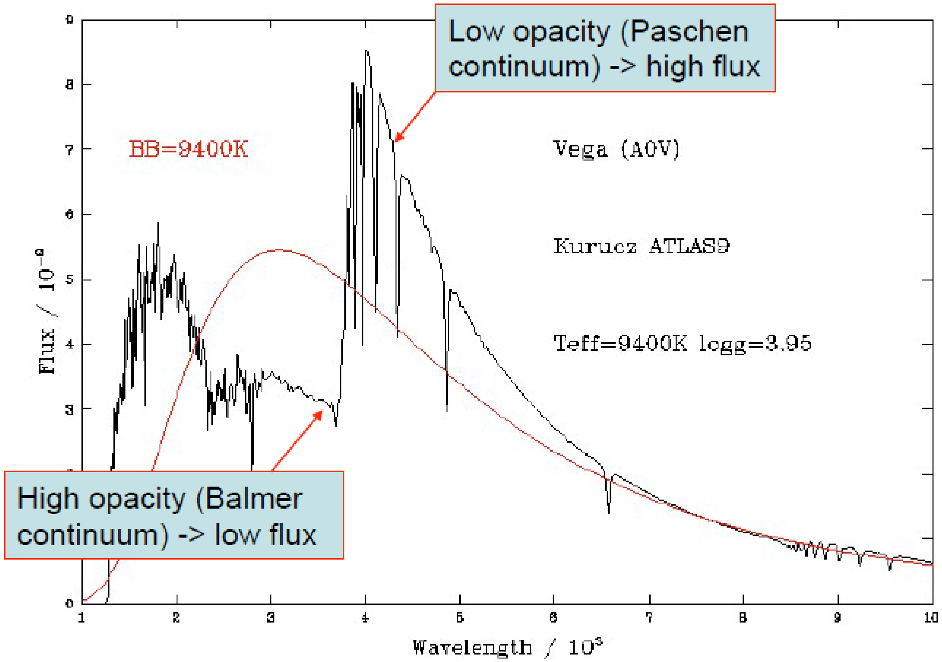
Eddington approximation

The grey approximation is very good for $\tau_{Ross} \gg 1$.

However, the atmosphere is NOT grey

- Opacity depends strongly on wavelengths →
  the atmosphere is **NOT** grey.

- Non-greyness changes the temperature structure.



Dominant sources of opacity

- The most important transitions for the continuous absorption are those which ionise atoms (with a continuum of final states).
- For H and He the line spectra do not greatly affect radiative transport. Some metals, with very complex line spectra do contribute to the continuum.
- New stellar opacities have been recalculated in the past 20-30 years by two groups – OPAL (Iglesias et al., 1996) and The Opacity Project/OP (Seaton et al., 1994; Badnell et al., 2005) which have led to a factor of 3 increase in opacity under some temperature-density conditions via improved treatment of atomic data.