The speed-averaged cross-section

- The number of reactions per second in a unit volume is

$$r_{ij} = \frac{n_i n_j}{1 + \delta_{ij}} \sigma v$$

where n_i and n_j are number densities of particles, and v is a relative velocity.
If a reaction between identical particles is considered (i.e., protons on protons) then r needs to be divided by 2, to avoid double counting , i.e. $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$

- In reality, the nuclei in a gas will have a distribution of velocities (a range of kinetic energies), so every velocity has some probability of occurring. Hence (after some algebra, which we will not describe)

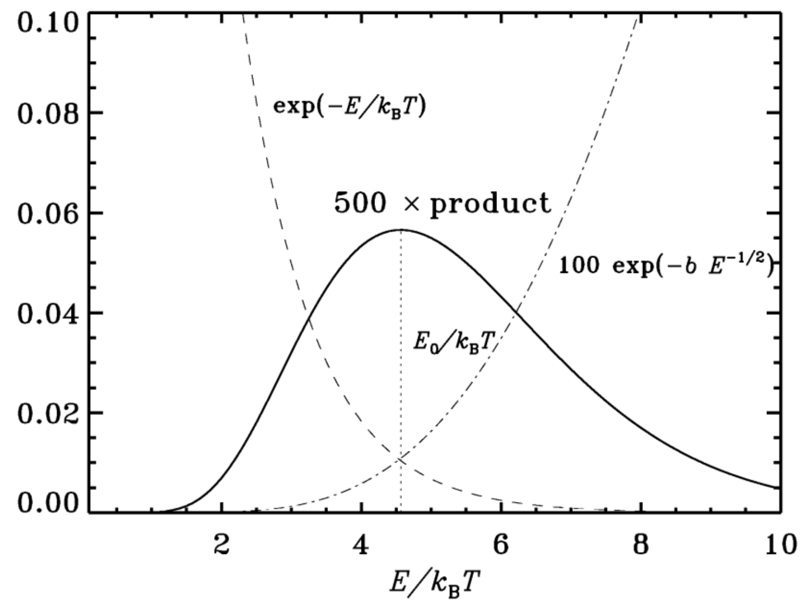
$$\langle \sigma v \rangle = \left(\frac{8}{\pi m_r} \right)^{1/2} \frac{S(E_0)}{(kT)^{3/2}} \int_0^\infty e^{-E/kT} e^{-\sqrt{E_G/E}} dE$$

- The integrand in this expression, $f(E) = e^{-E/kT} e^{-\sqrt{E_G/E}}$, is composed of the product of two exponential functions, one (from the Boltzmann distribution) falling with energy, and the other (due to the Gamow factor embodying the Coulomb repulsion) rising with energy. Obviously, $f(E)$ will have a narrow maximum at some energy E_0, at which most of the reactions take place:

$$E_0 = \left(\frac{kT}{2} \right)^{2/3} E_G^{1/3}$$

The Gamow peak

- The Gamow peak is the product of the Maxwellian distribution and tunnelling probability. The area under the Gamow peak determines the reaction rate.
- The higher the electric charges of the interacting nuclei, the greater the repulsive force, hence the higher the E_k and T are needed for reaction to occur.
- Highly charged nuclei are obviously the more massive, so reactions between light elements occur at lower T than reactions between heavy elements.



The Boltzmann probability distribution for $kT = 1$ keV, the Gamow factor for the case of two protons, with $E_G = 500$ keV, and their product. Scaled up by large factors for display purposes.

Nuclear Reaction Rates

$Q \equiv [Zm_p + Nm_n - m(Z,N)]c^2$

- Each reaction releases an amount of energy Q_{ij}, so then $Q_{ij}r_{ij}$ is the energy generated per unit volume and per second. The energy generation rate per unit mass from the reaction between nuclei of type i and j is then

$n_i = \frac{X_i \rho}{A m_H}$

$$\varepsilon_{ij} = \frac{Q_{ij} r_{ij}}{\rho} = \frac{Q_{ij}}{(1+\delta_{ij}) A_i A_j m_H^2} \rho X_i X_j \langle \sigma v \rangle_{ij}$$

$r_{ij} = \frac{n_i n_j}{1+\delta_{ij}} \langle \sigma v \rangle$

- The final expression for the power density due to a given nuclear reaction:

$$\varepsilon_{ij} = \frac{2^{\frac{5}{3}} \sqrt{2}\, Q_{ij}}{(1+\delta_{ij})\sqrt{3}} \frac{\rho X_i X_j}{m_H^2 A_i A_j \sqrt{m_r}} S_0 \frac{E_G^{1/6}}{(kT)^{2/3}} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{\frac{1}{3}}\right]$$

where $S_0 = S(E_0)$.

- The total power density at a point in a star with a given temperature, density, and abundance will be the sum of the power densities due to all the possible nuclear reactions, each described by this equation.
- Because of the exponential term in the Eqn, there will be a strong preference for reactions between species with low atomic number, and hence small E_G.
- Furthermore, the higher the Gamow energy, the more strongly will the reaction rate depend on temperature.

Timescale of a nuclear reaction

- The typical timescale of a nuclear reaction is inversely proportional to the reaction rate r_{ij}. The mean time it takes for a particular nucleus of type i to undergo fusion with a nucleus of type j is

$$\tau_i = \frac{n_i}{(1+\delta_{ij})\, r_{ij}}$$

- The extremely high sensitivity of nuclear reaction rates to temperature leads to the concept of "ignition" of a nuclear fuel: each reaction (or nuclear process) has a typical narrow temperature range over which its rate increases by orders of magnitudes, from negligible values to very significant ones.
- Around this range, the temperature dependence of the reaction rate may be well approximated by a power law (with a high power) and an ignition or threshold temperature may be defined. Hence, the equation of the power density ε should characteristically have

$$\varepsilon \approx \varepsilon_0 (T/T_0)^n$$

  H-burning: n = 5÷15; He-burning n = 40.
- The process of creation of new nuclear species by fusion reactions is called nucleosynthesis. Since the kinetic energy of particles is that of their thermal motion, the reactions between them are called thermonuclear.

Electron shielding (1)

- We found that the repulsive Coulomb force between nuclei plays a crucial role in determining the rate of a thermonuclear reaction. In our derivation of the cross-section, we have ignored the influence of the surrounding free electrons, which provide overall charge neutrality in the gas.
- In a dense medium, the attractive Coulomb interactions between atomic nuclei and free electrons cause each nucleus to be effectively surrounded by a cloud of electrons. This electron cloud reduces the Coulomb repulsion between the nuclei at large distances, and may thus increase the probability of tunneling through the Coulomb barrier. This effect is known as **electron screening** or **electron shielding**.

Electron shielding (2)

- According to the weak screening approximation, which applies to relatively low densities and high temperatures such as found in the centre of the Sun and other main-sequence stars, clouds of negatively charged electrons can increase r_{jk} by about 10%.
- The description of electron screening becomes complicated at high densities and relatively low temperatures, where the weak screening approximation is no longer valid. A general result is that with increasing strength of electron screening, the temperature sensitivity of the reaction rate diminishes, and the density dependence becomes stronger. At very high densities, $\rho > 10^6$ g/cm^3, the screening effect is so large that it becomes the dominating factor in the reaction rate.

Nuclear reactions in stellar interiors

ENERGY GENERATION
PP-CHAINS
CNO-CYCLE
HELIUM BURNING
CARBON BURNING AND BEYOND
IRON AND HEAVIER ELEMENTS
COMPOSITION CHANGES

Notations for nuclear reactions

The general description of a nuclear reaction is

- $I(A_i, Z_i) + J(A_j, Z_j) \rightarrow K(A_k, Z_k) + L(A_l, Z_l)$
- $I(A_i, Z_i) + J(A_j, Z_j) \rightarrow K(A_k, Z_k) + L(A_l, Z_l) + e^+ + \nu$
- $I(A_i, Z_i) + J(A_j, Z_j) \rightarrow K(A_k, Z_k) + L(A_l, Z_l) + \gamma$

e^+ – positron, γ – photon, ν – neutrino

Recall that in any nuclear reaction the following must be conserved:

- The baryon number – heavy particles (protons, neutrons and their anti-particles)
- The lepton number – light particles (electrons, positrons, neutrinos, and antineutrinos)
- Charge

Note also that the anti-particles have the opposite baryon/lepton number to their particles.

Examples of nuclear reactions

Deuterium is a stable isotope of hydrogen, which, unlike "normal" hydrogen atoms, also contains a neutron. The nucleus of a deuterium atom, called a deuteron, contains one proton and one neutron.

1. The first, and most obvious reaction is

$$^{1}H + {}^{1}H \rightarrow {}^{2}D + e^{+} + \nu$$

S_0 factor for this reaction is 22 (!) orders of magnitude smaller than that for other reactions. It proceeds via weak, not strong, interaction. Therefore, reaction proceeds extremely slowly.
Note that all three conservation laws are obeyed – baryon number, lepton number, and charge.

2. $$^{1}H + {}^{2}D \rightarrow {}^{3}He + \gamma$$

Fast reaction (strong interaction), the first reaction when gas is getting hotter.
But the abundance of deuterium is extremely small, and it burns away rapidly.

3. Reaction $^{2}D + {}^{2}D \rightarrow$ is not important, as D burns on H faster (τ smaller and abundance of H is larger)

4. $$^{1}H + {}^{4}He \rightarrow ?$$

There are no nuclei with A=5 in nature. H does not burn on He.

The main nuclear burning cycles

In principle, many different nuclear reactions can occur simultaneously in a stellar interior.

For a very precise analysis (i.e., for deriving the detailed isotopic abundances), a large network of reactions must be calculated.

However, for the calculation of the structure and evolution of a star usually a much simpler procedure is sufficient, for the following reasons:

- The very strong dependence of nuclear reaction rates on the temperature, combined with the sensitivity to the Coulomb barrier $Z_1 Z_2$, implies that nuclear fusions of different possible fuels – hydrogen, helium, carbon, etc. – are well separated by substantial temperature differences.
- The evolution of a star therefore proceeds through several distinct nuclear burning cycles. For each nuclear burning cycle, only a handful of reactions contribute significantly to energy production and/or cause major changes to the overall composition.
- In a chain of subsequent reactions, often one reaction is by far the slowest and determines the rate of the whole chain. Then only the rate of this bottleneck reaction needs to be taken into account.

Hydrogen burning

The most important series of fusion reactions are those converting H to He (H-burning). This dominates ~90% of lifetime of nearly all stars. Since a simultaneous reaction between four protons is extremely unlikely, a chain of reactions is always necessary for hydrogen burning. Hydrogen burning in stars takes place at temperatures ranging between 8×10^6 K and 50×10^6 K, depending on stellar mass and evolution stage.

We will consider here the main ones: the **PP-chain** and the **CNO cycle**.

Let's first discuss the most obvious, so-called p-p reaction: $^1H + {^1H} \rightarrow {^2H}$

- The resulting particle would be, however, unstable and it would immediately disintegrate back into two separate protons.
- Hans Bethe in 1939: The Pauli principle prevents two protons with parallel spins from occupying the same position, and even a very close approach between them (10^{-13} cm) becomes improbable.
- For this reaction to proceed, one proton should be transformed to neutron (weak force). However, $m_n > m_p$, energy is taken from binding energy of deuterium (2.24 MeV).
- On distance of 10^{-13} cm, the particles exist for just 10^{-21} s, and in that time β-decay should occur.
- Probability is very small, and cross-section is extremely small, 10^{-47} cm^2, impossible to measure in the laboratories. Computations give S-factor:

$$S_{pp}(E_0) = S_0 = 3.88 \times 10^{-22} \text{ keV barn} \quad (1 \text{ barn} = 10^{-24} \text{ cm}^2).$$

$$\sigma = \frac{S(E)}{E} g(E)$$

- Therefore, the following reaction proceeds extremely slowly:

$$^1H + {^1H} \rightarrow {^2D} + e^+ + \nu \qquad (\text{ or } \quad p + p \rightarrow d + e^+ + \nu\ )$$

You will show at home that the typical time a proton has to wait until it reacts with another proton $\tau_{pp} \sim 2 \times 10^{10}$ yr, the lifetime of the Sun. But this is sufficient to power the Sun.

The p-p chains

The first reaction is the p-p reaction. After some deuterium is produced, it rapidly reacts with another proton to form ^3He. Subsequently three different branches are possible to complete the chain towards ^4He:

$$p + p \rightarrow d + e^+ + \nu$$
$$d + p \rightarrow {}^3He + \gamma$$

$$^3He + {}^3He \rightarrow {}^4He + 2\,p$$

pp1

$$^3He + {}^4He \rightarrow {}^7Be + \gamma$$

$$^7Be + e^- \rightarrow {}^7Li + \nu$$
$$^7Li + p \rightarrow {}^4He + {}^4He$$

pp2

$$^7Be + p \rightarrow {}^8B + \gamma$$
$$^8B \rightarrow {}^8Be + e^+ + \nu \quad \leftarrow \beta \text{ decay}$$
$$^8Be \rightarrow {}^4He + {}^4He$$

pp3

The p-p chains

$$p + p \rightarrow d + e^+ + \nu$$
$$d + p \rightarrow {}^3He + \gamma$$

$${}^3He + {}^3He \rightarrow {}^4He + 2\,p$$

pp1

$${}^3He + {}^4He \rightarrow {}^7Be + \gamma$$

$${}^7Be + e^- \rightarrow {}^7Li + \nu$$
$${}^7Li + p \rightarrow {}^4He + {}^4He$$

pp2

$${}^7Be + p \rightarrow {}^8B + \gamma$$
$${}^8B \rightarrow {}^8Be + e^+ + \nu \quad \leftarrow \textit{β decay}$$
$${}^8Be \rightarrow {}^4He + {}^4He$$

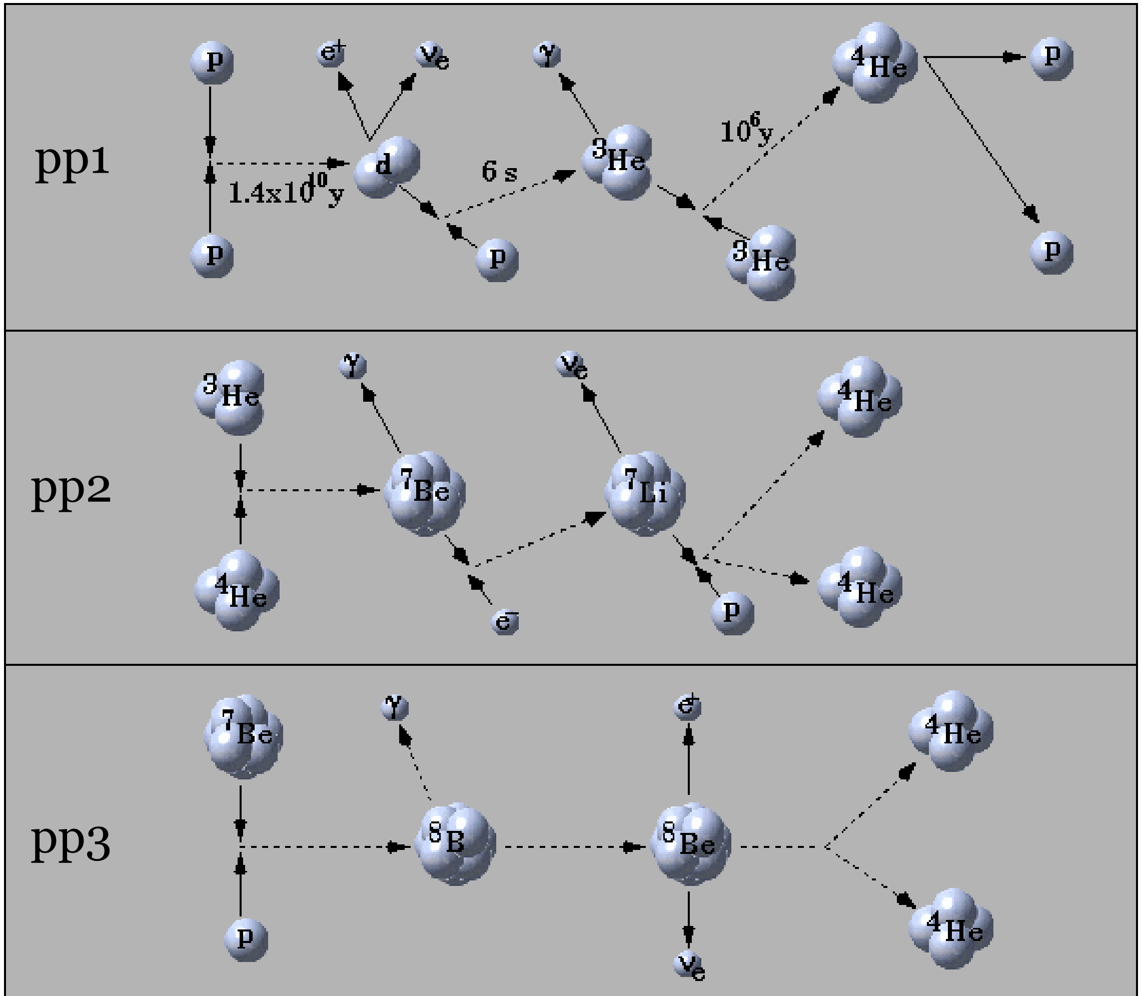
pp3

The **pp1** branch requires two ^3He nuclei, so the first two reactions in the chain have to take place twice.
The alternative **pp2** and **pp3** branches require only one ^3He nucleus and an already existing ^4He nucleus (either present primordially or produced previously by hydrogen burning).
The resulting ^7Be nucleus is unstable and can either capture an electron or fuse with another proton, giving rise to the second branching into **pp2** and **pp3**.

Three of the reactions in the chains are accompanied by neutrino emission, and the (average) neutrino energy is **different** in each case.

Therefore, the total energy release Q_H to produce one ^4He nucleus is different for each chain:
26.20 MeV (**pp1**), 25.66 MeV (**pp2**) and only 19.76 MeV for **pp3**.

Importance of different branches of the p-p chains



The relative importance of the **pp1** and **pp2** chains (branching ratios) depends on conditions of H-burning (T, ρ, abundances). The transition from **pp1** to **pp2** occurs at temperatures in excess of 1.3×10^7 K. Above 3×10^7 K the **pp3** chain dominates over the other two, but another process takes over in this case.

In **pp2** and **pp3** ^{4}He plays a role of a catalyst, it accelerates synthesis of itself. Abundance of ^{4}He changes, and therefore, the role of these branches grows even for T=const. **pp1** needs two pp-reactions, and therefore, is slow.

The overall rate of the whole reaction chain is set by the rate of the bottleneck p-p reaction, r_{pp}. In this steady-state or "equilibrium" situation the rate of each subsequent reaction adapts itself to the pp rate.

The **pp3** chain is never very important for energy generation, but it does generate abundant high energy neutrinos.

Neutrino emission

- A neutrino is released by weak interactions ($p \rightarrow n + e^{+} + \nu$), which escape without interacting with the stellar matter. It is customary **not** to include the neutrino energies into the overall energy release Q, but to take into account only the energy that is used to heat the stellar gas. This includes energy released in the form of γ–rays (including the γ-rays resulting from pair annihilation after e^{+} emission) and in the form of kinetic energies of the resulting nuclei.
- Thus, the effective Q-value of hydrogen burning is therefore somewhat smaller than 26.734 MeV and depends on the reaction in which the neutrinos are emitted.
- Assuming that neutrinos take away a small fraction of energy and knowing the solar luminosity, we can get the total formation rate of helium.

- In fact, it is these neutrinos that directly confirm the occurrence of nuclear reactions in the interior of the Sun. **No other direct observational test of nuclear reactions is possible.** The mean neutrino energy is ~0.26 MeV for a deuterium creation (**pp1 /2**) and ~7.2 MeV for β decay (**pp3**). But as **pp3** is negligible, the energy released for each He nucleus assembled is ~26 MeV (or 6×10^{18} erg g^{-1})

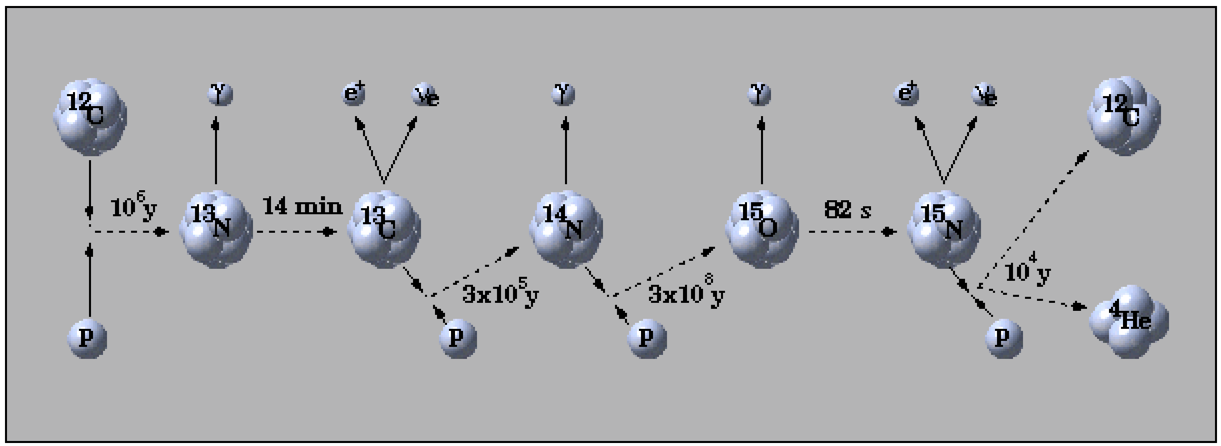
The CNO Cycle

At birth (most) stars contain a small (2%) mix of heavy elements, some of the most abundant of which are carbon, nitrogen and oxygen (CNO). If the temperature is sufficiently high, these nuclei may induce a chain of H-burning reactions in which they act as catalysts. The process is known as the CNO Cycle. This is a cyclical sequence of reactions that typically starts with a proton capture by a ^{12}C nucleus:

$$^{12}C + p \rightarrow {}^{13}N + \gamma$$
$$^{13}N \rightarrow {}^{13}C + e^+ + \nu$$
$$^{13}C + p \rightarrow {}^{14}N + \gamma$$
$$^{14}N + p \rightarrow {}^{15}O + \gamma$$
$$^{15}O \rightarrow {}^{15}N + e^+ + \nu$$
$$^{15}N + p \rightarrow {}^{12}C + {}^{4}He$$

$$\rightarrow {}^{16}O + \gamma$$
$$^{16}O + p \rightarrow {}^{17}F + \gamma$$
$$^{17}F \rightarrow {}^{17}O + e^+ + \nu$$
$$^{17}O + p \rightarrow {}^{14}N + {}^{4}He$$

a small probability ($<10^{-3}$)



Main cycle (CN cycle). Carbon is a catalyst in this cycle because it is not destroyed by its operation and it must be present in the original material of the star for the CNO cycle to operate. At high enough temperatures, $T \geq 1.5\times10^7$ K, all reactions in the cycle come into a steady state or "equilibrium" where the rate of production of each nucleus equals its rate of consumption. In this situation, the speed of the whole CNO cycle is controlled by the slowest reaction which is the capture of a proton by ^{14}N. As a result, most of ^{12}C is converted to ^{14}N before the cycle reaches equilibrium and this is the source of most of the nitrogen in the Universe.

Temperature dependence of PP chain and CNO Cycle

The rates of two reactions p+p → and ^{14}N + p → have very different temperature dependences:

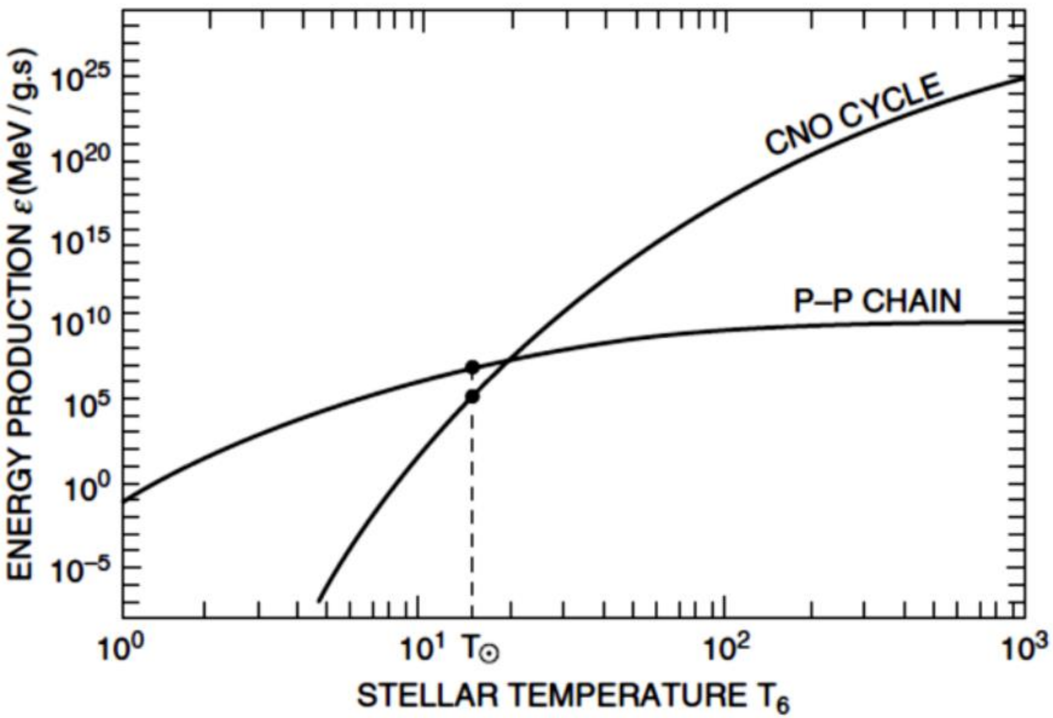
$$\varepsilon_{pp} \propto \left(\frac{T}{10^7 K}\right)^{4.53}$$

$$\varepsilon_{CNO} \propto \left(\frac{T}{2.5 \times 10^7 K}\right)^{16.7}$$

The rates are about equal at

$T{\approx}1.7{\times}10^7$ K

Below this temperature the **pp** chain is most important, and above it the **CNO** cycle dominates. This occurs in stars slightly more massive than the Sun, $1.2{\div}1.5 M_{\odot}$.



The **pp** chain is the least temperature-sensitive of **all** nuclear burning cycles.

Helium Burning: the triple α-reaction

When there is no longer any hydrogen left to burn in the central regions of a star, gravity compresses the core until the temperature T_c reaches the point where helium burning reactions become possible.

Simplest reaction in a helium gas should be the fusion of two helium nuclei, e.g. $\mathbf{^4He + {}^4He \rightleftharpoons {}^8Be}$

However, there is no stable configuration with A=8! For example, the beryllium isotope ^{8}Be has a lifetime of only 2.6×10^{-16} s and it rapidly decays to two ^{4}He nuclei again. While extremely short, this time is long enough to build up a very small equilibrium concentration of ^{8}Be, which increases with temperature and reaches about 10^{-9} at $T \approx 10^8$ K. Thus, a third helium nucleus can be added to ^{8}Be before it decays, forming ^{12}C by the so-called triple-alpha reaction:

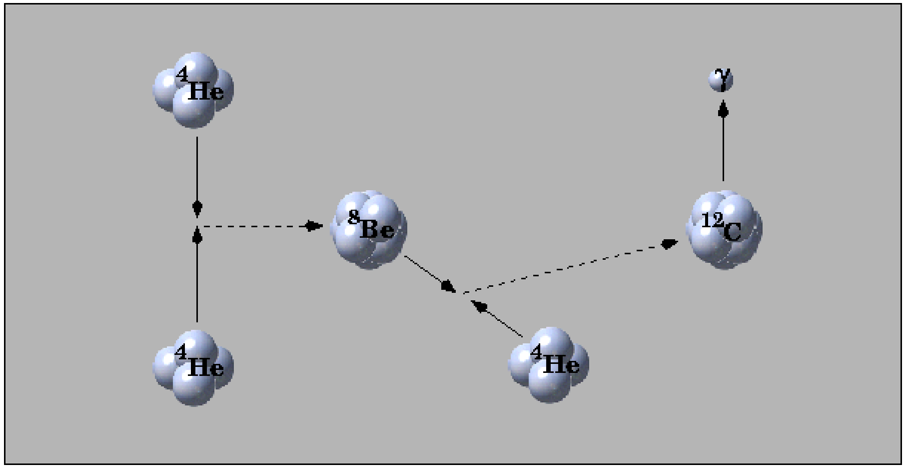
$^4He + {}^4He \to {}^8Be$

$^8Be + {}^4He \to {}^{12}C + \gamma$

Since the two reactions need to occur almost simultaneously, the 3α-reaction behaves as if it were a three-particle reaction:

$3\ {}^4He \to {}^{12}C + \gamma$

which has Q = 7.275MeV. The energy release per unit mass is $q = Q/m(^{12}C) = 5.9 \times 10^{17}$ erg/g, which is about 1/10 smaller than for H-burning.

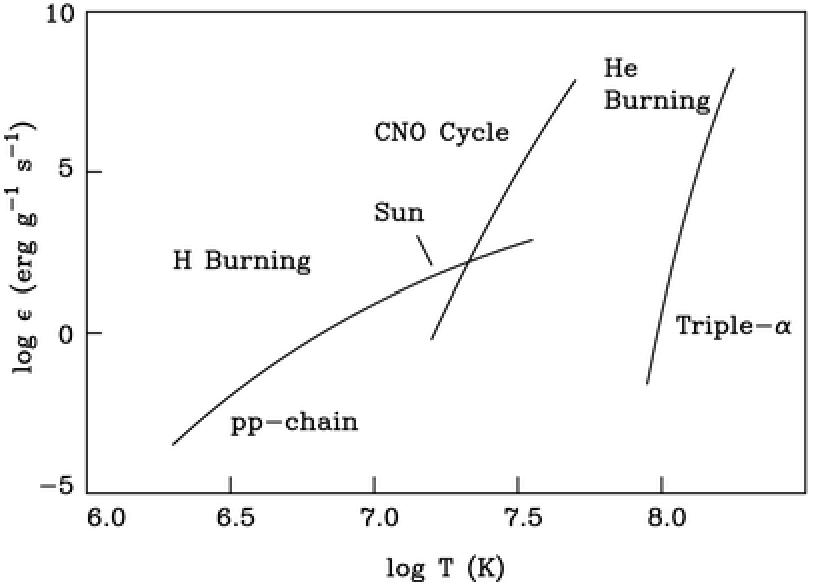


Helium Burning

In addition to the short-lived beryllium state, another factor that helps the 3α-reaction to go is the existence of a **resonance** in the ^{12}C nucleus that coincides closely in energy with that produced by colliding another helium nucleus with ^{8}Be.

This greatly enhances the rate at which the second step in the reaction chain takes place.

The temperature sensitivity of the 3α-reaction rate is extremely high, $\varepsilon_{3\alpha} \propto T^{40}$.



When a sufficient amount of ^{12}C has been created by the 3α-reaction, it can capture a further α-particle to form ^{16}O: $^{12}C + {}^{4}He \rightarrow {}^{16}O + \gamma$

It has $Q = 7.162$ MeV, or $q_{\alpha C} = 4.32 \times 10^{17}$ erg per gram of produced ^{16}O.

In principle further captures on ^{16}O are also possible, forming ^{20}Ne, but during normal helium burning become increasingly unlikely due to the increasing Coulomb barrier.

Thus, stars in which the triple-α process takes place wind up containing a mixture of carbon and oxygen, with the exact ratio depending on their age, density, and temperature.

Carbon burning and beyond

At even higher temperatures, the Coulomb barrier for oxygen and carbon can be overcome, creating yet heavier nuclei. Carbon burning (fusion of 2 carbon nuclei) requires temperatures above 5×10^8 K, and oxygen burning in excess of 10^9 K.

Many reaction paths are possible. We will not discuss these reactions here as the majority of the possible energy release by nuclear fusion reactions has occurred by the time that hydrogen and helium have been burnt.

However, examples of these reactions can be found in textbooks.

Silicon to Iron, photodisintegration

- At still higher temperatures, around 3×10^9 K, the typical photon becomes energetic enough that it can disrupt nuclei, knocking pieces off them in a process known as **photodisintegration**. The chemical balance in the star is then determined by a competition between this process and reactions between nuclei.
- However, as we might expect, the net effect is to drive the chemical balance ever further toward the most stable nucleus, iron. Once the temperature around 3×10^9 K, more and more nuclei begin to convert to ^{56}Fe, and its close neighbor's cobalt and nickel.
- Things stay in this state until the temperature is greater than about 7×10^9 K, at which point photons have enough energy to destroy even iron, and the entire process reverses: all elements are converted back into its constituent protons and neutrons, and photons reign supreme.

$$^{56}Fe \rightarrow 13\ ^{4}He + 4\ n$$

Major nuclear burning processes

Common feature is release of energy by consumption of nuclear fuel. Rates of energy release vary enormously. Nuclear processes can also absorb energy from radiation field, we shall see consequences can be catastrophic.

Nuclear Fuel	Process	$T_{threshold}$ 10^6K	Products	Energy per nucleon (MeV)
H	PP	~4	He	6.55
H	CNO	15	He, N	6.25
He	3α	100	C,O	0.61
C	C+C	600	O,Ne,Na,Mg	0.54
O	O+O	1000	Mg,S,P,Si	~0.3
Si	Nuc eq.	3000	Co,Fe,Ni	<0.18

Burning times of burning phases:

H: 10^{10} (yrs)

He: 10^8

C: 10^4

...

Si: hrs