Radiative Equilibrium

GREY ATMOSPHERE
THERMAL (RADIATIVE) EQUILIBRIUM
THE DEPTH DEPENDENCE OF THE SOURCE FUNCTION
EDDINGTON APPROXIMATION
TEMPERATURE STRUCTURE OF THE GREY ATMOSPHERE

Grey atmosphere

- Above we assumed that the opacity can be independent of λ, i.e. $\kappa_\lambda = \kappa$. We call such a (hypothetical) grey atmosphere.
- In the theory of stellar atmospheres, much of the technical effort goes into iteration schemes using equations of radiative equilibrium (which we will discuss today) to find the source function S_λ.
- Often, a starting point for such iterations is the **grey** case.

Thermal (radiative) equilibrium

- In stellar atmospheres, radiation dominates transfer of energy, so we can discuss (three) conditions of radiative equilibrium, which can be used to derive the temperature structure in the photosphere.

- The radiation we see from the Sun comes from a layer of geometrical height of a few hundred km.

- In a column of 100 km height and 1 cm^2 cross-section there are 10^{24} particles (since $n \sim 10^{17}/cm^3$ in Sun), each of which has a thermal energy of $3kT/2$ (10^{-12} erg). The total thermal energy of this column is therefore 10^{12} erg/cm^2. The observed radiative energy loss (per cm^2) of the solar surface is $F_{\odot}=6.3\times10^{10}$ erg cm^{-2} s^{-1}.

- If the Sun shines at a constant rate, the energy content of the solar photosphere can only last for 15 seconds without being replenished from below.

- Exactly the same amount of energy must be supplied or else the photosphere would quickly change temperature.

First equation of radiative equilibrium

- Since this does **not** happen, $dF/dt=0$ or $dF/dx=0$ or $dF/d\tau=0$, i.e. the total flux must be constant at all depths of the photosphere (**conservation of energy**) – the **1st equation of radiative equilibrium**

$$F(x) = F(0) = const = \sigma T_{eff}^4$$

- When all the energy is carried by radiation, we have

$$F(x) = \int_0^\infty F_\lambda(\tau_\lambda) d\lambda = F(0)$$

Although the shape of F_λ can be expected to change very significantly with depth, its integral remains invariant.

- If other sources of energy transport are significant, then a more general expression of flux constancy must be applied:

$$\Phi(x) + \int_0^\infty F_\lambda(\tau_\lambda) d\lambda = F(0)$$

$\Phi(x)$ is, for example, the convective flux

Radiative equilibrium

- We may integrate the plane-parallel transfer equation over solid angle ω.

$$\int \cos\theta \frac{dI_\lambda(\tau_\lambda,\theta)}{d\tau_\lambda} d\omega = \int I_\lambda(\tau_\lambda,\theta)\, d\omega - \int S_\lambda(\tau_\lambda)\, d\omega$$

$$\frac{d}{d\tau_\lambda}[F_\lambda(\tau_\lambda)] = 4\pi[J_\lambda(\tau_\lambda)] - \int S_\lambda(\tau_\lambda)\, d\omega$$

Based on the definition of mean intensity and flux:

$$J_\lambda = \frac{1}{4\pi}\oint I_\lambda d\omega \quad \text{and} \quad F_\lambda = \oint I_\lambda \cos\theta\, d\omega$$

- Finally, assuming S_λ to be isotropic we obtain,

$$\frac{1}{4\pi}\frac{d}{d\tau_\lambda}[F_\lambda(\tau_\lambda)] = J_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

Second equation of radiative equilibrium

- In the grey case, for which the opacity κ is independent of wavelength

$$\frac{1}{4\pi}\frac{d}{d\tau}F(\tau) = -S(\tau) + J(\tau) = 0$$

Since $dF/d\tau = 0$, the Source function must be equal the mean intensity J.

- If the atmosphere is **not** grey, which is the situation for most stars, let's incorporate the opacity κ into the RHS, and integrating over wavelength

$$\frac{1}{4\pi\rho}\frac{d}{ds}\left[\int_0^\infty F(\tau_\lambda)d\lambda\right] = \int_0^\infty (-\kappa_\lambda S_\lambda + \kappa_\lambda J_\lambda)d\lambda = 0$$

$$\tau_\lambda = \int_0^S \kappa_\lambda \rho ds$$

Since dF/ds =0, we get the **radiative balance equation (energy conservation)**

$$\int_0^\infty \kappa_\lambda S_\lambda d\lambda = \int_0^\infty \kappa_\lambda J_\lambda d\lambda$$

- This is **the second equation of radiative equilibrium** and can be understood as the total energy absorbed (RHS) must equal the total energy re-emitted (LHS) if no heating or cooling is taking place.

Third equation of radiative equilibrium

The third radiative equilibrium condition is obtained by multiplying the transfer equation by $\cos\theta$ and integrating over solid angle and then wavelength

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

$$\oint \cos^2\theta \frac{dI_\lambda(\tau_\lambda,\theta)}{d\tau_\lambda} d\omega = \oint \cos\theta\, I_\lambda(\tau_\lambda,\theta) d\omega - \oint \cos\theta\, S_\lambda(\tau_\lambda,\theta) d\omega$$

$$K_\lambda(\tau_\lambda) = \frac{1}{4\pi}\oint I_\lambda \cos^2\theta\, d\omega$$

$$F_\lambda = \oint I_\lambda \cos\theta\, d\omega$$

0 (S_λ is isotropic)

$$4\pi \int \frac{dK_\lambda}{d\tau_\lambda} d\lambda = \int F_\lambda\, d\lambda = F(\tau)$$

The third radiative equilibrium condition:

$$\int_0^\infty \frac{dK_\lambda}{d\tau_\lambda} d\lambda = \frac{F(\tau)}{4\pi}$$

Equations of radiative equilibrium

- All the three radiative equilibrium conditions are not independent. S_λ that is a solution of one will be the solution of all three.
- The flux constant $F(0)$ is often expressed in terms of an effective temperature $F(0) = \sigma T_{eff}^4$.
- When model photospheres are constructed using flux constancy as a condition to be fulfilled by the model, the effective temperature becomes one of the fundamental parameters characterizing the model.
- In real stars, energy is created or lost from the radiation field through e.g. convection, magnetic fields,
  plus in supernovae atmospheres energy conservation is not valid (radioactive decay of Ni to Fe),
  so the energy constraints are more complicated in reality.

Recap: Equations of radiative equilibrium

- **The 1st equation of radiative equilibrium:**

$$F(x) = F(0) = const = \sigma T_{eff}^4$$

i.e. the total flux must be constant at all depths of the photosphere (**conservation of energy**):
$dF/dt = 0$ or $dF/dx = 0$ or $dF/d\tau = 0$

- **The 2nd equation of radiative equilibrium:**
the total energy absorbed (RHS) must equal the total energy re-emitted (LHS) if no heating or cooling is taking place:

$$\int_0^\infty \kappa_\lambda S_\lambda d\lambda = \int_0^\infty \kappa_\lambda J_\lambda d\lambda$$

- **The 3rd radiative equilibrium condition:**

$$\int_0^\infty \frac{dK_\lambda}{d\tau_\lambda} d\lambda = \frac{F(\tau)}{4\pi}$$

- All the three radiative equilibrium conditions are not independent.
S_λ that is a solution of one will be the solution of all three.

The depth dependence of the source function

- In a **grey** atmosphere, with $K(\tau) = \int_0^\infty K_\lambda \, d\lambda$, the 3rd equation implies:

a new unknown function $K(\tau)$

$$\frac{dK(\tau)}{d\tau} = \frac{F(\tau)}{4\pi}$$

- We can differentiate this, and insert our earlier result:

$$\frac{d^2K(\tau)}{d\tau^2} = \frac{1}{4\pi}\frac{dF(\tau)}{d\tau} = J(\tau) - S(\tau) = 0 \qquad [1]$$

- Integration of the equation with respect to τ gives $K(\tau) = c_1\tau + c_2$ [2]

  where $dK/d\tau = c_1 = F/4\pi$

- For a given F, we now have two equations, [1] and [2], to determine the three unknowns: J, S and K (or c_2). We need an additional relation between two of these variables in order to determine all three.

Eddington approximation (1)

- Previously we have seen that for the determination of the flux the anisotropy in the radiation field is very important because in the flux integral the inward-going intensities are subtracted from the outward-going ones, due to the factor $\cos\theta$.
- But for K, a small anisotropy is unimportant because the intensities are multiplied by the factor $\cos^2\theta$, which does **not** change sign for inward and outward radiation.

- To evaluate K or c_2, we can approximate the radiation field by an isotropic radiation field of the mean intensity J: $I = J$ (by definition). From the definition of K_λ we obtain

$$4\pi\, K_\lambda = \oint I_\lambda(\tau_\lambda, \theta) \cos^2\theta \, d\omega = J_\lambda(\tau_\lambda) \oint \cos^2\theta \, d\omega = \frac{4\pi}{3} J_\lambda(\tau_\lambda)$$

or after division by 4π,

$d\omega = \sin\theta \, d\theta \, d\varphi$

$$K_\lambda(\tau_\lambda) = \frac{1}{3} J_\lambda(\tau_\lambda)$$

This approximation for the K-function is known as the **Eddington approximation**.

Eddington approximation (2)

- Inserting the Eddington approximation into the above equation we find

$$\frac{dK(\tau)}{d\tau} = \frac{F(\tau)}{4\pi}$$

$$\frac{dK(\tau)}{d\tau} = \frac{1}{3}\frac{dJ(\tau)}{d\tau} = \frac{F(\tau)}{4\pi} = c_1 \qquad \frac{dJ(\tau)}{d\tau} = \frac{3}{4\pi}F(\tau)$$

- Since the mean intensity J equals the source function S in a grey atmosphere, integrating the latter result we obtain

$$F(x) = F(0) = const$$

$$S(\tau) = \frac{3}{4\pi}\tau F(0) + C = J(\tau)$$

- From the conditions of radiative equilibrium, we finally obtained **the law for the depth dependence of the source function** (**for a grey atmosphere assuming the Eddington approximation**). We can evaluate C using boundary condition for the known emerging flux (there is no flux going into the star), plus we assume the outward intensity does not depend upon θ:

Eddington approximation (3)

- Boundary condition: there is no flux going into the star, i.e. $I(0,\theta) = I^- = 0$ for $\pi/2 < \theta < \pi$
- We also assume that the outward intensity does not depend upon θ, i.e. $I(0,\theta) = I^+ = \text{const}$ for $0 < \theta < \pi/2$

$$J_\lambda = \frac{1}{4\pi}\oint I_\lambda d\omega \quad \text{and} \quad F_\lambda = \oint I_\lambda \cos\theta \, d\omega$$

$$d\omega = \sin\theta \, d\theta \, d\varphi$$

- It gives $$J(0) = \frac{1}{2} I^+ = \frac{1}{2\pi} F(0)$$

$$S(\tau) = \frac{3}{4\pi}\tau F(0) + C = J(\tau)$$

$\tau=0$

- Hence $C = J(0) = F(0)/2\pi$ so:

$$S(\tau) = \frac{1}{\pi}\left(\frac{3}{4}\tau + \frac{1}{2}\right)F(0)$$

$$S(\tau) = \frac{3}{4\pi}\left(\tau + \frac{2}{3}\right)F(0)$$

- To find the depth dependence of T, we also need to assume **LTE**.

Temperature structure of the grey atmosphere

In LTE, the source function is the Planck function, $S(\tau)=B(\tau)=\sigma T^4/\pi$

$$B(\tau) = \frac{\sigma}{\pi} T^4(\tau) = \frac{3}{4\pi}(\tau + \frac{2}{3})F(0)$$

$$S(\tau) = \frac{3}{4\pi}(\tau + \frac{2}{3})F(0)$$

Recall that $F(0)=\sigma T^4_{\mathrm{eff}}$, by definition, so

$$\frac{1}{\pi}\sigma T^4(\tau) = \frac{3}{4\pi}(\tau + \frac{2}{3})\sigma T_{eff}^4$$

or

$$T^4(\tau) = \frac{3}{4}(\tau + \frac{2}{3})T_{eff}^4$$

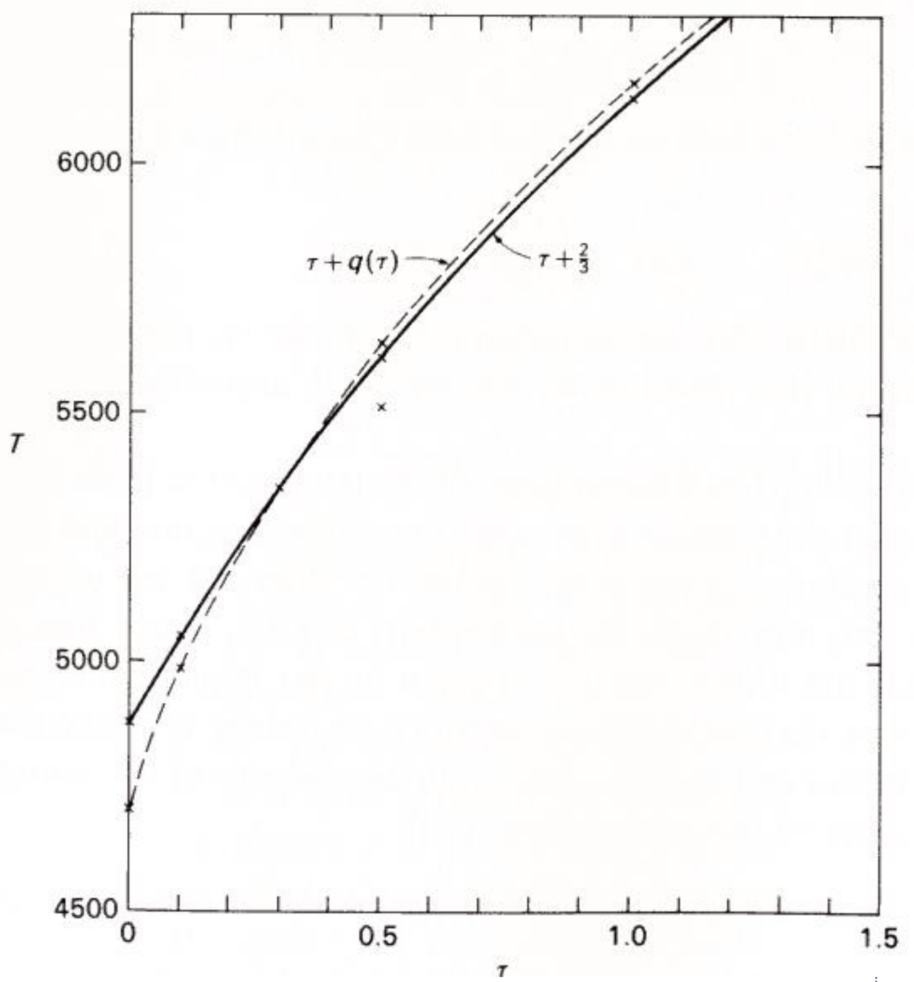
We derived the **temperature dependence on optical depth**.
Note $T(\tau=2/3) = T_{\mathrm{eff}}$ as we obtained earlier, and $T^4(\tau=0)=T_{\mathrm{eff}}^4 / 2$

A complete solution of the **grey** case, using accurate boundary conditions, without Eddington approximation, leads to a solution only slightly different from this, usually expressed as

$$T^4(\tau) = \frac{3}{4}[\tau + q(\tau)]T_{eff}^4$$

Here $q(\tau)$ is a slowly varying function (Hopf function), with $q = 1/\sqrt{3} = 0.577$ at $\tau=0$ to $q=0.710$ at $\tau=\infty$.

Grey Temperature Structure



Comparison between $T(\tau)$ in the Solar atmosphere using the simplifying Eddington assumption (solid) versus the exact grey case (dashed) using the Hopf function, $q(\tau)$:

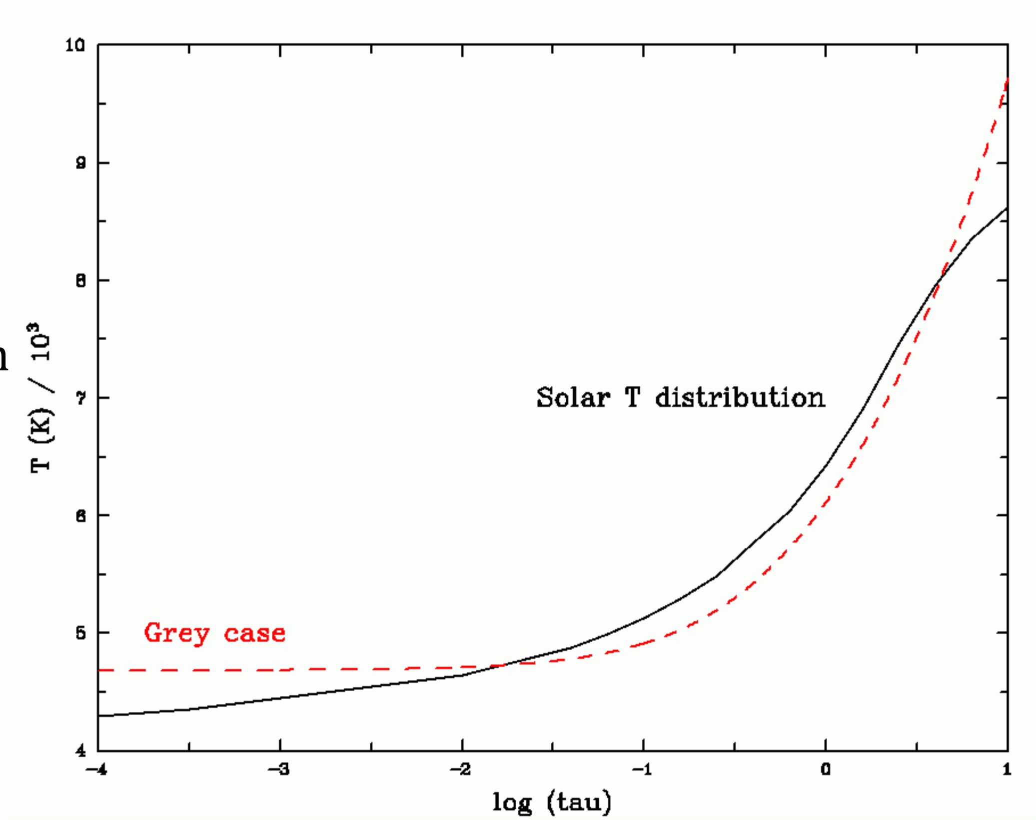
$$q(\tau) \approx 0.710 - 0.133e^{-2\tau}$$

How realistic is this?

- How good an approximate is the grey atmosphere? Next we must look at the frequency dependence of the sources of opacity.

- The grey temperature distribution is shown here versus the observed Solar temperature distribution as a function of optical depth τ at 5000Å (D. Gray, Table 9.2)

- The poor match is because the opacity is **wavelength dependent**, as we shall see next lectures.



Summary

- Three equations of radiative equilibrium can be derived:
  (a) constant flux with depth;
  (b) energy absorbed equals energy emitted;
  (c) the K-integral is linear in τ.

- From these, the **grey** temperature distribution $T(\tau)$ may be derived, assuming:
  (a) the Eddington approximation and
  (b) LTE, in reasonable agreement with the exact case.

- On the next lectures, we will discuss LTE in more detail.

Local Thermodynamic Equilibrium (LTE)

MAXWELLIAN VELOCITY DISTRIBUTION
BOLTZMANN EQUATION
SAHA EQUATION

Thermodynamic Equilibrium (TE)

- Interaction of radiation and matter is the most important physical process in stellar atmospheres.
- To find I_λ we need to know α_λ and ε_λ (or k_λ and j_λ) – absorption and emission coefficients.
- To find α_λ and ε_λ, density ρ, temperature T, and chemical composition X are **not** enough. We need to know distributions of atoms over levels and ionization states, which depend on radiation I_λ.
- In TE, ρ, T, and X fully determine α_λ and ε_λ.

Local Thermodynamic Equilibrium

In **Thermodynamic Equilibrium**:

1. All particles have Maxwellian distribution in velocities (with the same temperature T).
2. Atom populations follow Boltzmann law (same T).
3. Ionization is described by Saha formula (same T).
4. Radiation intensity is given by the Planck function (same T).
5. The principle of detailed equilibrium is valid (the number of direct processes = number of inverse processes).

In **Local thermodynamic equilibrium (LTE)**,
1-3 are applied **locally**.

The radiation spectrum can in principle be very far from Planck function.

LTE

In the study of stellar atmospheres, the assumption of Local Thermodynamic Equilibrium (**LTE**) is described by:

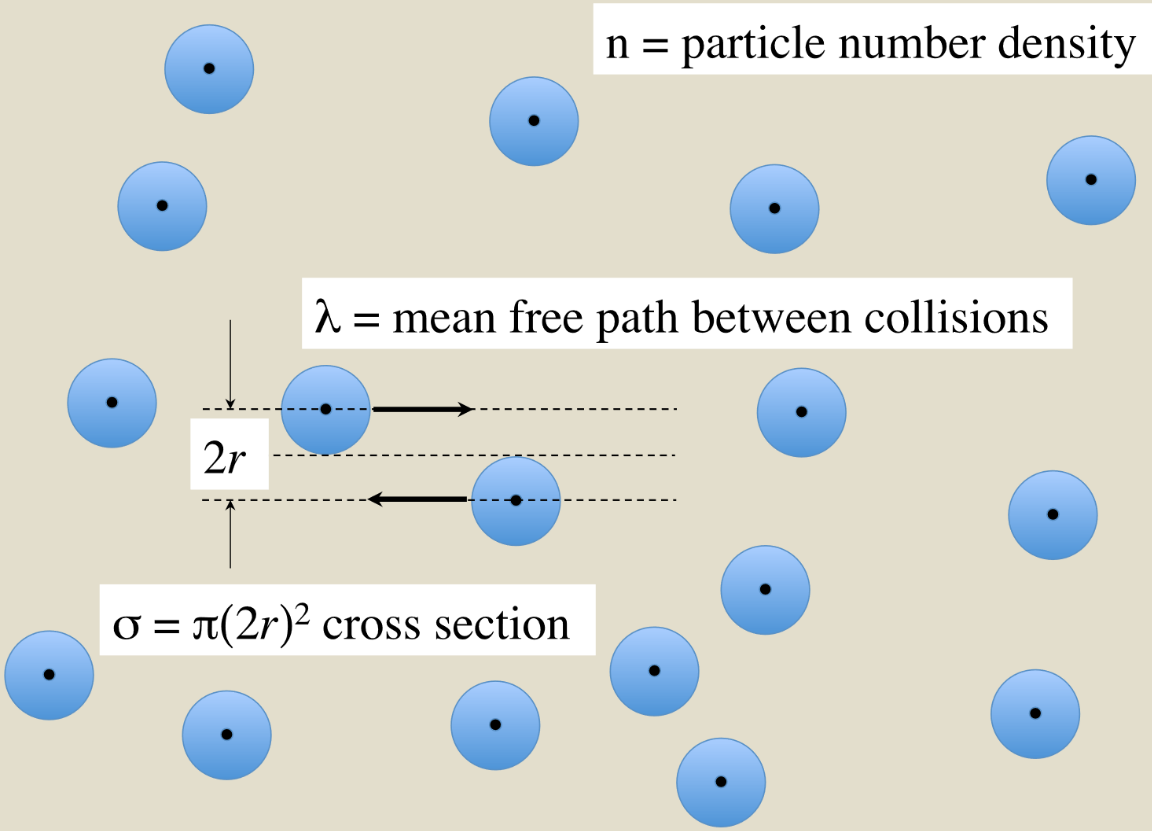
1. Electron and ion velocity distributions are Maxwellian.
2. Excitation equilibrium is given by Boltzmann equation (introduced today).
3. Ionization equilibrium is given by Saha equation (introduced today).
4. The source function is **given** by the Planck function

   $S_\lambda = I_\lambda = B_\lambda(T)$ i.e. Kirchoff's law $j_\lambda = \kappa_\lambda B_\lambda(T)$

Is LTE a valid assumption?

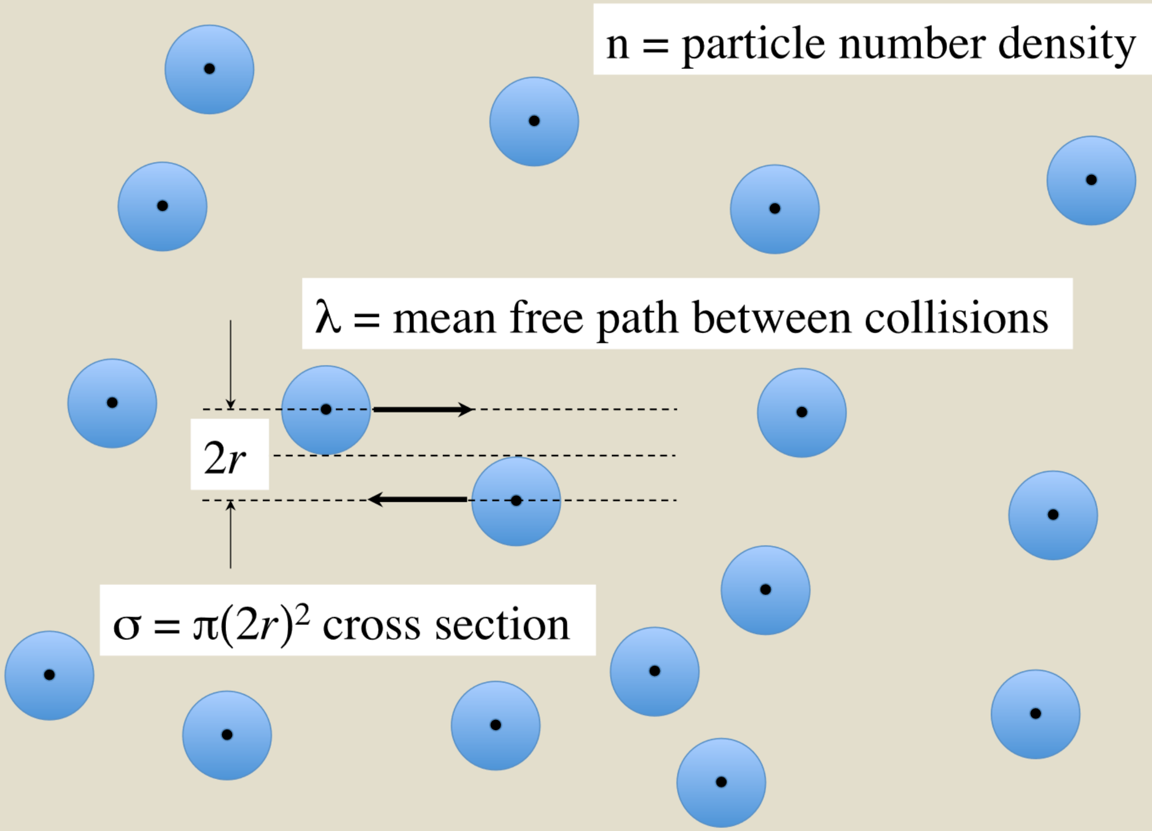
- For LTE to be valid, the photon and particle mean free paths need to be much smaller than the length scale over which these temperature changes significantly.
- Radiation cannot play a role in defining atom populations and ionization state. Collisions should dominate.
- Generally, when **collisional** processes dominate over radiative processes in the excitation and ionization of atoms, the state of the gas is close to LTE.
- Consequently, **LTE is a good assumption in stellar interiors, but may break down in the atmosphere**. If LTE is no longer valid, all processes need to be calculated in detail via **non-LTE**. This is much more complicated, but needs to be considered in some cases (see later in course).

Mean Free Path



- In the Sun, the characteristic distance over which the temperature varies (the temperature scale height) is ~500km.
- **How does this scale compare with the average distance travelled by an atom before hitting another atom?**
- Two hydrogen atoms will collide if their centres pass within a radius of 2 Bohr radii ($2a_o$) of each other. The collision cross-section of the H atom is $\sigma=\pi(2a_o)=3.5\text{x}10^{-16}\ \text{cm}^2$.
- The mean free path between collisions is $\lambda=1/(\sigma n(\text{H}))$.

Mean free path in the solar photosphere



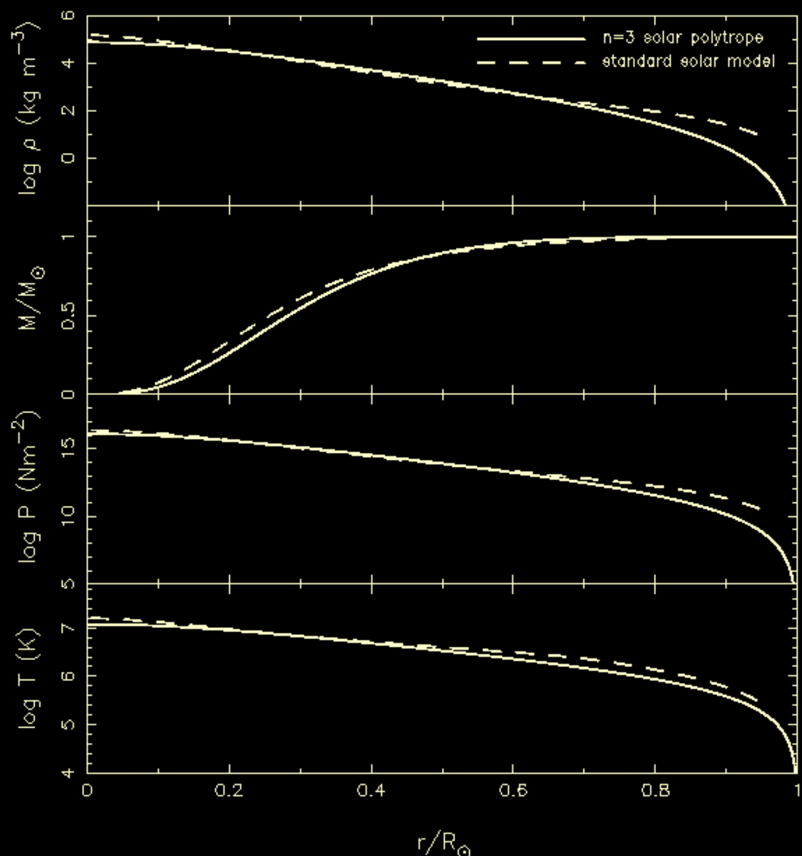
- The density of the Solar photosphere is $\rho = 2.5 \text{x} 10^{-7}$ g/cm^3 so the number of H atoms/cm^3 is $n(\text{H})=\rho/m_\text{H}=1.5\text{x}10^{17}$ cm^3 where m_H is the mass of the H atom.
- Then the mean free path between collisions is $\lambda=1/(\sigma n(\text{H}))=0.02$ cm. i.e. atoms are confined within a limited volume of space in the photosphere at effectively fixed temperature (relative to the temperature scale height).

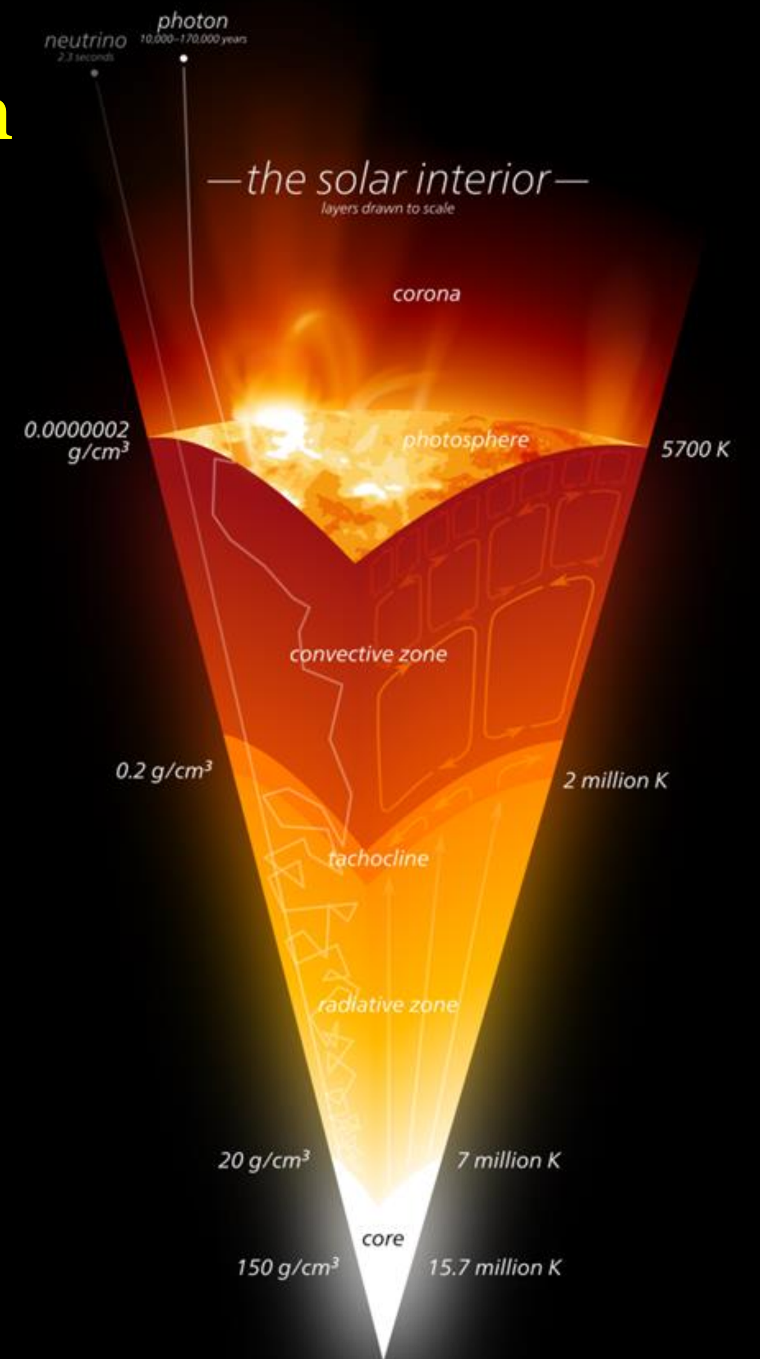
In the upper layers, $\rho \rightarrow 0$, $\lambda \uparrow$, radiation dominates over collisions → **out of LTE**

Mean Free Path in the Sun

Since the photosphere is the layer visible from Earth, photons must be able to escape freely into space. After ~10^{21} scatterings and re-emissions (thousands years!) from the centre.

Calculate the time needed for a photon to escape!





The Random Walk

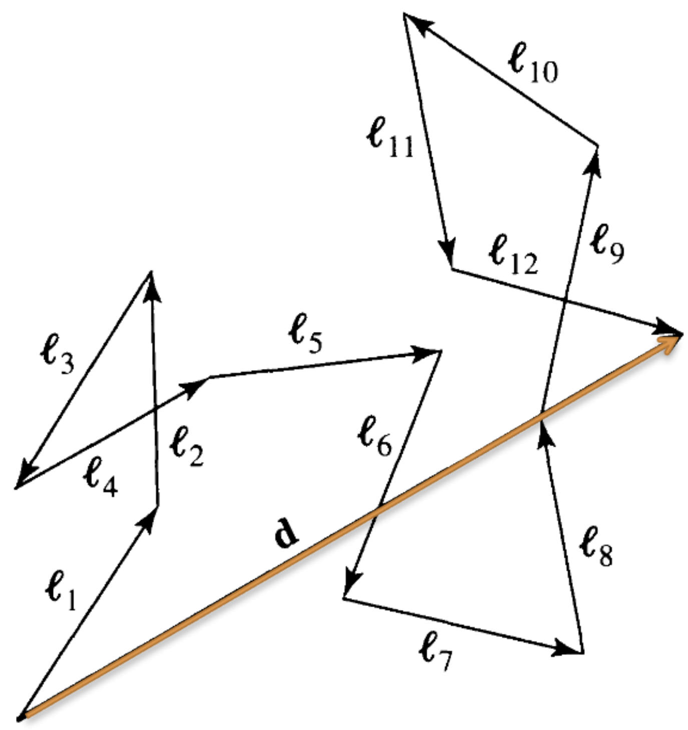
- As the photons diffuse upward through the stellar material, they follow a haphazard path called a random walk. Figure shows a photon that undergoes a net vector displacement d as the result of making a large number N of **randomly** directed steps, each of length l (=λ, the mean free path).
- It can be shown that for a random walk, the displacement d is related to the size of each step, l, by

$$d = l\sqrt{N}.$$

- This implies that the distance from the centre of a star to the surface is

$$D = l \times N$$

- This is why the transport of energy through a star by radiation may be extremely **inefficient**.



LTE

As noticed above, **LTE** is described by:

1. Maxwellian electron and ion velocity distributions.
2. Excitation equilibrium given by Boltzmann equation.
3. Ionization equilibrium given by Saha equation.

Let's discuss them.

Maxwellian velocity distribution

Gas pressure is produced by the motions of the gas particles. The velocities of particles are distributed in a Maxwellian distribution (also called the Maxwell–Boltzmann distribution).

$$\frac{\mathrm{d}N(v)}{N_{\text{total}}} = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{kT}\right)^{3/2} v^2 \mathrm{e}^{-mv^2/2kT}\,\mathrm{d}v$$

Because the particles produce Doppler shifts, the line-of-sight velocities have a distribution that is an important special case for spectroscopy:

$$\frac{\mathrm{d}N(v_{\mathrm{R}})}{N_{\text{total}}} = \left(\frac{m}{2\pi kT}\right)^{3/2} \mathrm{e}^{-mv_{\mathrm{R}}^2/2kT}\,\mathrm{d}v_{\mathrm{R}}$$

where v_{R} is the radial (line of sight) velocity component.

Maxwellian velocity distribution

The maximum of the speed distribution occurs at $\boldsymbol{v_1}$ (the most probable velocity):

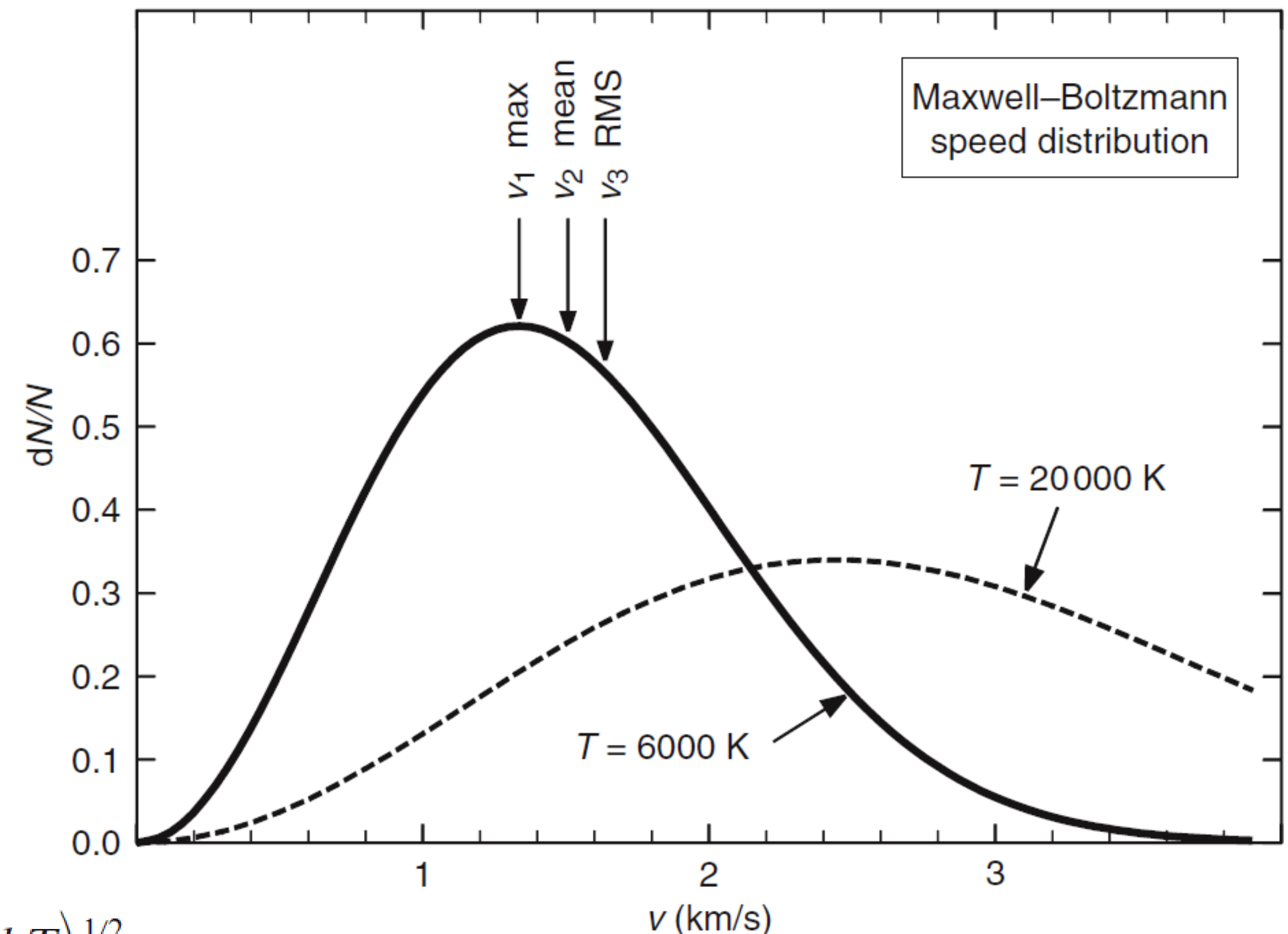
$$v_1 = \left(\frac{2kT}{m}\right)^{1/2}$$

The average velocity, $\boldsymbol{v_2}$, is

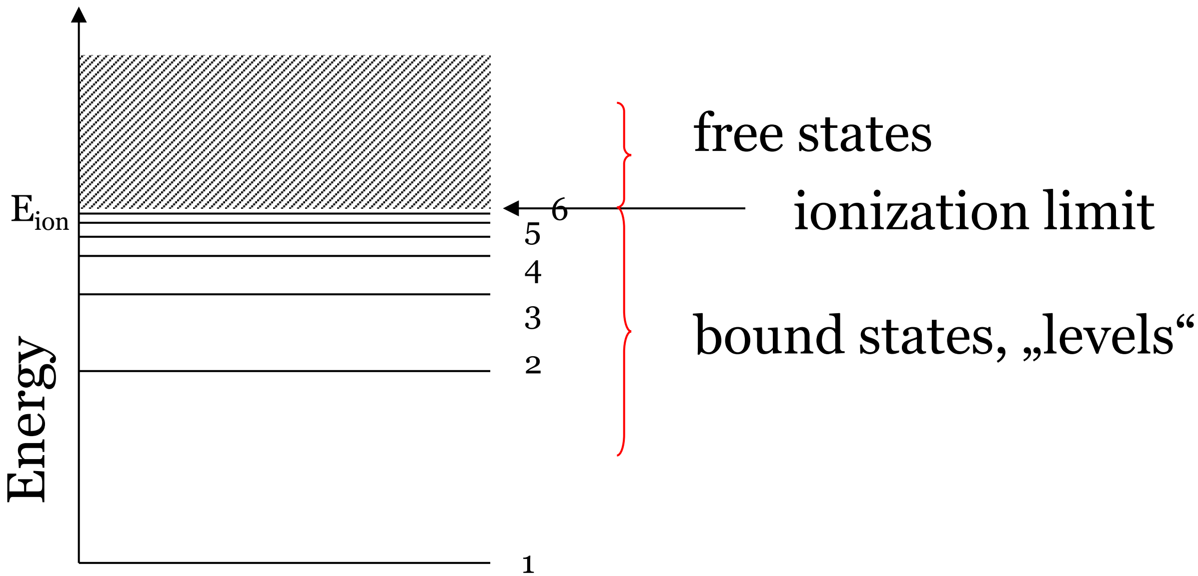
$$v_2 = \left(\frac{8}{\pi}\frac{kT}{m}\right)^{1/2} = 1.128 v_1$$

The root mean square velocity, $\boldsymbol{v_3}$, is

$$v_3 = \left(\frac{3kT}{m}\right)^{1/2} = 1.225 v_1$$



Boltzmann equation



For excited levels *u* and *l* of e.g. atomic hydrogen, the **Bolzmann equation** relates their population (occupation) numbers as follows:

$$\frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-(E_u - E_l)/kT}$$

where $\chi_{ul} = E_u - E_l$ is the energy difference between the levels, g_u & g_l are their statistical weights (see next slide), $k = 8.6174 \times 10^{-5}$ eV/K is the Boltzmann constant.

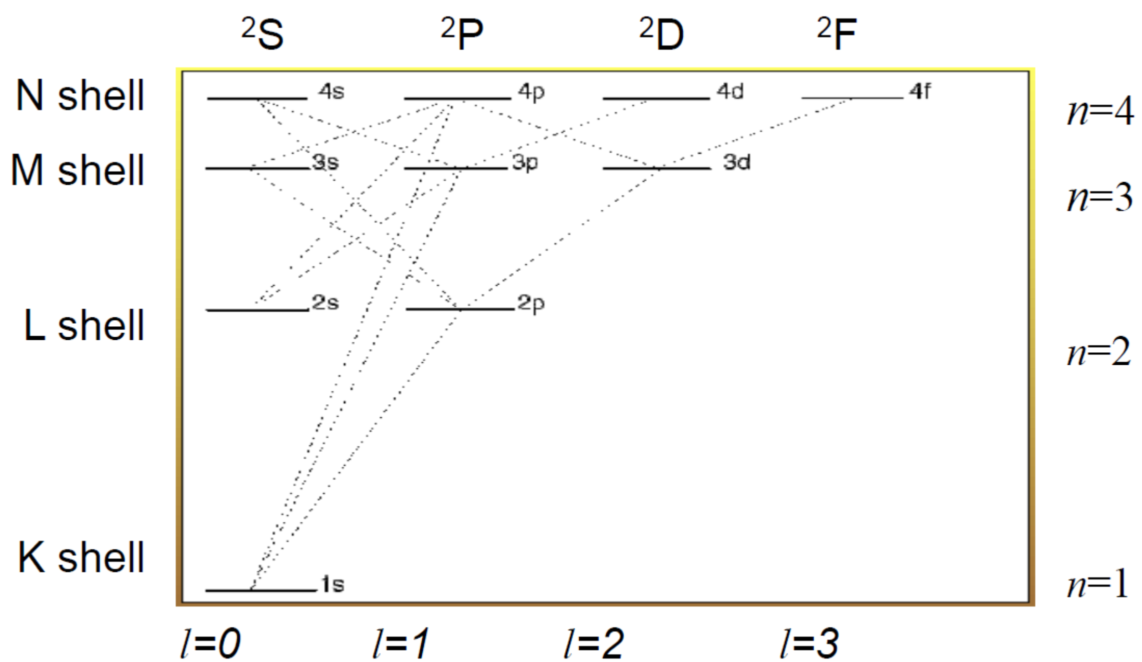
Boltzmann equation may also be written as:

$$\log \frac{N_u}{N_l} = \log \frac{g_u}{g_l} - \frac{5040}{T} \chi_{ul}(eV)$$

$\Theta = 5040/T$

In the "ground state" (n=1), "first excited state" (n=2), and all other excited states of H more than one quantum state may have the same energy.

The number of these for orbital n is the statistical weight, g_n, (also known as the degeneracy).



Hydrogen

For H, orbital n has a statistical weight of $g_n=2n^2$ – the various permutations for n=1 and n=2 are listed here, with statistical weights g_1=2 and g_2=8, respectively.

l=0…n-1 azimuthal quantum number
m_l=magnetic quantum number with $-l \leq m_l \leq l$
m_s=electron "spin" angular momentum ±1/2

Transition energy between levels u and l:

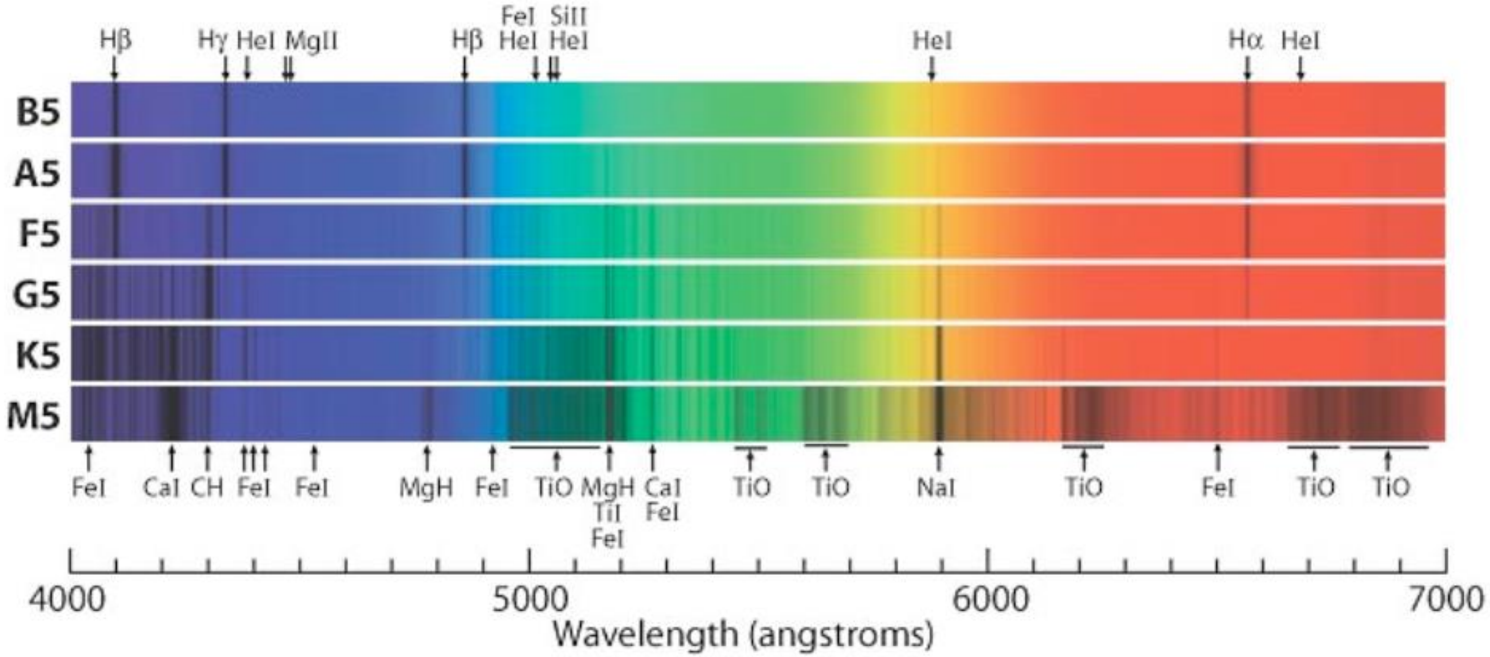
$$\chi_{ul} = C\left(\frac{1}{u^2} - \frac{1}{l^2}\right)$$

where $C=\chi_{\text{ion}}$=-13.6 eV

Ground States s_1				Energy E_1 (eV)
n	ℓ	m_ℓ	m_s	
1	0	0	+1/2	−13.6
1	0	0	−1/2	−13.6
First Excited States s_2				**Energy E_2 (eV)**
n	ℓ	m_ℓ	m_s	
2	0	0	+1/2	−3.40
2	0	0	−1/2	−3.40
2	1	1	+1/2	−3.40
2	1	1	−1/2	−3.40
2	1	0	+1/2	−3.40
2	1	0	−1/2	−3.40
2	1	−1	+1/2	−3.40
2	1	−1	−1/2	−3.40

Balmer lines

An exceptionally high *T* is required for a significant number of H atoms to have electrons in their 1st excited states. The Balmer lines (involving an upward transition from *n*=2 orbital) reach a peak strength at spectral class A (≈10000K)



so why do the Balmer lines **diminish** in strength at higher temperatures?

We need Saha equation to answer this question.

Balmer lines