Opacity (1)

Concept of opacity was introduced when deriving the equation of radiation transport.
It will be discussed extensively in the Stellar atmospheres part of this course.
Opacity is the resistance of material to the flow of radiation through it. In most stellar interiors it is determined by all the processes which scatter and absorb photons.

Four main processes:

- Bound-bound absorption:
  - is related to photon-induced transitions of a (bound) electron in atoms or ions to a higher energy state by the absorption of a photon. The atom is then de-excited either spontaneously or by collision with another particle, whereby a photon is emitted. Although this is limited to certain transition frequencies, the process can be efficient because the absorption lines are strongly broadened by collisions.
- Bound-free absorption:
  - which is another name for photoionization - the removal of an electron from an atom (ion) caused by the absorption of a photon. The inverse process is radiative recombination.
- Free-free absorption:
  - the absorption of a photon by a free electron, which makes a transition to a higher energy state by briefly interacting with a nucleus or an ion. The inverse process, leading to the emission of a photon, is known as bremsstrahlung.
- [Electron] Scattering:
  - the scattering of a photon by a free electron, the photon's energy remaining unchanged (known as Thomson scattering).

Which process is most important in the deep stellar interiors?

Opacity (2)

Which process is most important in the deep stellar interiors?

- Temperatures are very high there, thus the last two processes are dominant, simply because the material is almost completely ionized, there are very few bound electrons.
- Furthermore, the energy of most photons in the Planck distribution is of the order of keV, whereas the separation energy of atomic levels is only a few tens eV. Hence most photons interacting with bound electrons would set them free. Thus bound-bound (and even bound-free) transitions have extremely low probabilities, interactions occurring predominantly between photons and free electrons.
- Thus, the most dominant process in the deep stellar interiors is electron scattering.

The opacity per unit mass of material in this case is $\kappa_e = \frac{n_e \sigma_T}{\rho}$,

where $\sigma_T = 6.65 \times 10^{-25}$ cm^2 is the Thomson cross section. The opacity is therefore:

$$\kappa_e \cong 0.2(1 + \mathrm{X})\ \mathrm{cm}^2\mathrm{g}^{-1}$$

$$n_e = \frac{\rho}{\mu_e m_H}$$

The opacity resulting from electron scattering is temperature and density **independent**!

Opacity (3)

We need an expression for opacity to solve the equations of stellar structure. For stars in thermodynamic equilibrium with only a slow outward flow of energy, the opacity should have the form $\kappa = \kappa(\rho, T, \text{chemical composition})$

Opacity coefficients may be calculated, taking into account all possible interactions between the elements and photons of different frequencies. This requires an enormous amount of calculation and is beyond the scope of this course. Such calculation are done e.g. by the OPAL opacity project at Lawrence Livermore National Laboratory.

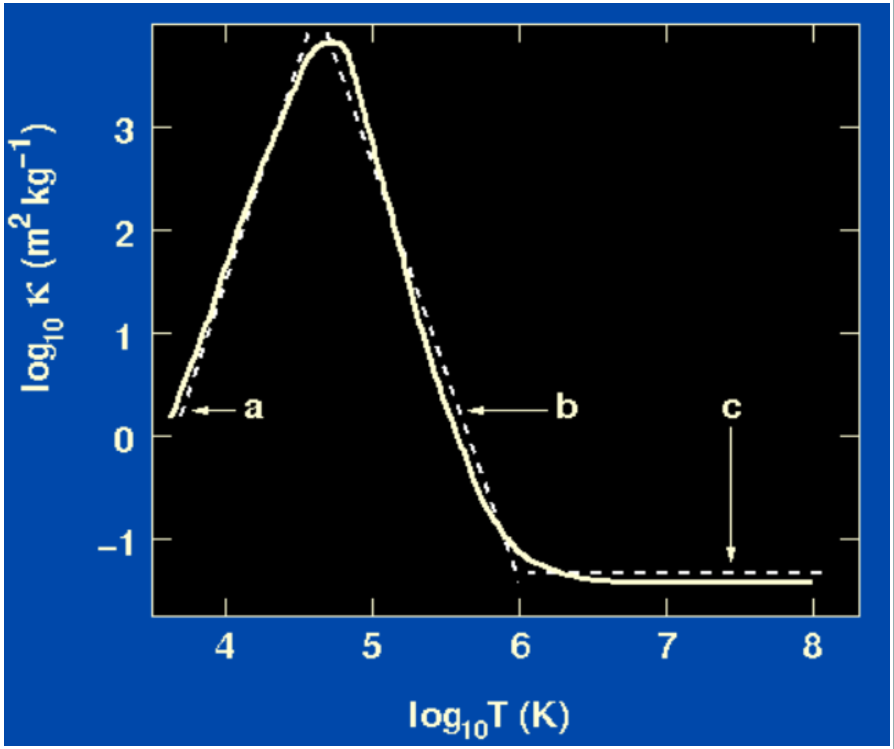
When it is done, the results are usually approximated by the relatively simple formula:

$$\kappa = \kappa_0 \rho^\alpha T^\beta$$

where α, β are slowly varying functions of density and temperature and κ_0 is a constant for a given chemical composition.

Opacity (4)

Figure shows opacity as a function of temperature for a star of given ρ (10^{-4} g cm^{-3}). Solid curve is from detailed opacity calculations. Dotted lines are approximate power-law forms.



- At high T: κ is low and remains constant. Most atoms are fully ionized, high photon energy, hence free-free absorption unlikely. Dominant mechanism is electron scattering, independent of T, $\alpha=\beta=0$:

$$\kappa = \kappa_1 = \sigma_T / m_H \mu_e \quad \text{(curve c)}$$

- Opacity is low a low T, and increases towards higher T. Most atoms are not ionized, few electrons available to scatter photons or for free-free absorption. Approximate analytical form is α=1/2 , β=4:

$$\kappa = \kappa_2 \rho^{0.5} T^4 \quad \text{(curve a)}$$

- At intermediate T, κ peaks, when bound-free and free-free absorption are very important, then decreases with T (Kramers opacity law):

$$\kappa = \kappa_3 T^{-7/2} \quad \text{(curve b)}$$

- κ_1, κ_2, κ_3 are constants for stars of given chemical composition but all depend on composition.

Summary and conclusions

- We have learned the approximate forms of the equation of state and the opacity.
- Next lecture: A method of simplifying the solution of the stellar structure equations.
- After that we will discuss nuclear reactions and move on to discussing the output of full numerical solutions of the equations and realistic predictions of modern theory

Nuclear Energy Production

BASICS ON NUCLEAR REACTIONS
THE BINDING ENERGY
QUANTUM TUNNELLING
REACTION CROSS-SECTION
THE GAMOW PEAK
NUCLEAR REACTION RATES
ELECTRON SHIELDING

Introduction

- We have seen that the 4 equations of stellar structure must be supplemented with expressions for P, κ, ε.
- We have discussed that
  - P is given by the equation of state of the stellar matter.
  - κ is determined by the atomic physics of the stellar material.
- The last function we still need to describe is the power density $\varepsilon = \varepsilon(\rho, T,$ chemical composition$)$.
- In Lecture 4 we have concluded that the energy source behind ε must be nuclear burning.
- Thus, ε is defined by the nuclear energy source in the interiors. We need to develop a theory and understanding of nuclear physics and reactions.

We now move from the studies of a stellar body to studies of its "soul" (V.V. Ivanov).

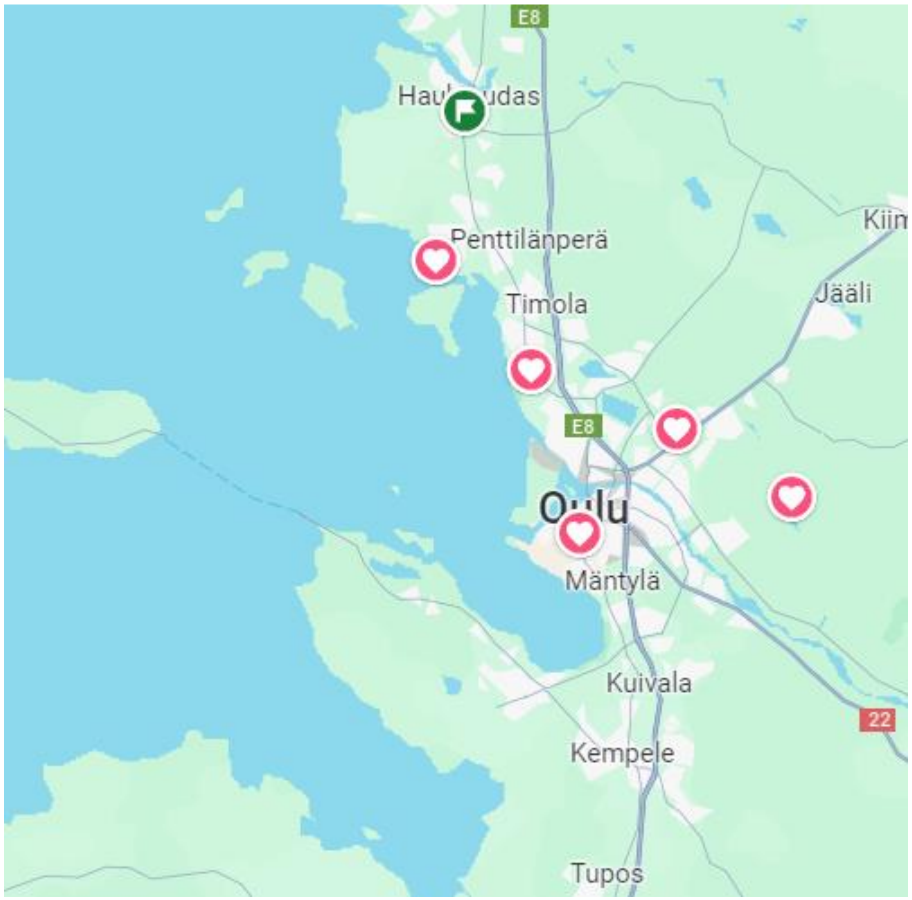
How to compute ε?

- Our goal is to compute the rate of energy generation ε per unit mass per unit time.
- The computation can be separated into three parts:
  - the cross section for a reaction between a pair of nuclei, which is determined predominantly by the properties of the nuclei;
  - the amount of energy generated per reaction, which again is a property of the nuclei;
  - the total reaction rate which, beside the cross section, also depends on the statistics of the motion of the nuclei.
- An additional consequence of the nuclear processes is a gradual change of the chemical composition, which controls the evolution of the star. Hence, we must determine the rate of change of the abundances.

Nuclear dimensions

- A convenient unit for measuring the radius of an atom is the Bohr radius, $\sim 0.5 \times 10^{-10}$ m $= 0.5$ Å.
- Atomic radii are typically a few to a few tens of Bohr radii, or of order 10^{-10} m, or a few Å.
- Proton radius $= 0.83 \times 10^{-15}$ m $\approx 10^{-13}$ cm $= 10^{-5}$ Å.
- The radius R of a nucleus containing A nucleons, i.e. having an atomic mass number of A, is $R \approx r_0 A^{1/3}$, where $r_0 = 1.2 \times 10^{-15}$ m.
- Nuclei are therefore of order a few $\times 10^4$ times smaller in radius than an atom.
- How compact the nucleus compared to the atom within which it resides?

Nuclear dimensions



~40 km

~1 m

Thus, atoms are very empty, but nuclei are very compact and incompressible dense.

Nuclear processes in stars

- The most important series of fusion reactions are those converting hydrogen into helium (H-burning). As we will see later, this dominates ~90% of lifetime of nearly all stars.
- Mass of nuclei with several protons and/or neutrons does not exactly equal mass of the constituents - slightly smaller because of the binding energy of the nucleus.
- Hence, there is a decrease of mass, and from the Einstein mass-energy relation **$E=mc^2$** the mass deficit is released as energy.

Example: $4\times{}^{1}H \rightarrow {}^{4}He$

4 protons × each of mass 1.0078 atomic mass units: 4.0313 amu

mass of helium nucleus: 4.0026 amu

Mass difference: 0.0287 amu = 4.76×10^{-26} g

$$\Delta E = \Delta M c^2 = 4.278\times10^{-5}\ \text{erg} = 26.7\ \text{MeV}$$

- Since binding energy differs for different nuclei, it can release or absorb energy when nuclei either fuse or fission.

The binding energy

- General calculation: define the binding energy of a nucleus as the energy required to break it up into constituent protons and neutrons.
- Suppose nucleus has:
  - Proton number Z
  - N neutrons
  - Atomic mass number A (number of protons + neutrons, $Z+N$), or the baryon number, or nucleon number, or nuclear mass
  - Nucleus mass $m_{nuc} = m(Z,N)$
- Binding energy of the nucleus (also known as the difference between the masses of nucleons and nucleus) is:

$$Q(Z,N) \equiv [Z m_p + N m_n - m(Z,N)]c^2$$

- Some useful units:
  - 1 eV $= 1.602 \times 10^{-12}$ erg $= 11.65 \times 10^3$ K
  - 1amu $= 1/12\ m(^{12}\mathrm{C}) = 931.49$ MeV/c^2 $= 1.660 \times 10^{-24}$ g
  - $m_e c^2 = 0.511$ MeV
  - $m_p = 1.007825$ amu $m_p c^2 = 938.72$ MeV
  - $m_n = 1.008665$ amu $m_n c^2 = 939.56$ MeV
  - $m(^4\mathrm{He}) = 4.002603$ amu

The binding energy per nucleon (1)

- For our purposes, a more useful quantity for considering which nuclear reactions yield energy is the total binding energy per nucleon:

$$q = \frac{Q(Z,N)}{A}$$

$\boxed{{}^{A}_{Z}X}$

- We can then consider this number relative to the hydrogen nucleus.
- For reaction $4\times{}^{1}\mathrm{H} \rightarrow {}^{4}\mathrm{He}$

$$q = \frac{28.30}{4} = 7.07\ \mathrm{MeV}$$

- This corresponds to 7.07 MeV / 931 MeV = 0.7% of the rest mass converted to energy when hydrogen burns. The time for the Sun to radiate away just 10% of the energy available from this source is

$$\tau_{nuc} = \frac{0.1 \times 0.007 \times M_{\odot} \times c^2}{L_{\odot}} \approx 10^{10}\ \mathrm{yr}$$

In terms of energy budget, hydrogen fusion can easily produce the solar luminosity over the age of the Solar System.

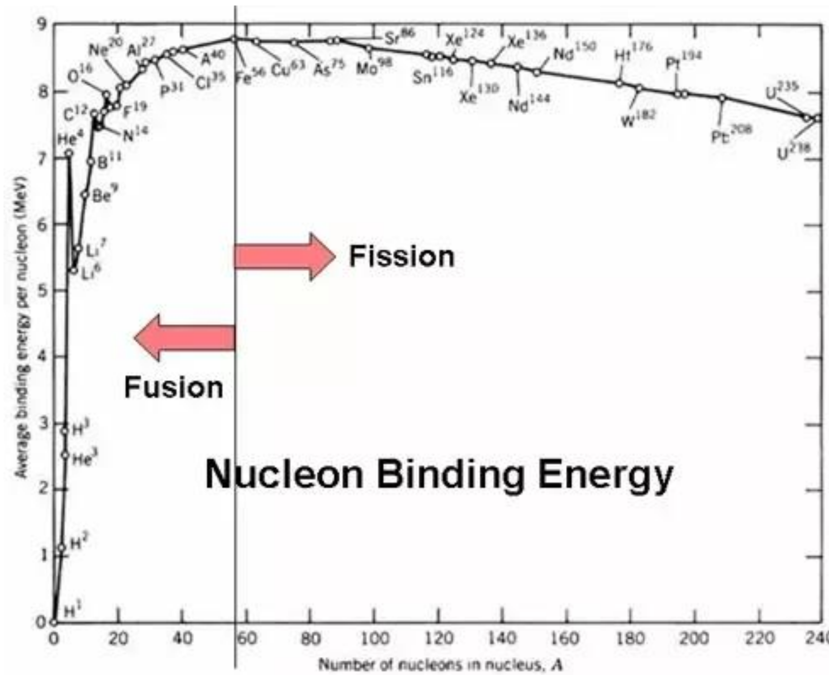
- Compare with fusion of hydrogen to iron ${}^{56}\mathrm{Fe}$: q=8.8 MeV per nucleon. Most of this is already obtained in forming helium (7.07 MeV).

The binding energy per nucleon (2)

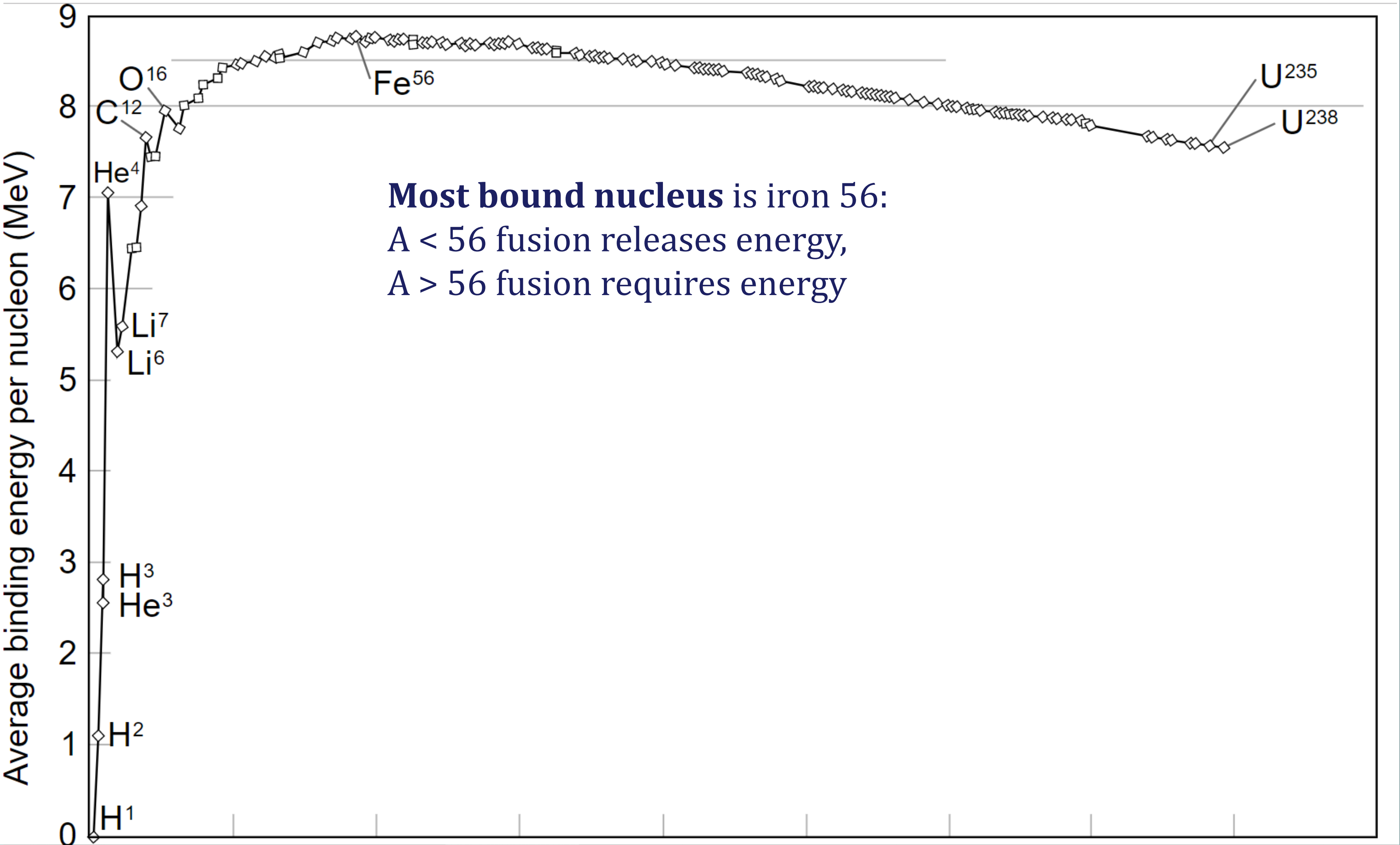
On average, the binding energy per nucleon is about $q \approx 8$ MeV/nucleon

More accurately it depends on the baryon number A.

- General trend is an increase of q with atomic mass up to $A = 56$ (Fe). Then slow monotonic decline.
- There is steep rise from H through ^{2}H, ^{3}He, to ^{4}He $\rightarrow$ fusion of H to He should release larger amount of energy *per unit mass* than say fusion of He to C.
- Energy may be gained by *fusion* of light elements to heavier, up to iron
- Or from *fission* of heavy nuclei into lighter ones down to iron.



The binding energy per nucleon (3)



Most bound nucleus is iron 56:
A < 56 fusion releases energy,
A > 56 fusion requires energy

Occurrence of fusion reactions (1)

Let's now discuss the conditions under which these fusion reactions can occur and whether such conditions exist in stellar interiors.

Consider two nuclei with proton numbers Z_1 and Z_2. Nuclei interact through four forces of physics, but only electromagnetic and strong nuclear important here:

- Two positively charged nuclei repel each other and must overcome a repulsive Coulomb potential

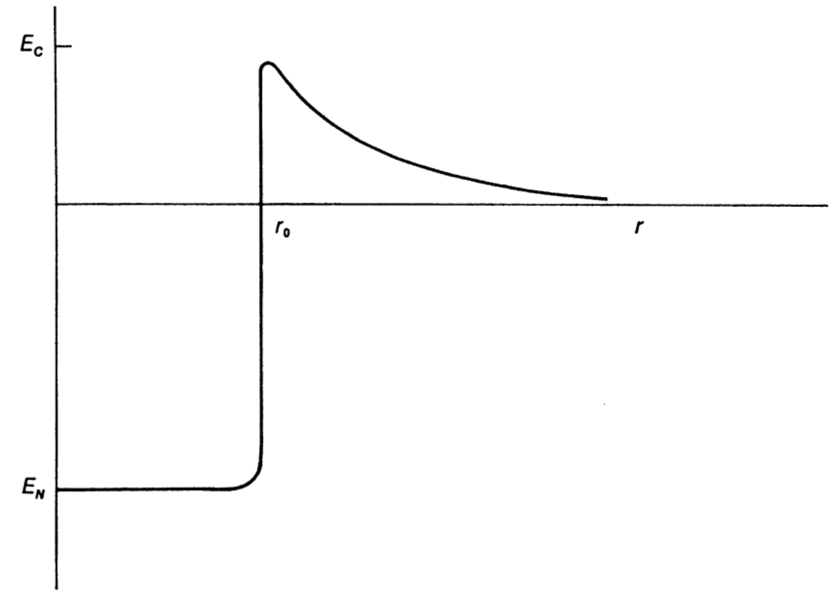
$$V_C = \frac{Z_1 Z_2 e^2}{r} = 1.44 \frac{Z_1 Z_2}{r[\mathrm{fm}]} \text{ MeV}$$

r[fm] – r in fermis ($\times 10^{-13}$ cm)

- to reach separation distances where strong force dominates (typical size of nucleus)

$$r_0 \approx 1.44 \times 10^{-13} A^{1/3} \text{ cm}$$

A – Atomic mass number



Schematic potential energy between two nuclei. For $r < r_0$ the attractive nuclear forces dominate; for $r > r_0$ Coulomb repulsion dominates.

At $r = r_0$, height of the Coulomb barrier is:

$$V_0 = \frac{Z_1 Z_2 e^2}{r_0[fm]} = Z_1 Z_2 \text{ MeV}$$

i.e. of the order of 1 MeV for two protons…

Femtometer **[fm]** (derived from the Danish and Norwegian word *femten* 'fifteen' equal to 10^{-15} m. This distance is also called **fermi**.

Classical approach

At $r = r_0$, height of the Coulomb barrier is $V_0 = Z_1 Z_2$ MeV, of the order of 1 MeV for two protons.

Class task:

Compare this to the average kinetic energy of a particle ($3kT/2$). What T do we require for fusion? How does this compare with the minimum mean T of the Sun? Any comments on these two temperatures?

- At a typical internal stellar temperature of 10^7 K, the kinetic energy of a nucleus is $3kT/2 \sim 1$ keV. The characteristic kinetic energy is thus of order 10^{-3} of the energy required to overcome the Coulomb barrier.
- Typical nuclei will approach each other only to a separation $r \sim 10^{-10}$ cm, 1000 times larger than the distance at which the strong nuclear binding force operates.
- Can nuclei that are in the high-energy tail of the Maxwell–Boltzmann distribution overcome the barrier? The fraction of nuclei with such energies is

$$e^{E/kT} \approx e^{-1000} \approx 10^{-434}$$

- Even considering the high-energy tail of the Maxwell-Boltzmann distribution, the fraction of particles with $E > E_C$ is **vanishingly small**.
- With purely classical considerations nuclear reactions have **no chance of happening** at such temperatures.

Boltzmann's constant $k = 1.38 \times 10^{-16}$ cm^2 g s^{-2} K^{-1} = 8.62×10^{-5} eV K^{-1}

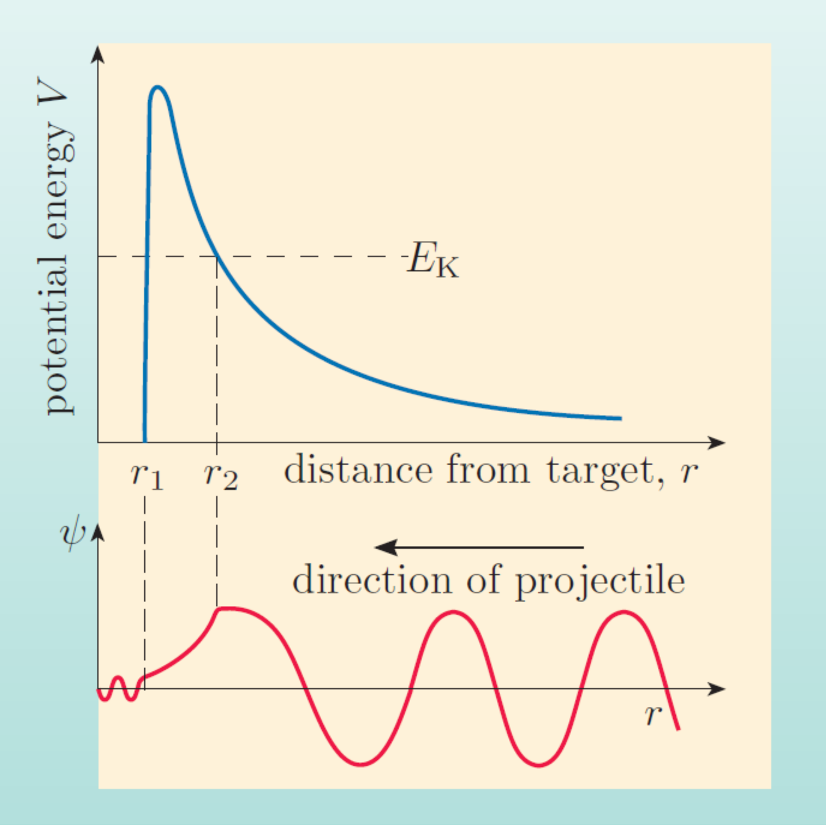
Occurrence of fusion reactions (2)

At $r = r_0$, height of the Coulomb barrier is $V_0 = Z_1 Z_2$ MeV, i.e. of the order of 1 MeV for two protons…

If we imagine what will happen as the nuclei approach one another with a certain kinetic energy E_K then, ignoring quantum mechanics, they will simply come to rest temporarily when their kinetic energy has been converted into electrical potential energy, before 'bouncing' apart again.

The distance of least separation is given by

$$r_2 = \frac{Z_1 Z_2 e^2}{E_K}$$



Thus, **classically**, a particle with kinetic energy E_K cannot get closer to the origin than $r = r_2$ because of the potential barrier of height $V > E_K$. **Quantum-mechanically**, however, the wave function shows that there is a **non-zero probability** of finding the particle beyond the barrier, i.e. at $r < r_1$.

Quantum tunnelling

- We need to turn to quantum mechanics to see how nuclear reactions are possible at stellar temperatures.
- As was discovered by George Gamow (1928), there is a finite non-zero probability for a particle to penetrate the repulsive Coulomb barrier even if $E_K \ll V_C$ as if "tunnel" existed.
- For the Coulomb barrier, the penetration probability can be expressed in terms of the particle energy $E=E_K$, and the Gamow energy E_G which depends on the atomic number (and therefore charge) of the interacting nuclei, and hence the size of the Coulomb barrier:

$P_{pen} = g(E) \approx e^{-\sqrt{E_G/E}}$ with the Gamow energy $E_G = 2m_r c^2(\pi\alpha Z_1 Z_2)^2$, where $m_r = m_1 m_2/(m_1+m_2)$ is the reduced mass of particles, α is the fine structure constant $\approx 1/137$.

  - P_{pen} which is also called the Gamow factor $g(E)$ increases rapidly with E
  - P_{pen} decreases with $Z_1 Z_2$ - lightest nuclei can fuse more easily than heavy ones
  - Higher energies / temperatures needed to fuse heavier nuclei, so different nuclei burn in well-separated phases during stellar evolution.

Quantum tunnelling for two protons

$$E_G = 2m_r c^2 (\pi \alpha Z_1 Z_2)^2$$

$$P_{pen} \approx e^{-\sqrt{E_G/E}}$$

$m_r = m_1 m_2/(m_1+m_2)$ is the reduced mass of particles, α is the fine structure constant ≈ 1/137.

- **Class task:** calculate E_G for two protons and find P_{pen} for the typical kinetic energy of particles in the Sun's core, E ~ 1 keV.
  It is convenient to remember that the rest energy of a proton, $m_p c^2$, is 938 MeV.

  - For two protons, $E_G \approx 500$ keV
  - $P_{pen} \approx e^{-22} \approx 2 \times 10^{-10}$. It is very small, but considerably larger than the classical probability.

Reaction cross-section

- Even if tunneling occurs, and two nuclei are within the strong force's interaction range, the probability of a nuclear reaction will still depend on a nuclear cross section, which will generally depend inversely on the kinetic energy. Thus, the total cross-section for a nuclear reaction can be written as

The cross-section is defined as

$$\sigma = \frac{\text{number of reactions per sec}}{\text{number of incident particles per sec per cm}^2}$$

$$\sigma = \frac{S(E)}{E} g(E) = \frac{S(E)}{E} e^{-\sqrt{E_G/E}}$$

where $S(E)$ is a **slowly varying** S-factor:

- $S(E)$ contains all remaining effects, i.e. the intrinsic nuclear properties of the reaction including possible resonances.
- Its evaluation requires laboratory data.
- Generally, it is insensitive to particle energy or velocity.
- In some cases, however, $S(E)$ can vary quite rapidly, peaking at specific energies. These energies correspond to energy levels within the nucleus, analogous to the orbital energy levels of electrons. It is a resonance between the energy of the incoming particle and differences in energy levels within the nucleus that accounts for these strong peaks.