Radiative transfer III

RADIATIVE TRANSFER EQUATION IN PLANE-PARALLEL ATMOSPHERE.
LIMB DARKENING.

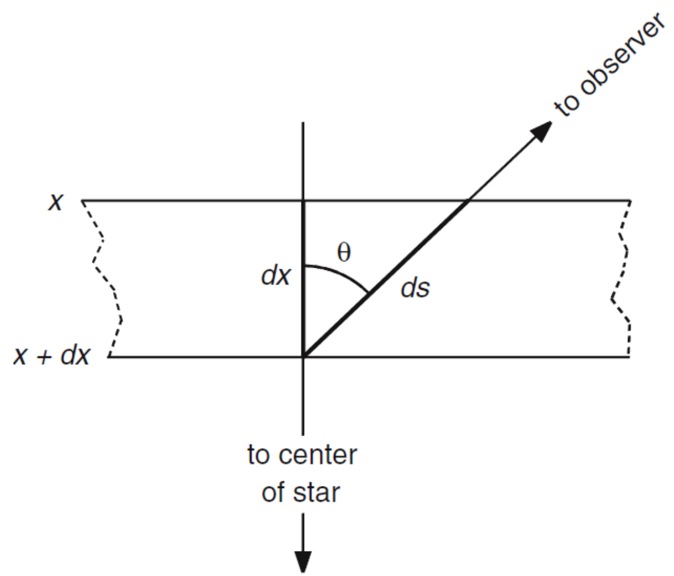
Solar limb darkening

Transfer Equation for Stars

From lecture 6 (slide I-167):

The plane-parallel transfer equation
(for stars with thin photospheres)

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$



The cos(θ) term is because the optical depth is measured along the radial direction *x* and not along the line of sight, i.e $d\tau_\lambda = -\kappa_\lambda \rho\, dx$

We are looking from the outside in, along direction x

Surface Intensity

- To derive the intensity at the **surface**, we can multiply the plane-parallel transfer equation by an integrating factor $e^{-\tau/\cos\theta} = e^{-u}$, $\tau = u\cos\theta$

$$\frac{dI_\lambda(\theta)}{du} e^{-u} - I_\lambda(\theta)\ e^{-u} = -S_\lambda e^{-u}$$

- This can be written as

$$\frac{d(I_\lambda(\theta) e^{-u})}{du} = -S_\lambda e^{-u}$$

- Integrating *du* from 0 to infinity

$$[I_\lambda(\theta) e^{-u}]_0^\infty = -\int_0^\infty S_\lambda\,(\tau_\lambda) e^{-u} du$$

$$I_\lambda(0, \theta) = \int_0^\infty S_\lambda\,(\tau_\lambda) e^{-u} du$$

Limb darkening

Let us assume a linear source function: $S_\lambda(\tau_\lambda) = a_\lambda + b_\lambda \tau_\lambda$

We then derive: $I_\lambda(0,\theta) = \int_0^\infty S_\lambda(\tau_\lambda) e^{-u} du = \int_0^\infty a_\lambda e^{-u} du + \int_0^\infty b_\lambda \tau_\lambda e^{-u} du$

Recall $u = \tau/\cos(\theta)$, so $\tau = u\cos(\theta)$ and $I_\lambda(0,\theta) = a_\lambda \int_0^\infty e^{-u} du + b_\lambda \cos\theta \int_0^\infty u\, e^{-u} du$

Using the standard integral $\int_0^\infty u^n e^{-u} du = n!$

we obtain

$$I_\lambda(0,\theta) = a_\lambda + b_\lambda \cos\theta = S_\lambda(\tau_\lambda = \cos\theta)$$

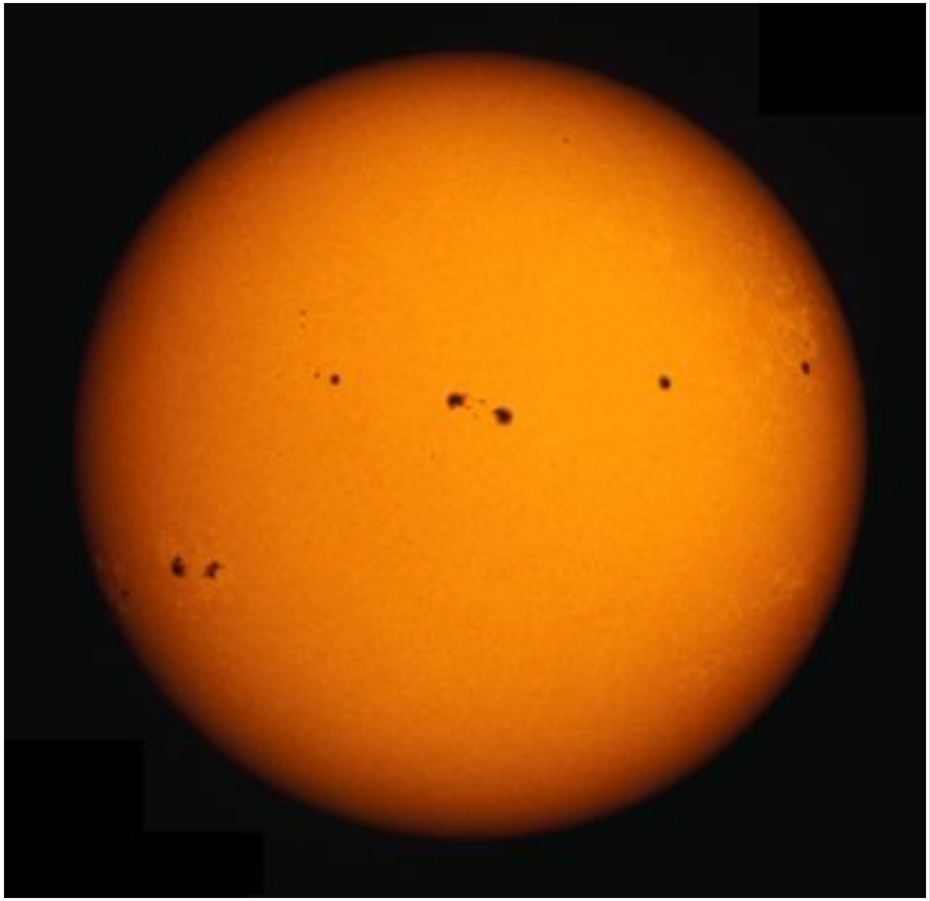
Thus, in the linear approximation for the Source function, the optical depth lies between 0 and 1. From the centre of the star we see radiation leaving the star perpendicular to the surface: $I_\lambda(0,0°) = a_\lambda + b_\lambda$, whilst at the limb the starlight leaves the surface at an angle $I_\lambda(0,90°) = a_\lambda$.

Limb darkening (less light from the limb versus the centre, if $b_\lambda > 0$).

Solar limb darkening

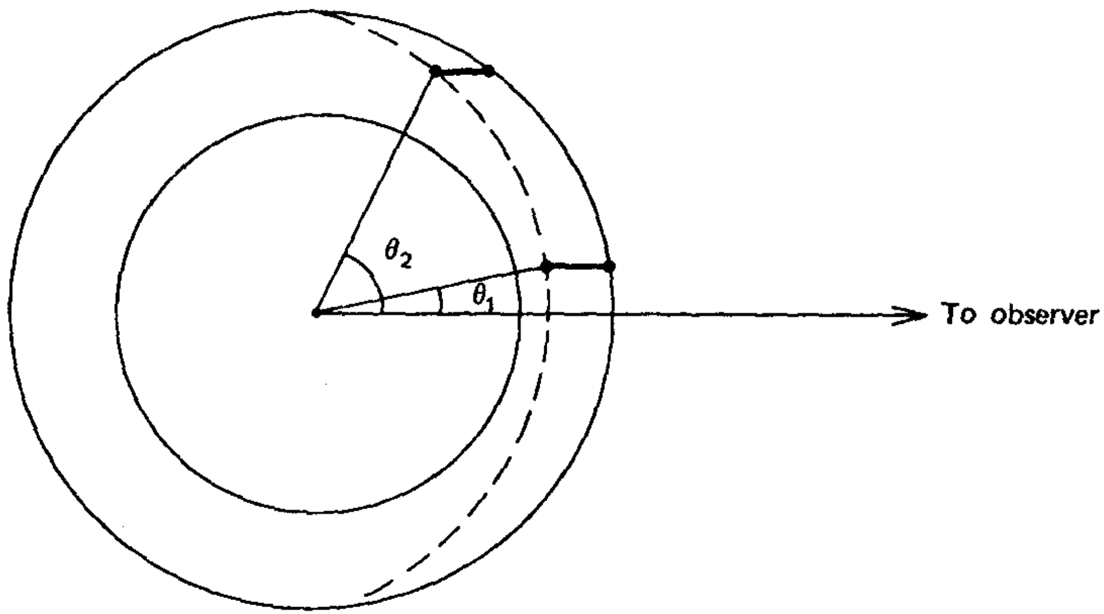
- This optical image of the Sun clearly shows limb darkening. We see into the atmosphere down to a depth of $\tau=1$.
- Limb darkening exists because the continuum source function decreases outward:
  $S_\lambda(\tau_\lambda) = a_\lambda + b_\lambda \tau_\lambda$,
  both a_λ and $b_\lambda > 0$.

- As we look towards the limb, we see higher photospheric layers, which are less bright.



Schematic of limb darkening

Schematic illustration of limb darkening – penetration of different lines of sight (thick lines) to "unit optical depth" (dashed lines) corresponds to different depths in the photosphere, depending on θ. Radiation seen at θ_2 is characteristic of higher (cooler) layers than the radiation seen at position θ_1



Linear vs Quadratic source function

Up to now we assumed a linear source function. More generally, if:

$$S_\lambda(\tau_\lambda) = \sum_{n=0} a_{n\lambda}\,\tau_\lambda^n$$

Then

$$I_\lambda(0,\theta) = \sum_{n=0} A_n \cos^n\theta \qquad A_n = a_{n\lambda}\int_0^\infty u^n\, e^{-u} du = a_{n\lambda} n$$

We still get $S_\lambda(0)$ at the limb, but a more complicated result at the centre. For example, a quadratic term requires the solution of

$$S(\tau_\lambda) = a_{0\lambda} + a_{1\lambda}\tau_\lambda + a_{2\lambda}\tau_\lambda^2$$

$$I_\lambda(0,\theta) = a_{0\lambda} + a_{1\lambda}\cos\theta + 2a_{2\lambda}\cos^2\theta$$

At $\theta = 90°$, $\tau_\lambda=0$, whilst at $\theta=0°$, $\tau_\lambda \sim 1+2a_{1\lambda}/a_{2\lambda}$ providing $a_{2\lambda} << a_{1\lambda}$.
The ratio of the limb-to-centre intensity is

$$\boldsymbol{I_\lambda(0,90°)/I_\lambda(0,0°) = a_{0\lambda}/(a_{0\lambda} + a_{1\lambda} + 2a_{2\lambda})}$$

Example for Solar Case:

The measured centre to limb variation of the solar intensity is

$$I_\lambda(0,\theta)/I_\lambda(0,0) = a_{0\lambda} + a_{1\lambda}\cos\theta + 2a_{2\lambda}\cos^2\theta$$

λ(μm)	a_0	a_1	$2a_2$
0.3	0.06	0.74	0.20
0.4	0.14	0.91	-0.05
0.6	0.35	0.88	-0.23
0.8	0.49	0.73	-0.22
1.5	0.56	0.64	-0.20
2.0	0.70	0.48	-0.18

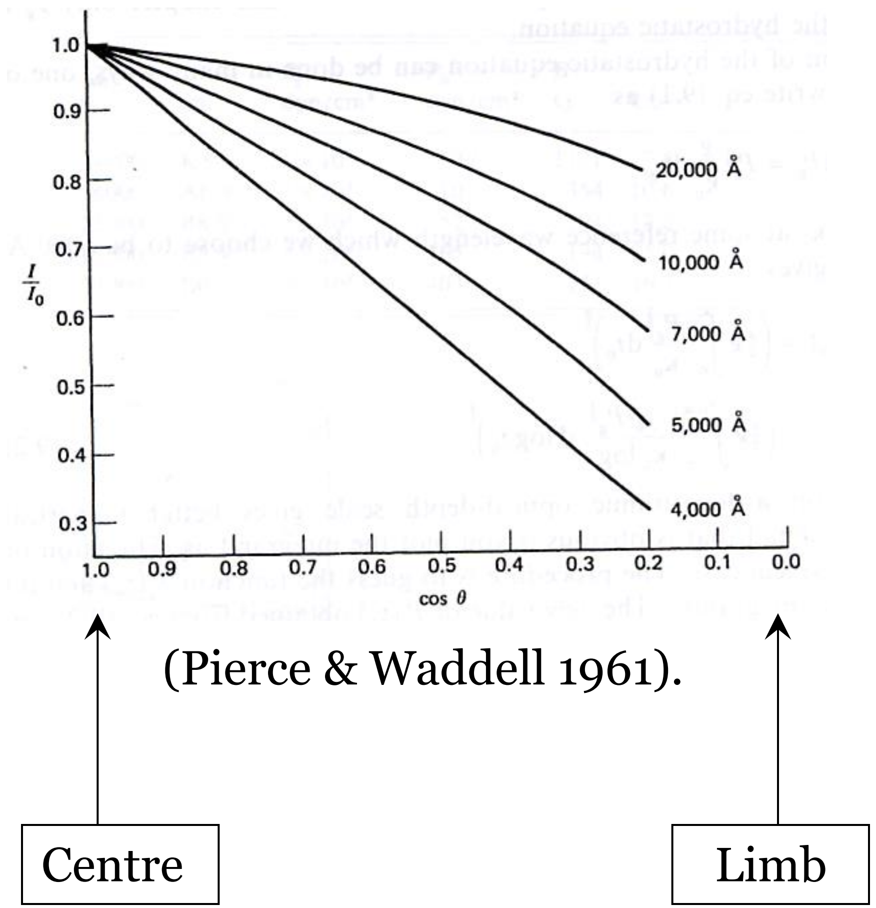
(Table 4.17, AQ 4th edition)

Wavelength dependence

Limb darkening is observed to be greatest at **shorter** wavelengths in the Sun. The temperature distribution of the upper atmosphere of the Sun can be obtained from limb darkening measurements, carried out via e.g. multi-filter images of the Solar continuum (between the lines).

Until recently, the Sun was the only star for which limb darkening was observed, since one needs to **spatially** resolve the disc (most other stars appear as point sources!) to measure limb darkening.

Other methods are now possible.



(Pierce & Waddell 1961).

Limb darkening for other stars

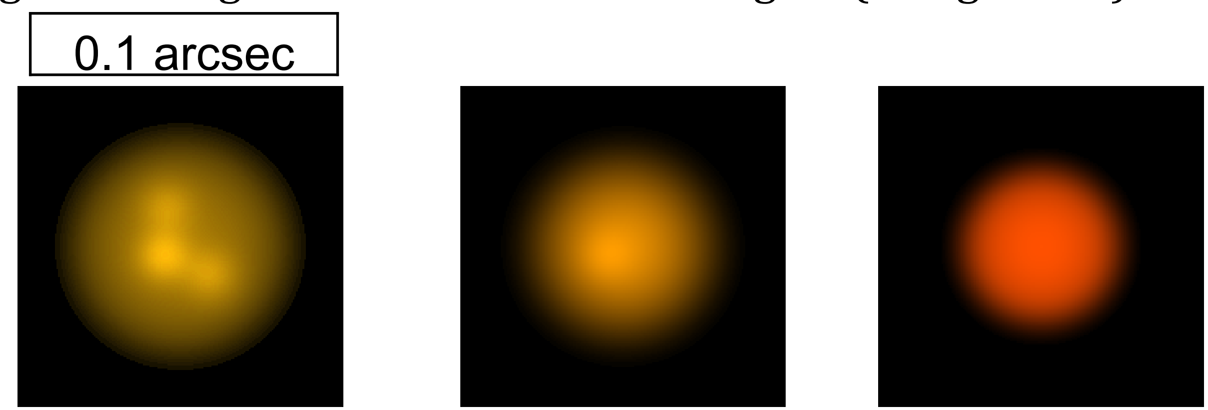
1. Direct interferometry, via high spatial resolution "imaging" – e.g. ESO/VLT interferometry or COAST array, providing a star is very large and nearby (a cool supergiant).

2. The light curve due to the gravitational micro-lensing of a background (generally Galactic bulge or Magellanic Cloud) star by a foreground source (e.g. PLANET team).

3. The light curve from an eclipsing binary system during secondary eclipse allows us to study limb darkening of the primary, although non-trivial! Similar approach followed by extra solar planets occulting parent star (e.g. HD209458).

Limb darkening from interferometry

COAST (Cambridge Optical Aperture Synthesis Telescope) spatial resolution of 20-30 milli-arcsec) has made limb darkening observations of M supergiant Betelgeuse at different wavelengths (using filters).



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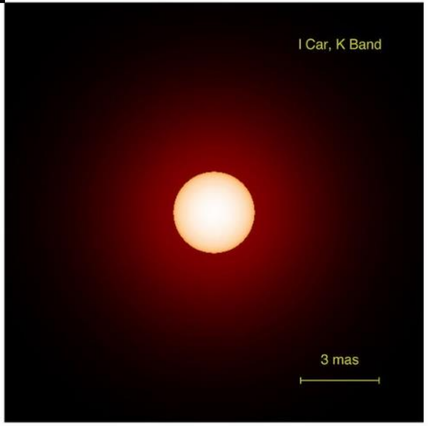
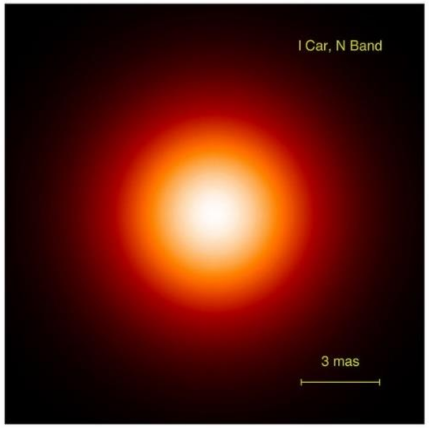
Limb darkening from interferometry

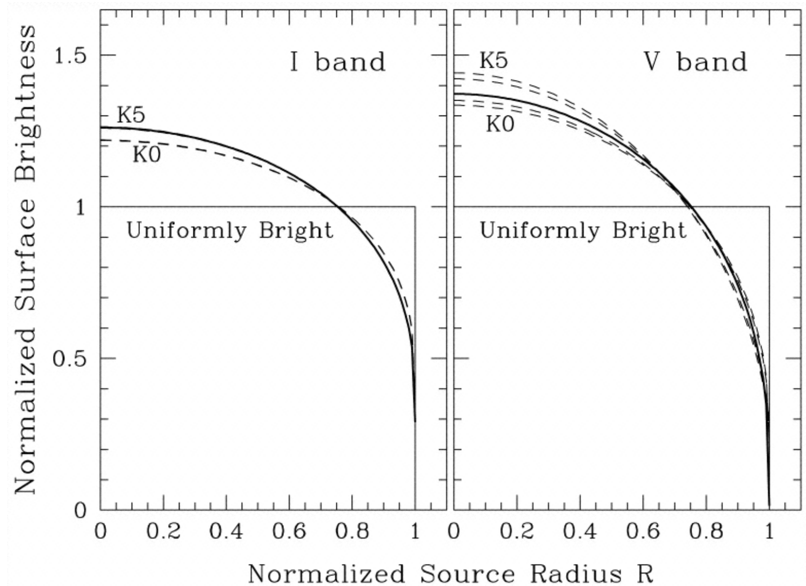
ESO's Very Large Telescope Interferometer (VLTI) is possible to achieve a resolution of 0.001 arcsec or even less. It has resolved the disc of the cepheid L Carinae.





Limb darkening from microlensing

- Galactic gravitational micro-lensing occurs when a foreground object (lens) passes in front of a background star (source). The gravitational deflection of light by the lens causes the flux from the source to be amplified.
- Microlensing surveys (e.g. PLANET, MACHO) have identified hundreds of such events towards the Galactic bulge and Magellanic Clouds.
- One such event, MACHO 97-BLG-28 was studied to reveal limb darkening information for the background K giant (Albrow et al. 1999).

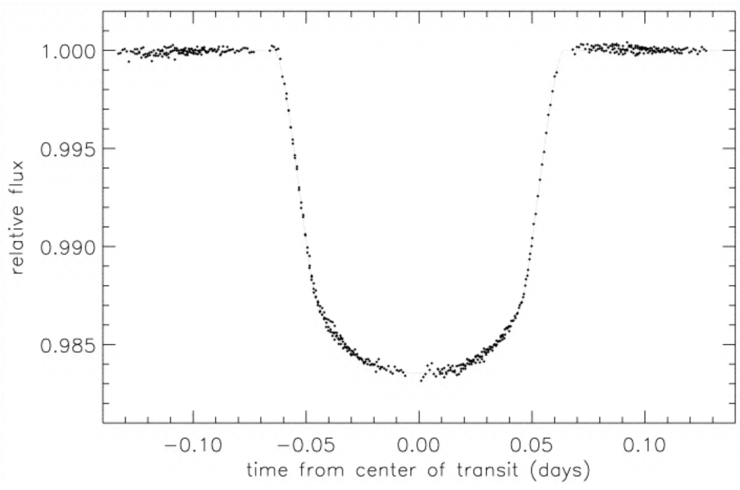


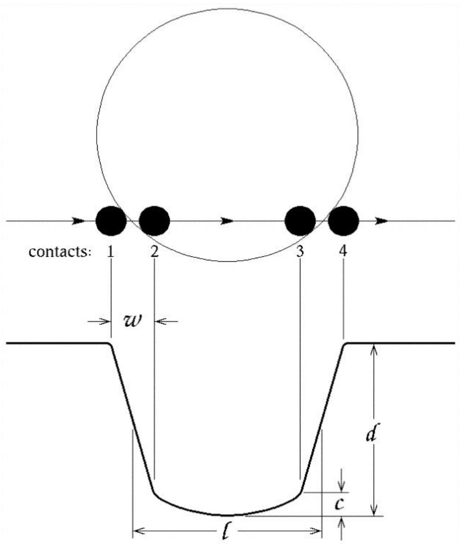
Thick lines show how much fainter the K giant becomes at its edges in the red I (left) and blue-green V filter (right). If the star emitted a uniform amount of light across its whole stellar disk, the profile would look like the straight solid black line instead

Limb darkening from eclipsing systems

- HD209458 is the first system in which extra-solar planet (P=3.5d, $0.6M_J$) has been observed to transit its (F8V) primary, allowing determination of limb darkening (Brown et al. 2001).

- More generally eclipsing binaries are problematic due to degeneracy with other parameters (Grygar et al. 1972). Accurate light curves needed for linear limb darkening parameters.





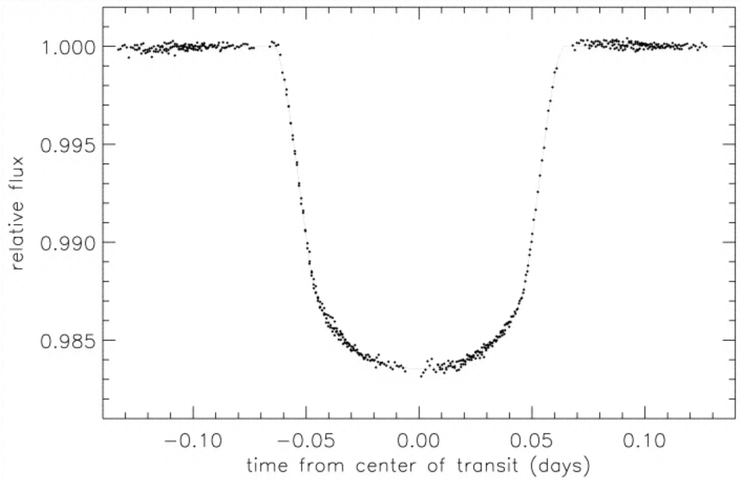
Limb darkening: current state

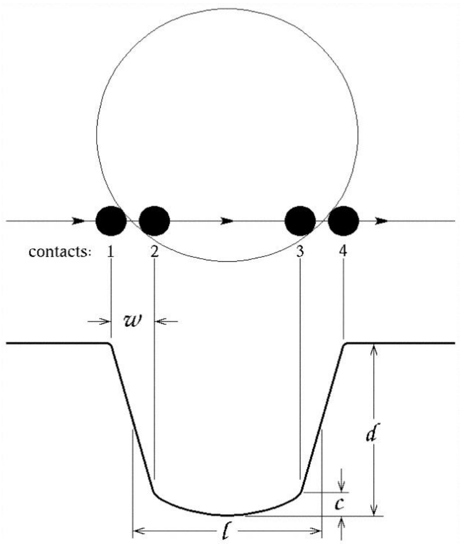
- Stars appear darker at their limbs than at their disk centers because at the limb we are viewing the higher and cooler layers of stellar photospheres.

- Limb darkening derived from state-of-the-art stellar atmosphere models systematically **fails** to reproduce recent transiting exoplanet light curves from the Kepler, TESS, and JWST telescopes – stellar brightness obtained from measurements drops less steeply towards the limb than predicted by models.

- Possible explanation: magnetic fields on the stellar surface are not taken into account:

  Kostogryz et al. (2024, NatAst):
  stellar atmosphere models computed with the use of a 3D radiative magneto-hydrodynamic code show that small-scale concentration of magnetic fields on the stellar surface affect limb darkening at a level allowing the authors to explain the observations.





Eddington-Barbier relation

FORMAL SOLUTION TO THE PLANE-PARALLEL TRANSFER EQUATION.

EDDINGTON-BARBIER RELATION.

GREY ATMOSPHERE.

Formal Solution to RTE (1)

The **plane-parallel** transfer equation (for stars with thin photospheres)

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

The integrated form of the RTE is
[See D. Gray (page 127-129, 131) for more detail]:

$$I_\lambda(\tau_\lambda) = -\int_c^{\tau_\lambda} S_\lambda(t_\lambda) e^{-(t_\lambda - \tau_\lambda)\sec\theta} \sec\theta \, dt_\lambda$$

Here, the integration limit c (*which complicates the integral*), replaces $I_\nu(0)$ in the parallel-ray transfer equation (Lecture 5, slide I-149):

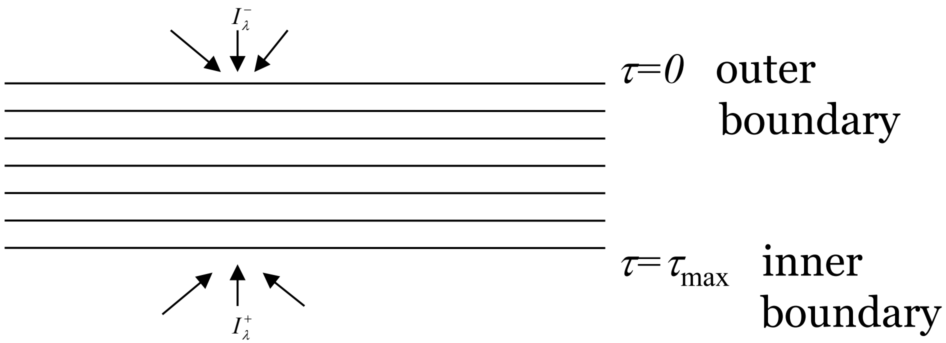
$$I_\lambda(\tau_\lambda) = \int_0^{\tau_\lambda} S_\lambda(t_\lambda) e^{-(\tau_\lambda - t_\lambda)} dt_\lambda + I_{\lambda 0} e^{-\tau_\lambda}$$

This is because the boundary conditions are different for radiation going in (θ>90°) and coming out (θ<90°) →

Formal Solution to RTE (2)

- The full intensity at the position τ_λ on the line of sight through the photosphere is

$$I_\nu(\tau_\nu) = I_\nu^{\text{out}}(\tau_\nu) + I_\nu^{\text{in}}(\tau_\nu)$$
$$= \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu$$
$$- \int_0^{\tau_\nu} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu$$



- An important special case occurs at the stellar surface. In this case

$$I_\nu^{\text{in}}(0) = 0$$
$$I_\nu^{\text{out}}(0) = \int_0^{\infty} S_\nu e^{-t_\nu \sec\theta} \sec\theta \, dt_\nu$$

where we assumed that the external radiation is **completely negligible** compared to the star's own radiation. **This Equation is the expression we need to compute the spectrum**.

- However, since the discs of most stars are spatially unresolved, we must deal with flux rather than intensity, so we will not deal with this equation any further.

Emergent Flux

From our lecture 6 (slide I-161), the flux is [If there is no azimuthal (ϕ) dependence in I_λ]:

$$F = 2\pi \int_{-1}^{1} I(\mu)\, \mu\, d\mu \qquad \mu = \cos\theta$$

Netto = Outwards – Inwards.

Decomposition into two half-spaces:

$$F = 2\pi \int_0^1 I(\mu)\mu\, d\mu + 2\pi \int_{-1}^0 I(\mu)\, \mu\, d\mu$$

$$= 2\pi \int_0^1 I(\mu)\, \mu\, d\mu - 2\pi \int_0^1 I(-\mu)\mu\, d\mu = \boldsymbol{F^+} - \boldsymbol{F^-}$$

Eddington-Barbier relation

Special case: at the surface of a star $F^{-} = 0$, so that $F = F^{+}$

$$F_\lambda(0) = 2\pi \int_0^1 I_\lambda\,(0,\theta)\,\mu\,d\mu$$

From earlier, assuming a linear source function $S_\lambda(\tau_\lambda) = a_\lambda + b_\lambda \tau_\lambda$ yields

$$I_\lambda(0,\theta) = a_\lambda + b_\lambda\,cos\,\theta = a_\lambda + b_\lambda \mu$$

In this case we obtain the "Eddington-Barbier" relation:

$$F_\lambda(0) = \pi(a_\lambda + 2/3\,b_\lambda) = \pi S_\lambda(\tau_\lambda = 2/3)$$

The emergent flux from the stellar surface is π times the Source function at an optical depth of $2/3$

Grey atmosphere (1)

If we assume Local TE (LTE), then

$$F_\lambda(0) = \pi S_\lambda(\tau_\lambda = 2/3) = \pi B_\lambda[T(\tau_\lambda = 2/3)]$$

Let us assume the opacity is independent of λ, i.e. $\kappa_\lambda = \kappa$. We call such a (hypothetical) atmosphere a **grey atmosphere**. Then

$$F_\lambda(0) = \pi B_\lambda[T(\tau = 2/3)]$$

The energy distribution of F_λ is that of a blackbody corresponding to the temperature at the optical depth $\boldsymbol{\tau = 2/3}$.

The black body intensity is defined (following discovery by Max Planck in 1900) as either

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/\lambda kT} - 1} \qquad \text{or} \qquad B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT} - 1}$$

where c=2.99x10^{10} cm, h=6.57x20^{-27} erg s, k=1.38x10^{-16} erg/s.

Let's compute the Bolometric flux.

Bolometric flux of Black Body

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

Note that: $$B_\nu(T)\mathrm{d}\nu = B_\lambda(T)\mathrm{d}\lambda \Rightarrow B_\lambda = B_\nu\left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = B_\nu\frac{c}{\lambda^2}$$

Let us compute the bolometric flux:

$$F = \pi\int_0^\infty B_\nu(T)\mathrm{d}\nu = \pi\int_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}\mathrm{d}\nu = \pi\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\int_0^\infty\frac{x^3}{e^x-1}\mathrm{d}x = \pi\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\frac{\pi^4}{15} = \sigma_{SB}T^4$$

$$\sigma_{SB} = 2\frac{\pi^5k^4}{15c^2h^3} = 5.67\ 10^{-5}\ \mathrm{erg\ cm^{-2}s^{-1}\ K^{-4}}$$ – Stefan-Boltzmann constant

Planck function is monotonic with temperature: $$\frac{\partial B_\nu(T)}{\partial T} = \frac{2h^2\nu^4}{c^2kT^2}\frac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2} > 0$$

$$F = \sigma T^4$$

Grey atmosphere (2)

If we assume Local TE (LTE), then

$$F_\lambda(0) = \pi S_\lambda(\tau_\lambda = 2/3) = \pi B_\lambda[T(\tau_\lambda = 2/3)]$$

Let us assume the opacity is independent of λ, i.e. $\kappa_\lambda = \kappa$. We call such a (hypothetical) atmosphere a grey atmosphere. Then

$$F_\lambda(0) = \pi B_\lambda[T(\tau = 2/3)]$$

The energy distribution of F_λ is that of a blackbody corresponding to the temperature at the optical depth $\tau = 2/3$.

Thus, integrating over λ

$$F(0) = \int_0^\infty F_\lambda(0) d\lambda = \pi \int_0^\infty B_\lambda(T(\tau = 2/3)] d\lambda = \sigma T^4(\tau = 2/3)$$

From Stefan-Boltzmann, $F(0) = \sigma T_{eff}^4$, by definition, we find $T_{eff} = T(\tau = 2/3)$.
The "surface" of a star, which has temperature T_{eff} (by definition) is not at the very top of the atmosphere (where $\tau = 0$), but lies deeper down, at $\tau = 2/3$.

This can be considered as an *average* point of origin from the observed photons.

Summary

- Solution to plane-parallel transfer equation at surface explains limb darkening in Sun.
- Limb darkening in other stars can be estimated from interferometry, eclipsing binaries, microlensing.
- Eddington-Barbier relation.
- Grey atmosphere.
- Assuming a grey atmosphere , we found that the "surface" of a star, which has temperature T_{eff} (by definition) is not at the very top of the atmosphere (where $\tau=0$), but lies deeper down, at $\tau = 2/3$.