

# The r- and s-processes (1)

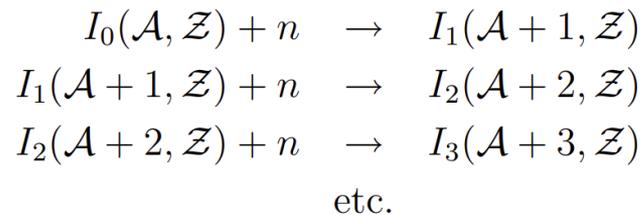
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We have already seen how elements up through [iron](#) are built, but we have not yet mentioned how even heavier elements can be created. The answer is that they are not made in stars under normal circumstances, because when the only forces at work are electromagnetism and nuclear forces, it is never energetically favorable to create such elements in any significant number.

[Creating such elements requires the intervention of another force: gravity.](#)

When stars are in the process of being crushed by gravity, right before they explode as supernovae (which we will discuss toward the end of the course), gravity drives a process that converts most of the protons to neutrons. This creates a neutron-rich environment unlike any found at earlier stages of stellar evolution, when the lack of neutrons was often the rate-limiting step.

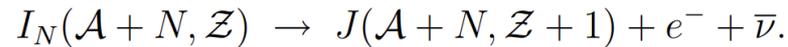
Neutron capture by heavy [nuclei is not limited by the Coulomb barrier](#) – so could proceed at relatively low temperatures. In a neutron-rich environment, it becomes possible to create heavy nuclei via the absorption of neutrons by existing nuclei. Since the neutrons are neutral, there is no Coulomb barrier to overcome, and the reaction proceeds as quickly as the neutron supply allows. Reactions look like this:



# The r- and s-processes (2)

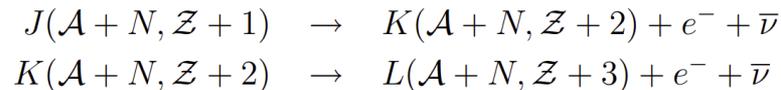
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This continues until it produces a nucleus that is unstable and undergoes  $\beta$  decay, converting one of the neutrons back into a proton:



anti-neutrino

If the new element produced in this way is stable, it will begin neutron capturing again. If not, it will keep undergoing  $\beta$  decays until it becomes stable:



etc.

$\beta$  decays therefore increase  $Z$  at constant  $A$

These processes together lead to the build-up of elements heavier than iron. The chain **stops** if at any point it reaches a nucleus that is stable against  $\beta$  decay, and is also not able to capture neutrons.

The neutron capture reactions may proceed more **slowly** or more **rapidly** than the competing  $\beta$  decays. Elements that are build up by reaction chains in which  $\beta$  decays occur faster are called **r-process**, for rapid. Elements where  $\beta$  decays are slower are called **s-process**, for slow.

Knowing which process produces which elements requires knowing the stability, binding energy, and  $\beta$  decay lifetimes of the various elements, which must be determined experimentally.

# Abundance changes (1)

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The rate of change in the number density  $n_i$  of nuclei of type  $i$  owing to reactions with nuclei of type  $j$  is

$$\left(\frac{dn_i}{dt}\right)_j = -(1 + \delta_{ij})r_{ij} = -n_i n_j \langle \sigma v \rangle_{ij}$$

One can define the nuclear lifetime of a species  $i$  owing to reactions with  $j$  as

$$\tau_{ij} = \frac{n_i}{|(dn_i/dt)_j|} = \frac{1}{n_j \langle \sigma v \rangle_{ij}}$$

which is the timescale on which the abundance of  $i$  changes as a result of this reaction.

The overall change in the number  $n_i$  of nuclei of type  $i$  in a unit volume can generally be the result of different nuclear reactions. Some reactions (with rate  $r_{ij}$  as defined above) consume  $i$  while other reactions, e.g. between nuclei  $k$  and  $l$ , may produce  $i$ . If we denote the rate of reactions of the latter type as  $r_{kl,i}$ , we can write for the total rate of change of  $n_i$ :

$$\frac{dn_i}{dt} = - \sum_j (1 + \delta_{ij})r_{ij} + \sum_{k,l} r_{kl,i}$$

# Abundance changes (2)

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The number density  $n_i$  is related to the mass fraction  $X_i$  as  $n_i = X_i\rho/(A_i m_H)$ , so that we can write the rate of change of the mass fraction due to nuclear reactions as

$$\frac{dX_i}{dt} = A_i \frac{m_H}{\rho} \left( - \sum_j (1 + \delta_{ij}) r_{ij} + \sum_{k,l} r_{kl,i} \right)$$

For each nuclear species  $i$  one can write such an equation, describing the composition change at a particular mass shell inside the star (with density  $\rho$  and temperature  $T$ ) resulting from nuclear reactions. In the presence of internal mixing (in particular of convection) the redistribution of composition between different mass shells should also be taken into account.

Note the **similarity between the expressions for the nuclear energy generation rate and the equation for composition changes**, both of which are proportional to  $r_{ij}$ . Combining them together, we can obtain a useful expression for a simple case where only one reaction occurs, or a reaction chain in which one reaction determines the overall rate.

An example is the fusion of 4  $^1\text{H}$  into  $^4\text{He}$ . One can show that

$$\frac{dY}{dt} = - \frac{dX}{dt} = \frac{\varepsilon_H}{q_H}$$

$$\varepsilon_{ij} = \frac{Q_{ij} r_{ij}}{\rho} = \frac{Q_{ij}}{(1 + \delta_{ij}) A_i A_j m_H^2} \rho X_i X_j \langle \sigma v \rangle_{ij}$$

where  $\varepsilon_H$  is the energy generation rate by the complete chain of H-burning reactions, and  $q_H$  is amount of energy produced by converting 1 gram of 4  $^1\text{H}$  into  $^4\text{He}$ .

# Summary of lectures on nuclear reactions

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Recalling that we wanted to determine the physics governing  $\varepsilon$  will be defined by the nuclear energy source in the interiors. We needed to develop a theory and understanding of nuclear physics and reactions.

- We have covered the basic principles of energy production by fusion.
- The **PP chain** and **CNO** cycle have been described.
- He burning by the **triple-alpha** reaction was introduced.
- Later burning stages of the heavier elements (C,O, Si) were shortly discussed.
- The *r*- and *s*-processes – origin of the elements heavier than Fe.

# Solution of the Equations of Stellar Structure

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THE STELLAR STRUCTURE EQUATIONS AND HOW TO SOLVE THEM?

SIMPLE STELLAR MODELS

POLYTROPIC MODELS

LANE-EMDEN EQUATION

RELATIONSHIPS FOR POLYTROPIC STARS

CHANDRASEKHAR MASS

DYNAMICAL STABILITY OF STARS

# Introduction

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We have now derived all the four differential equations and the three additional functions that, together with boundary conditions, define uniquely the **equilibrium** properties of a star of a given mass and composition.

- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$
- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2} \rho(r)$
- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r) \varepsilon(r)$
- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma r^2} \frac{\rho(r)\kappa_R(r)}{T^3(r)} L(r)$
- $\frac{P}{T} \frac{dT}{dP} = \frac{\gamma-1}{\gamma}$

Energy transport

by radiation

by convection

- $r$  = radius
- $P$  = pressure at  $r$
- $m$  = mass of material within  $r$
- $\rho$  = density at  $r$
- $L$  = luminosity at  $r$
- $T$  = temperature at  $r$
- $\kappa_R$  = Rosseland mean opacity at  $r$
- $\varepsilon$  = energy release

Three supplement equations:

$$P = P(\rho, T, \text{chemical composition}) - \mathbf{EOS}$$

$$\kappa_R = \kappa_R(\rho, T, \text{chemical composition})$$

$$\varepsilon = \varepsilon(\rho, T, \text{chemical composition})$$

Plus, the equation of composition changes:

$$\frac{dX_i}{dt} = A_i \frac{m_H}{\rho} \left( -\sum_j (1 + \delta_{ij}) r_{ij} + \sum_{k,l} r_{kl,i} \right)$$

# How to solve the Stellar Structure equations?

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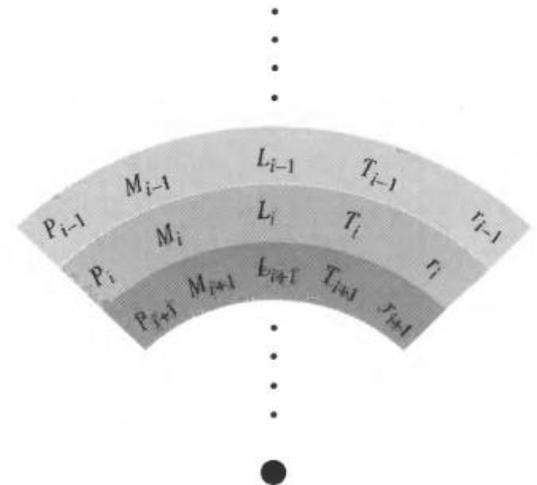
- “Solving” this system of coupled equations means finding the functions  $P(r)$ ,  $T(r)$ , and  $\rho(r)$ , which are the ones that are usually considered to describe the structure of the star.
- **The Vogt-Russell theorem:**  
“The mass and the composition structure throughout a star **uniquely** determine its radius, luminosity, and internal structure, as well as its subsequent evolution.”
  - This “theorem” has not been proven and is not even rigorously true; there are known exceptions. However, an actual star would probably adopt one unique structure as a consequence of its evolutionary history. In this sense, the Vogt-Russell “theorem” should be considered a general rule rather than a rigorous law.
- Unfortunately, unless some unrealistic assumptions are made, there is no analytic solution to the equations, given the complicated nature of the functions  $P$ ,  $\kappa$ , and  $\varepsilon$  when all relevant processes are included. Because the complete set of equations with two-point boundary values is highly non-linear and time-dependent, their full solution requires a complicated numerical procedure. This is what is done in detailed stellar evolution codes, the results of which we will discuss in the following few lectures.

# Numerical modeling of the equations

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We will not go into any detail about the numerical methods commonly used in such codes, but shortly they can be described as the following:

- In a numerical solution, the differentials in the equations are replaced by differences. For instance, by replacing  $dP/dr$  by  $\Delta P/\Delta r$ . The star is then imagined to be constructed of spherically symmetric shells.
- The numerical integration of the stellar structure equations may be carried out shell by shell from the surface toward the center, from the center toward the surface, or, as is often done, in both directions simultaneously.
- If the integration is carried out in both directions, the solutions will meet at some fitting point where the variables must vary smoothly from one solution to the other.
- Simultaneously matching the surface and central boundary conditions for a desired stellar model usually requires several **iterations** before a satisfactory solution is obtained. If the surface-to-center and center-to-surface integrations do not agree at the fitting point, the starting conditions must be changed. The initial conditions of the next integration are estimated from the outcome of the previous integration.



# Simple stellar models

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- Although today the standard approach is to hand the problem to a computer, insight into the structure of stars may be gained both by analyzing the equations, without actually solving them, and by seeking simple solutions based on additional simplifying assumptions.
- The main purpose of this lecture is to briefly analyze the differential equations of stellar evolution and their boundary conditions, and to see how the full set of equations can be simplified in some cases to allow simple or approximate solutions – so-called simple stellar models. We will concentrate on **polytropic models**.
- Although nowadays their practical use has mostly been superseded by more realistic stellar models, due to their simplicity **polytropic** models still give useful insight into several important properties of stars. Moreover, in some cases the **polytropic** relation is a good approximation to the real equation of state
- As the very first simplification, we assume that a star is in both **hydrostatic** and **thermal equilibrium**. In this case, the four partial differential equations for stellar structure reduce to ordinary, **time-independent** differential equations.

# Simplifying assumptions

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- The four equations of stellar structure divide naturally into two groups:
  - one describing the mechanical structure of the star  $\frac{dm}{dr} = 4\pi r^2 \rho(r); \frac{dP(r)}{dr} = -\frac{Gm}{r^2} \rho(r)$
  - and the other giving the thermal structure.  $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r) \epsilon(r); \frac{P}{T} \frac{dT}{dP} = \frac{\gamma-1}{\gamma}$
- However, the only contact between the mechanical variables and thermal equations is through the temperature dependence of the equation of state.
- If we can write the pressure in terms of the density alone, **without reference to the temperature**, then we can separate these two equations from the others and solve them by themselves. Solving two differential equations (plus one algebraic equation relating  $P$  and  $\rho$ ) is much easier than solving seven equations.
- We have already seen, that under certain circumstances, the pressure can indeed become independent of temperature, and only depend on density, i.e., **degeneracy pressure**, or the case where pressure and density are related adiabatically (**convection**).
- In the above examples we derived a relation of the form  $P = K\rho^\gamma = K\rho^{1+\frac{1}{n}}$  where  $K$  and  $\gamma$  are constants; this is called **a polytropic relation**, and the resulting models are called **polytropic models**.

# Polytropic models

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When the equation of state can be written in this form, the temperature does not enter at all into the equations and the calculations of stellar structure simplify enormously. There are even analytical solutions for certain values of  $\gamma$ .

- If we then take the equation for hydrostatic support, multiply it by  $r^2/\rho$ , differentiate with respect to  $r$ , and then divide by  $r^2$ , we get:

$$\left. \begin{aligned} \frac{r^2}{\rho} \frac{dP}{dr} &= -Gm & \rightarrow & \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -G \frac{dm}{dr} \\ \frac{dm}{dr} &= 4\pi r^2 \rho \end{aligned} \right\} \frac{1}{r^2} \frac{d}{dr} \left( \frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G \rho$$

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2} \rho; \frac{dm}{dr} = 4\pi r^2 \rho$$

What we have done is **exact**. Now we make our approximation. We approximate that the pressure and density are related by a power-law  $P = K\rho^\gamma = K\rho^{1+\frac{1}{n}}$  (it customary to adopt  $\gamma=1+1/n$ , or  $n=1/(1-\gamma)$ , where  $n$  is the polytropic index):

$$\frac{K(n+1)}{r^2 n} \frac{d}{dr} \left( \frac{r^2}{\rho} \rho^{1/n} \frac{d\rho}{dr} \right) = -4\pi G \rho$$

# Lane-Emden equation

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$$\frac{K(n+1)}{4\pi Gn} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \rho^{1/n-1} \frac{d\rho}{dr} \right) = -\rho$$

The solution  $\rho(r)$  for  $0 \leq r \leq R$  is called a polytrope and requires two boundary conditions.

Hence a polytrope is uniquely defined by three parameters:  $K$ ,  $n$ , and  $R$ . This enables calculation of additional quantities as a function of radius, such as pressure, mass or gravitational acceleration.

Let's define a dimensionless variable  $\theta$  in the range  $0 \leq \theta \leq 1$  by  $\rho = \rho_c \theta^n$ , where  $\rho_c$  is the central density. Then the equation becomes

$$\frac{K(n+1)\rho_c^{1/n-1}}{4\pi G} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\theta}{dr} \right) = -\theta^n$$

To simplify the equation further, we introduce the dimensionless radius  $\xi = r/\alpha$ , where

$$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G} \quad \leftarrow \text{constant having the dimension of length squared!}$$

The equation finally becomes

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n$$

This equation is called the *Lane-Emden equation*, and the solution  $\theta = \theta_n(\xi)$  is called the *Lane-Emden function*.

# Solving the Lane-Emden equation

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$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n$$

$$\xi = r/\alpha$$

$$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}$$

$$P = K\rho^\gamma = K\rho^{1+1/n}$$

$$\rho = \rho_c \theta^n$$

Since it is a second order differential equation, we need two boundary conditions.

The 1st is at the center: from spherical symmetry, the pressure gradient at the center ( $\theta = 1$ ) must be zero.

The 2nd condition comes from the surface,  $\xi = \xi_1$ , where the density should go to zero.

So, our boundary conditions are  $\frac{d\theta}{d\xi} = 0$ ,  $\theta = 1$  at  $\xi=0$  (the center), and  $\theta = 0$  at  $\xi = \xi_1$  (the surface).

Solving the equation for the dimensionless function  $\theta_n(\xi)$  in terms of  $\xi$  for a specific polytropic index  $n$  leads directly to the profile of density with radius  $\rho_n(r)$ . The polytropic equation of state provides the pressure profile. In addition, if the ideal gas law and radiation pressure are assumed for constant composition, then the temperature profile,  $T(r)$ , is also obtained.

$$P = \frac{\mathfrak{R}T\rho}{\mu} + \frac{aT^4}{3}$$

Unfortunately, the Lane-Emden equation does not have an analytic solution for arbitrary values of  $n$ .

In fact, there are only three analytic solutions, namely  $n=0$ ,  $1$ , and  $5$ :

$$n = 0, \theta = 1 - \frac{\xi^2}{6} \quad \xi_1 = \sqrt{6}$$

$$n = 1, \theta = \frac{\sin \xi}{\xi} \quad \xi_1 = \pi$$

$$n = 5, \theta = \left( 1 + \frac{\xi^2}{3} \right)^{-1/2} \quad \xi_1 \rightarrow \infty$$

Solutions for all other values of  $n$  must be solved **numerically**.

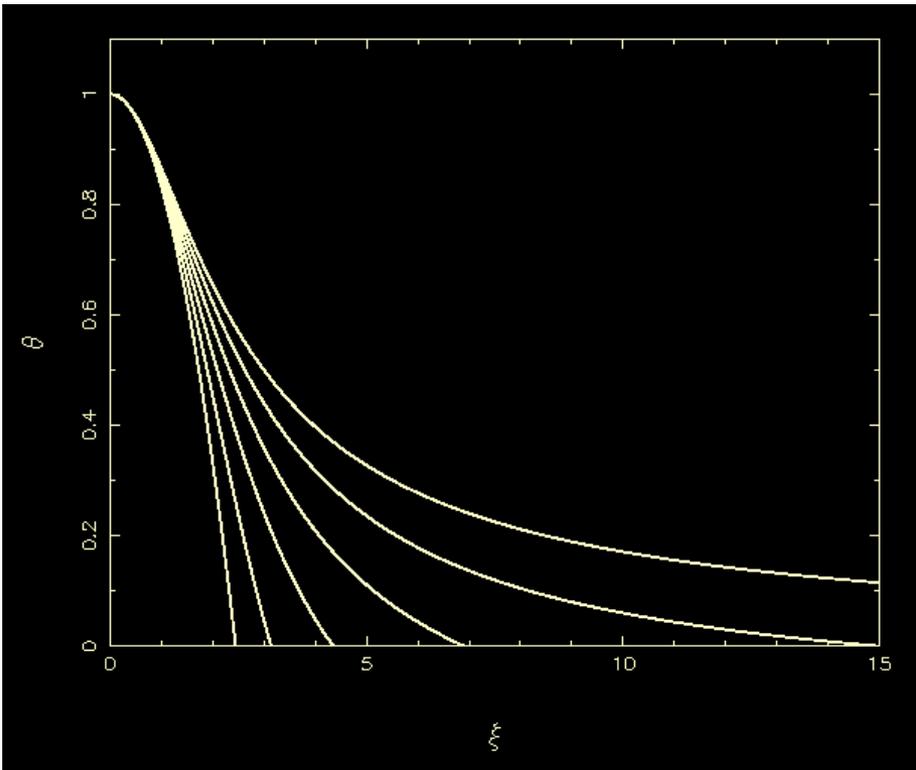
1. Solve the Eqn for  $n=0$

2. Find the dimensionless radius of these polytropic stars

# Solutions of the Lane-Emden equation

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Numerical solutions to the Lane-Emden equation for (left-to-right)  $n = 0, 1, 2, 3, 4, 5$ . Some key values resulting from the integration are shown in the table.



$n$	$\xi_1$	$-\xi_1^2 \left( \frac{d\theta}{d\xi} \right)_{\xi_1}$	$\rho_c / \bar{\rho}$
0.0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800.
5.0	$\infty$	1.73205	$\infty$

Solutions decrease **monotonically** and have  $\theta=0$  at  $\xi = \xi_1$  (i.e. the stellar radius).  
With **increasing** polytropic index, the star becomes more centrally **condensed**.

# Polytropic stars

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The surface radius of the polytropic model is

$$R = \left( \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G} \right)^{1/2} \xi_1$$

$$\xi = r/\alpha$$

$$\rho = \rho_c \theta^n$$

$$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}$$

The total mass  $M$  of a polytropic star is given by

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi \alpha^3 \rho_c \int_0^{\xi_1} \xi^2 \theta^n d\xi = -4\pi \alpha^3 \rho_c \int_0^{\xi_1} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) d\xi = -4\pi \alpha^3 \rho_c \left[ \xi^2 \left( \frac{d\theta}{d\xi} \right) \right]_{\xi_1}$$

From a polytropic model, we can derive other useful numbers and relationships. As one example, it is often convenient to know how centrally concentrated a star is, i.e. how much larger its central density is than its mean density. We define this quantity as

$$D_N \equiv \frac{\rho_c}{\bar{\rho}} = \frac{\rho_c 4\pi R^3}{3M} = \frac{4\pi}{3} \rho_c (\alpha \xi_1)^3 \left[ -4\pi \alpha^3 \rho_c \xi_1^2 \left( \frac{d\theta}{d\xi} \right)_{\xi_1} \right]^{-1} = \left[ -\frac{3}{\xi_1} \left( \frac{d\theta}{d\xi} \right)_{\xi_1} \right]^{-1}$$

Values in Table  
in slide 266

# Mass-Radius relationship for polytropic stars

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Another useful relationship is between mass and radius. We start by expressing the central density  $\rho_c$  in terms of the other constants and our length scale  $\alpha$ :

$$\rho_c = \left[ \frac{K(n+1)}{4\pi G \alpha^2} \right]^{n/(n-1)}$$

$$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}$$

Substitute this into the equation for the mass:

$$M = -4\pi\alpha^3 \rho_c \xi_1^2 \left( \frac{d\theta}{d\xi} \right)_{\xi_1} = -4\pi\alpha^3 \left[ \frac{K(n+1)}{4\pi G \alpha^2} \right]^{n/(n-1)} \xi_1^2 \left( \frac{d\theta}{d\xi} \right)_{\xi_1}$$

Making the substitution  $\alpha = R/\xi_1$  and re-arranging, we arrive at

$$\left[ \frac{GM}{-\xi_1^2 (d\theta/d\xi)_{\xi_1}} \right]^{(n-1)} \left( \frac{R}{\xi_1} \right)^{3-n} = \frac{[K(n+1)]^n}{4\pi G}$$

$n = 1$  is a special case, for which the radius is independent of mass and is uniquely determined by  $K$ :

$$R = \xi_1 \left( \frac{K}{2\pi G} \right)^{1/2}$$

Another important polytropic index is  $n=3$ , for which the  $R$  dependence disappears.

We find that

$$M = -\frac{4}{\sqrt{\pi}} \xi_1^2 \left( \frac{d\theta}{d\xi} \right)_{\xi_1} \left( \frac{K}{G} \right)^{3/2}$$

# *M-R* relationship for polytropic stars

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For other  $n$ , mass and radius are related by  $M \sim R^{(n-3)/(n-1)}$ .

Note what this means: for a polytropic index of  $n=1.5$  (the  $\gamma = 5/3$  case),  $R \sim M^{-1/3}$ . Thus, for a set of stars with the same  $K$  and  $n$  (i.e., white dwarfs), the stellar radius is inversely proportional to the mass. We will use it a few slides later.

# Eddington standard model (1)

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Another important polytropic index is  $n=3$ , "Eddington standard model" associated with a star in radiative equilibrium (slide 208). The contribution to the total pressure at a certain location in the star due to an ideal gas is given by

$$P = \frac{\mathfrak{R}T\rho}{\mu} + \frac{aT^4}{3}$$

$$P_g = \frac{\mathfrak{R}T\rho}{\mu} = \beta P$$

$$\mathfrak{R} = \frac{k}{m_p}$$

Then the contribution due to radiation pressure is

$$P_r = \frac{aT^4}{3} = (1 - \beta)P$$

Combining both equations to eliminate  $T$  we get:

$$P^3 = \frac{3(1 - \beta)}{a} \left( \frac{\mathfrak{R}\rho}{\mu\beta} \right)^4$$

This leads immediately to an expression for the total pressure in terms of the density, namely

$$P = K\rho^\gamma \quad \text{where} \quad K \equiv \left[ \frac{3(1 - \beta)}{a} \right]^{1/3} \left( \frac{\mathfrak{R}}{\mu\beta} \right)^{4/3}, \quad \gamma = 4/3, \quad \text{and} \quad n = 3$$

Thus, we have obtained a polytropic equation of state of index 3, which implies a unique relation between  $K$  and  $M$ . The Eddington quartic equation is

$$1 - \beta = 0.003 \left( \frac{M}{M_{sun}} \right)^2 \mu^4 \beta^4$$

# Eddington standard model (2)

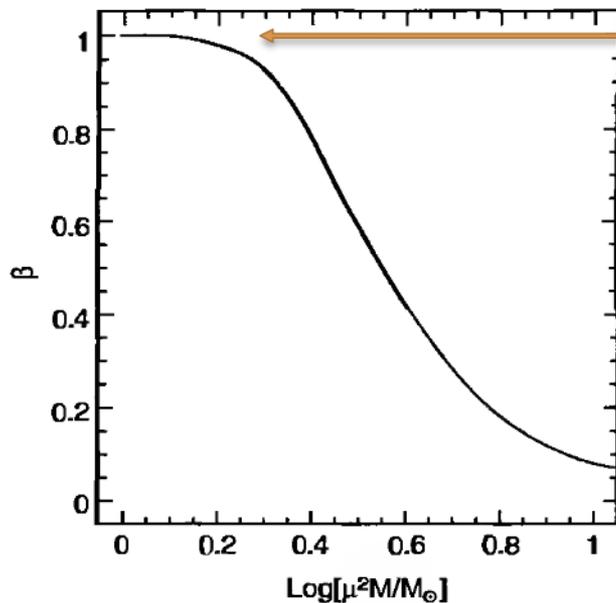
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$$P = K\rho^\gamma \quad \text{where} \quad K \equiv \left[ \frac{3(1-\beta)}{a} \right]^{1/3} \left( \frac{\mathfrak{R}}{\mu\beta} \right)^{4/3}, \quad \gamma = 4/3, \quad \text{and} \quad n = 3$$

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$$M = -\frac{4}{\sqrt{\pi}} \xi_1^2 \left( \frac{d\theta}{d\xi} \right)_{\xi_1} \left( \frac{K}{G} \right)^{3/2}$$



pure gas pressure

pure radiation pressure