Specific and mean Intensity

From the previous lecture:

$$I_\lambda = \frac{E_\lambda}{\cos\theta \; d\lambda \, d\sigma \, d\omega \, dt}$$

Let's try in another way:

- The (specific) intensity $\boldsymbol{I_\lambda}$ is a measure of brightness:

$$I_\lambda = \frac{dE_\lambda}{\cos\theta \; d\lambda \, d\sigma \, d\omega \, dt}$$

$d\lambda, d\sigma, d\omega, dt \rightarrow 0$ dE diminishes to zero as well

- In this way, we define the specific intensity at a "point" on the surface, at a given time, in a direction θ, at a wavelength λ - *brightness*.

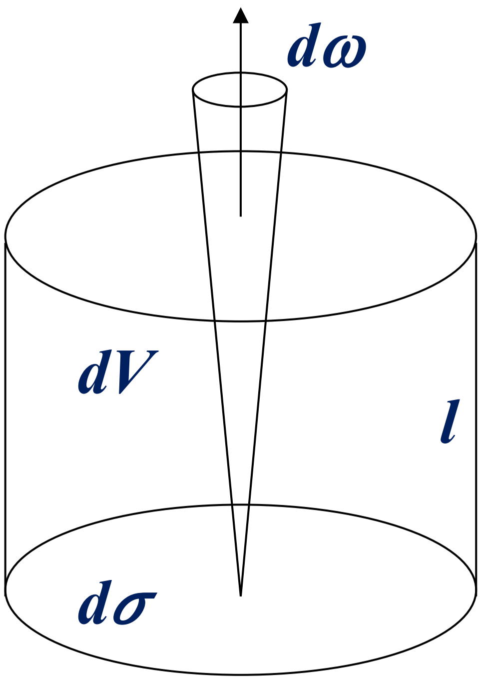
The *mean* intensity $\boldsymbol{J_\lambda}$ is the directional average of the specific intensity (over 4π steradians):

$$J_\lambda = \frac{1}{4\pi} \oint I_\lambda d\omega$$

Integrated over the whole unit sphere centered on the point of interest.

Mean intensity and Energy density

$$J_\lambda = \frac{1}{4\pi}\oint I_\lambda d\omega$$



- The *mean* intensity $\boldsymbol{J_\lambda}$ is related to the energy density $\boldsymbol{u_\lambda}$:
- Energy radiated through area element $d\sigma$ during time dt:

$$dE_\lambda = I_\lambda \, d\lambda \, d\sigma \, d\omega \, dt$$

$$l = c\,dt \Rightarrow dV = l\,d\sigma = c\,dt\,d\sigma$$

- Hence, the energy contained in volume element dV per wavelength interval is:

$$u_\lambda dV d\lambda = \oint I_\lambda \, d\omega \, d\lambda \, d\sigma \, dt = 4\pi J_\lambda \frac{dV}{c} d\lambda$$

$$u_\lambda = \frac{4\pi}{c} J_\lambda \quad \left[\frac{erg}{cm^3 \text{Å}}\right]$$

$$u = \int_0^\infty u_\lambda \, d\lambda = \frac{4\pi}{c}\int_0^\infty J_\lambda \, d\lambda \quad \left[\frac{erg}{cm^3}\right]$$

Total radiation emerge in volume element

Flux (1)

- From an observational point of view, we are generally more interested in the energy flux or flux (L_λ, L) and the flux density (F_λ, F).
  Flux density gives the power of the radiation per unit area and hence has dimensions of erg s^{-1} cm^{-2} $Å^{-1}$ (or erg s^{-1} cm^{-2} Hz^{-1}).
  Observed flux densities are usually extremely small and therefore (especially in radio astronomy) flux densities are often expressed in units of the Jansky (Jy), where 1 Jy=10^{-23} erg s^{-1} cm^{-2} Hz^{-1}.

- You should be aware - and beware - that different authors define the terms flux density, flux and intensity differently, and they are sometimes used interchangeably!

- We will often call flux density as just flux.

- Standard definition:
  Flux describes any effect that appears to pass or travel through a surface or substance. In transport phenomena (radiative transfer, heat transfer, mass transfer, fluid dynamics), flux is defined as the rate of flow of a property per unit area, which has the dimensions $[\text{quantity}]\times[\text{time}]^{-1}\times[\text{area}]^{-1}$.
  - For example, the magnitude of a river's current, i.e. the amount of water that flows through a cross-section of the river each second is a kind of flux.

Flux (2)

In radiative transfer, flux is related to the intensity ("specific" is often omitted):

- Flux F_λ is a measure of the net energy flow across an area $d\sigma$, over a time dt, in a $d\lambda$. The only directional significance is whether the energy crosses $d\sigma$ from the top or from the bottom. Then we can write:

The solid angle $d\omega$ appears for I_λ but not for F_λ

$$F_\lambda = \frac{\oint dE_\lambda}{d\lambda\, d\sigma\, dt}$$

Integrated over all directions.

substitute

$$I_\lambda = \frac{dE_\lambda}{\cos\theta\, d\lambda\, d\sigma\, d\omega\, dt}$$

$$F_\lambda = \oint \underbrace{I_\lambda \cos\theta\, d\omega}\ \left[\frac{erg}{\text{Å}\ cm^2\ s}\right]$$

Thus, flux F_λ is the projection of the specific intensity I_λ in the radial direction (integrated over all solid angles)

The amount of energy going through 1 cm² per second per 1Å into the solid angle dω in the direction inclined by the angle θ to the normal of the area.

Flux (3)

$d\omega = \sin\theta \, d\theta \, d\varphi$

Expressing $d\omega$ by means of θ and φ,

$$F_\lambda = \oint I_\lambda \cos\theta \, d\omega = \int_0^{2\pi} d\varphi \int_0^{\pi} I_\lambda \cos\theta \sin\theta \, d\theta$$

If there is no azimuthal dependence for I_λ then

$$F_\lambda = \oint I_\lambda \cos\theta \, d\omega = 2\pi \int_0^{\pi} I_\lambda \cos\theta \sin\theta \, d\theta$$

In the plane-parallel or spherical case, we do not find any dependence of I_λ on the longitude φ

$$F_\lambda = -2\pi \int_0^{\pi} I_\lambda \cos\theta \, d(\cos\theta)$$

Meaning of flux:

Radiation flux = **netto** energy going through area
Decomposition into two half-spaces:

$$F = -2\pi \int_0^{\pi} I_\lambda \cos\theta \, d(\cos\theta) = 2\pi \int_{-1}^{1} I(\mu)\,\mu\, d\mu \qquad \boldsymbol{\mu = \cos\theta}$$

$$= 2\pi \int_0^1 I(\mu)\,\mu\,d\mu + 2\pi \int_{-1}^{0} I(\mu)\,\mu\,d\mu$$

$$= 2\pi \int_0^1 I(\mu)\,\mu\,d\mu - 2\pi \int_0^1 I(-\mu)\mu\,d\mu = \boldsymbol{F^+ - F^-}$$

Netto = Outwards - Inwards

Special cases: at the surface of a star $F^- = 0$, so that $F = F^+$
at the centre of a star, isotropic radiation field: $F=0$

Intensity, Flux, and Luminosity

- I is independent of distance from the source and can only be measured directly if we resolve the radiating surface.
  In contrast, F obeys the inverse square law and is all that may be measured for most stars.

$dS \equiv r^2 \, d\omega$

- Indeed, if we consider a star as the source of radiation, then the flux emitted by the star into a solid angle $d\omega$ is $dL = d\omega r^2 F$, where F is the flux density observed at a distance r from the star. If the star radiates isotropically then radiation at a distance r will be distributed evenly on a spherical surface of area $4\pi r^2$ and hence we get the relationship:

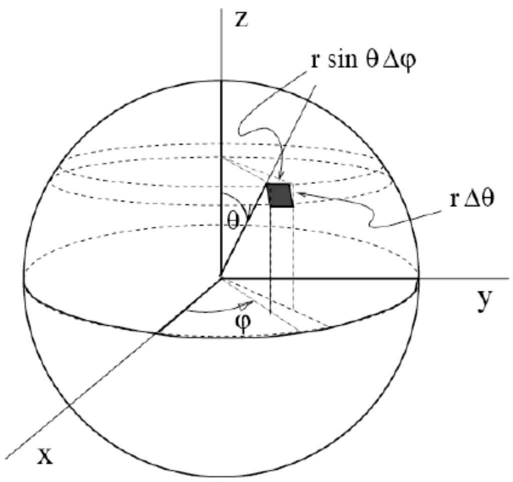
$$L = 4\pi r^2 F$$

- It is also usual to refer to the total flux from a star as the Luminosity, L.

Surface brightness

- Flux density arriving from a point source is inversely proportional to the distance. But what about an extended luminous object such as a nebula or galaxy? The situation is slightly more complicated.
- The surface brightness is defined as the flux density per unit solid angle. The geometry of the situation results in the interesting fact that the observed surface brightness is independent of the distance of the observer from the extended source.
- This slightly counter-intuitive phenomenon can be understood by realizing that although the flux density arriving from a unit area is inversely proportional to the square of the distance to the observer, the area on the surface of the source enclosed by a unit solid angle at the observer is directly proportional to the square of the distance.
- Thus, the two effects cancel each other out.

$$dS \equiv r^2\, d\omega$$



Mean Intensity, Flux and K-integral

- The *mean* intensity $\boldsymbol{J_\lambda}$ is the directional average of the specific intensity (over 4π steradians):

$$J_\lambda = \frac{1}{4\pi} \oint I_\lambda d\omega$$

- Flux $\boldsymbol{F_\lambda}$, is the projection of the specific intensity in the radial direction (integrated over all solid angles):

$$F_\lambda = \oint I_\lambda \cos\theta \, d\omega$$

- There is also a K-integral which we will use later:

$$K_\lambda = \frac{1}{4\pi} \oint I_\lambda \cos^2\theta \, d\omega$$

K-integral and radiation pressure

- K-integral is related to the radiation pressure: $K_\lambda = \frac{1}{4\pi}\oint I_\lambda \cos^2\theta \, d\omega$
- A photon has momentum $p_\lambda = E_\lambda / c$

- Consider photons transferring momentum to a solid wall.
  Force:

$$F = \frac{dp_{\lambda\perp}}{dt} = \frac{1}{c}\frac{dE_\lambda}{dt}\cos\vartheta$$

- Pressure: $dP_\lambda = \frac{F}{d\sigma} = \frac{1}{c}\frac{dE_\lambda}{dt}\frac{\cos\vartheta}{d\sigma} = \frac{1}{c} I_\lambda \cos^2\vartheta \; d\omega \, d\lambda$

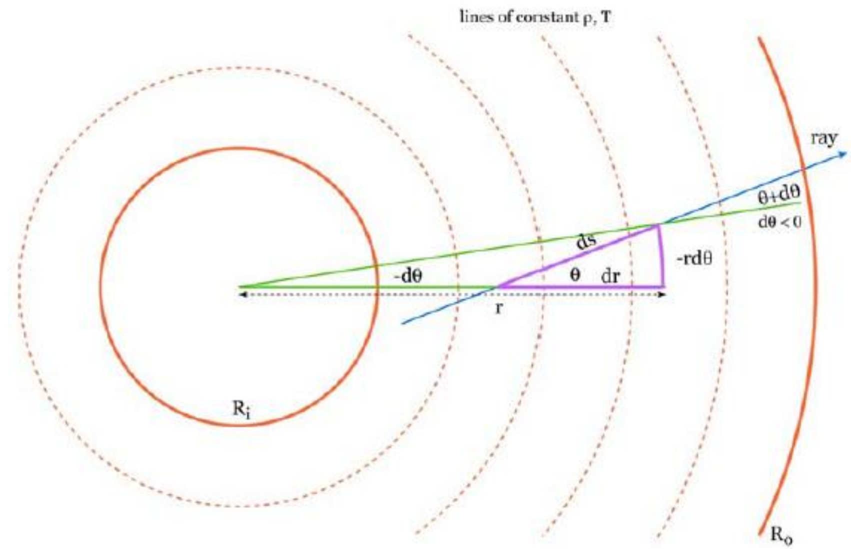
$$I_\lambda = \frac{dE_\lambda}{\cos\theta \, d\lambda \, d\sigma \, d\omega \, dt}$$

$$P_{rad}(\lambda) = \frac{1}{c}\oint_{4\pi} I_\lambda \cos^2\vartheta \; d\omega = \frac{4\pi}{c} K_\lambda$$

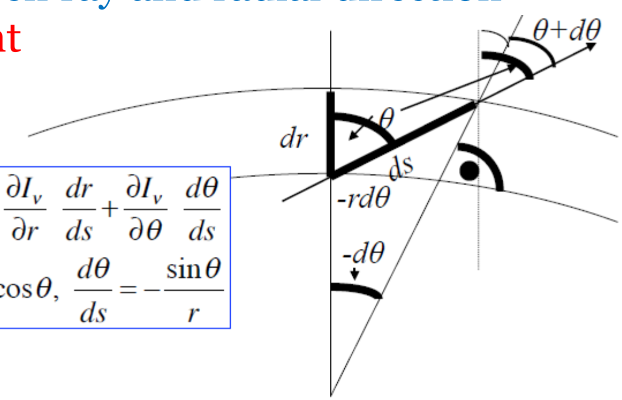
Plane-parallel vs spherical geometry

- Parallel-ray RTE is a very simple approach.
- In principal, we need to consider spherical geometry when calculating the transfer equation in stars.
- Fortunately, the geometrical thickness of most stellar photospheres is small compared to their radii, permitting the plane-parallel approximation, $r \rightarrow \infty$

$$\frac{dI_\lambda}{ds} = -\cos\vartheta \frac{\partial I_\lambda}{\partial r}$$



angle ϑ between ray and radial direction is not constant



$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \theta}\frac{d\theta}{ds}$$

$$\frac{dr}{ds} = \cos\theta, \quad \frac{d\theta}{ds} = -\frac{\sin\theta}{r}$$

Transfer Equation for Stars

The plane-parallel transfer equation
(for stars with thin photospheres)

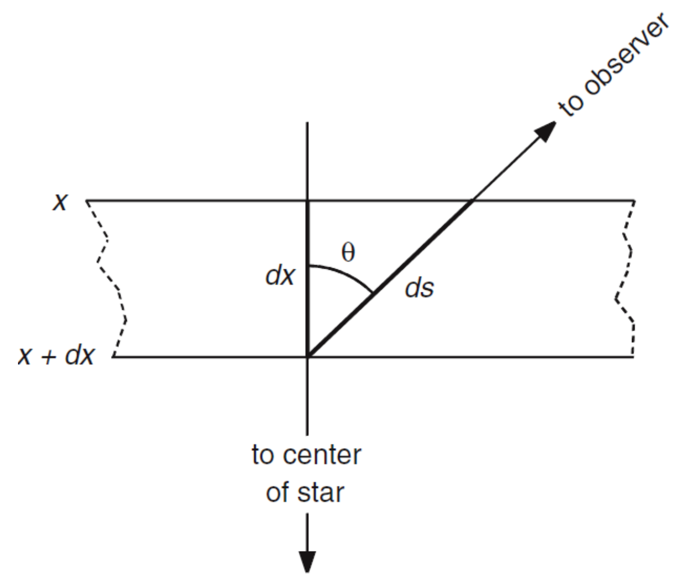
$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

is identical to the parallel-ray transfer equation
(for ISM studies),

$$\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda + S_\lambda$$

except for

1. the cos(θ) term, because the optical depth is measured along the radial direction *x* and not along the line of sight, i.e $d\tau_\lambda = -\kappa_\lambda \rho\, dx$
2. sign change, since we are now looking from the outside in, along direction x.



The full spherical geometry transfer equation is necessary for supergiants.

The plane-parallel RTE

- We will try to solve **the plane-parallel RTE** later when we start discussing stellar photospheres.
- But now let's concentrate on stellar interiors.

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

- The plane-parallel RTE leads to two particularly useful relations between the various quantities describing the radiation field.

- First, recall that S depends only on the local conditions of the gas, independent of direction. Then, integrating over all solid angles, we get

$$\frac{d}{d\tau_\lambda}\oint I_\lambda \cos\theta \, d\omega = \oint I_\lambda d\omega - S\oint d\omega$$

$$\frac{dF_\lambda}{d\tau_\lambda} = 4\pi(J_\lambda - S)$$

Radiative diffusion (1)

- The second relation: multiply the plane-parallel RTE by cos(θ) and again integrate over all solid angles:

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

$$\frac{d}{d\tau_\lambda}\oint I_\lambda \cos^2\theta \, d\omega = \oint I_\lambda \cos\theta \, d\omega - S_\lambda \oint \cos\theta \, d\omega$$

$d\omega = \sin\theta \, d\theta \, d\varphi$

$$P_{rad,\lambda} = \frac{1}{c}\oint_{4\pi} I_\lambda \cos^2\vartheta \; d\omega$$

$$\frac{dP_{rad,\lambda}}{d\tau_\lambda} = \frac{1}{c}F_\lambda$$

$$\oint \cos\theta \, d\omega = \int_0^{2\pi} d\varphi \int_0^{\pi} \cos\theta \sin\theta \, d\theta = 0$$

$d\tau_\lambda = -\kappa_\lambda \rho \, dx$

$$\frac{dP_{rad,\lambda}}{dr} = -\frac{\kappa_\lambda \rho}{c}F_\lambda$$

- Integrating the radiation pressure and flux over wavelengths, and replacing κ_λ by a weighted mean of opacity κ_R – the Rosseland mean opacity [we will introduce it later]:

$$\frac{dP_{rad}}{dr} = -\frac{\rho\kappa_R}{c}F$$

Radiative diffusion (2)

$$\frac{dP_{rad}}{dr} = -\frac{\rho\kappa_R}{c}F$$

- This relation can be interpreted as that the net radiative flux is driven by differences in the radiation pressure, with a "photon wind" blowing from high to low $P_{\rm rad}$.
- Thus, the transfer of energy by radiation is a process involving the slow upward diffusion of randomly walking photons, drifting toward the surface in response to tiniest differences in the radiation pressure.
- As we see, the description of a "ray" of light is in fact only a convenient fiction, used to define the direction of motion instantly shared by the photons that are continually absorbed and scattered into and out of the beam.
  - It can be shown that a photon generated near the centre of the Sun will be absorbed and re-emitted $\sim 10^{22}$ times before it escapes at the surface and the time it takes to do this is approximately equal to the thermal timescale of the Sun (a few $\times\ 10^7$ years). This means that when we observe energy radiated at the solar surface, we are usually seeing the results of nuclear reactions which occurred tens of millions of years ago.

The Radiative Temperature Gradient

- The radiation pressure gradient:

$$\frac{dP_{rad}}{dr} = -\frac{\kappa_R \rho}{c} F = \frac{4}{3} a T^3 \frac{dT}{dr}$$

- Then

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa_R \rho}{T^3} F$$

- Let's write Flux in terms of the local radiative luminosity of the star at radius r:

$$F(r) = \frac{L(r)}{4\pi r^2}$$

- The temperature gradient for radiative transport becomes:

$$\frac{dT}{dr} = -\frac{3}{4ac} \frac{\kappa_R \rho}{T^3} \frac{L(r)}{4\pi r^2} = -\frac{3}{64\pi \sigma_{SB} r^2} \frac{\kappa_R \rho}{T^3} L(r)$$

The fourth equation of stellar structure.

Recall: the pressure exerted by photons on the particles in a gas is:

$$P_{rad} = \frac{aT^4}{3}$$

where radiation density constant

$$a = \frac{4\sigma_{SB}}{c}$$

Summary of the lectures on RT

- In addition to the specific intensity I_λ, emission (j_λ and ε_λ) and absorption coefficients (κ_λ and α_λ), optical depth $d\tau_\lambda$, the source function S_λ, we defined the mean intensity J_λ and the energy density, radiative flux F_λ and luminosity L, K-integral and the radiation pressure F_{rad}.

- We derived the plane-parallel equation of radiative transfer (RTE):

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

- We have also derived the fourth differential equation of stellar structure (the temperature gradient for radiative transport):

$$\frac{dT}{dr} = -\frac{3}{64\pi\sigma_{SB}r^2}\frac{\kappa_R\rho}{T^3}L(r)$$

- Now we have all four equations, which govern the structure of stars. Let's now start searching for possible ways to solve them.

The equations of stellar structure

THE EQUATIONS OF STELLAR STRUCTURE AND POSSIBLE WAYS TO SOLVE THEM.

BOUNDARY CONDITIONS.

CONVECTION AND CONDITIONS FOR ITS OCCURRENCE

Solving the equations of stellar structure

Now we have all four differential equations, which govern the structure of stars
(Note! in the absence of convection)

- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$
- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$
- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$
- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma r^2}\frac{\rho(r)\kappa_R(r)}{T^3(r)}L(r)$

Where

- r = radius
- P = pressure at r
- m = mass of material within r
- ρ = density at r
- L = luminosity at r (rate of energy flow across sphere of radius r)
- T = temperature at r
- κ_R = Rosseland mean opacity at r
- ε = energy release per unit mass per unit time

We will consider the quantities:

- $P = P(\rho, T,$ chemical composition)
- $\kappa_R = \kappa_R(\rho, T,$ chemical composition)
- $\varepsilon = \varepsilon(\rho, T,$ chemical composition)

→ **The equation of state**

Boundary conditions

- Two of the boundary conditions are fairly obvious,
  at r=0, the centre of the star, m=0, L=0.

- At the surface of the star its not so clear, but we use approximations to allow solution.
  - There is no sharp edge to the star, but for the Sun ρ(surface)~10^{-7} g cm^{-3}. It is much smaller than mean density $\bar{\rho}$~1.4 g cm^{-3}.
  - We also know the surface temperature (T_{eff}=5780K) is much smaller than its minimum mean temperature (2x10^6 K).
- Thus, we make two approximations for the surface boundary conditions: ρ=0, T=0 at r=R, i.e. that the star does have a sharp boundary with the surrounding vacuum.

Use of mass as the independent variable

The above formulae would (in principle) allow theoretical models of stars with a given radius. However, from a theoretical point of view it is the mass of the star which is chosen, the stellar structure equations solved, then the radius (and other parameters) are determined. We observe stellar radii to change by orders of magnitude during stellar evolution, whereas mass appears to remain constant. Hence it is much more useful to rewrite the equations in terms of m rather than r.

We did it before: divide the equations by the equation of mass conservation:

- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$
- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$
- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$
- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma_{SB} r^2}\frac{\rho(r)\kappa_R(r)}{T^3(r)}L(r)$

- $\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$
- $\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$
- $\frac{dL}{dm} = \varepsilon$
- $\frac{dT}{dm} = -\frac{3\kappa_R L}{256\times\pi^2\sigma_{SB} r^4 T^3}$

We specify m and the chemical composition and now have a well-defined set of relations to solve. It is possible to do this analytically if simplifying assumptions are made, but in general these need to be solved numerically on a computer.

Stellar evolution (1)

- We have a set of equations that will allow the complete structure of a star to be determined, given a specified mass and chemical composition.
  However, what do these equations not provide us with?

- In deriving the equation for hydrostatic support, we have seen that provided the evolution of star is occurring slowly compared to the dynamical time t_d, we can ignore temporal changes (e.g. pulsations). Indeed, for the Sun $t_d \sim 30$ min, hence this is certainly true.

- And we have also assumed that time dependence can be omitted from the equation of energy generation, if the nuclear timescale (the time for which nuclear reactions can supply the stars energy) is greatly in excess of t_{th}.

Stellar evolution (2)

- If there are no bulk motions in the interior of the star, then any changes of chemical composition are localized in the element of material in which the nuclear reactions occurred. So, the star would have a chemical composition which is a function of mass $m(r)$.
- In the case of no bulk motions – the set of equations we derived must be supplemented by equations describing the rate of change of abundances of the different chemical elements. Let $C_{\mathrm{X,Y,Z}}$ be the chemical composition of stellar material in terms of mass fractions of hydrogen (X), helium (Y), and metals (Z) [e.g., for the Solar system X=0.7, Y=0.28, Z=0.02].

$$\frac{\partial (C_{X,Y,Z})_m}{\partial t} = f(\rho, T, C_{X,Y,Z})$$

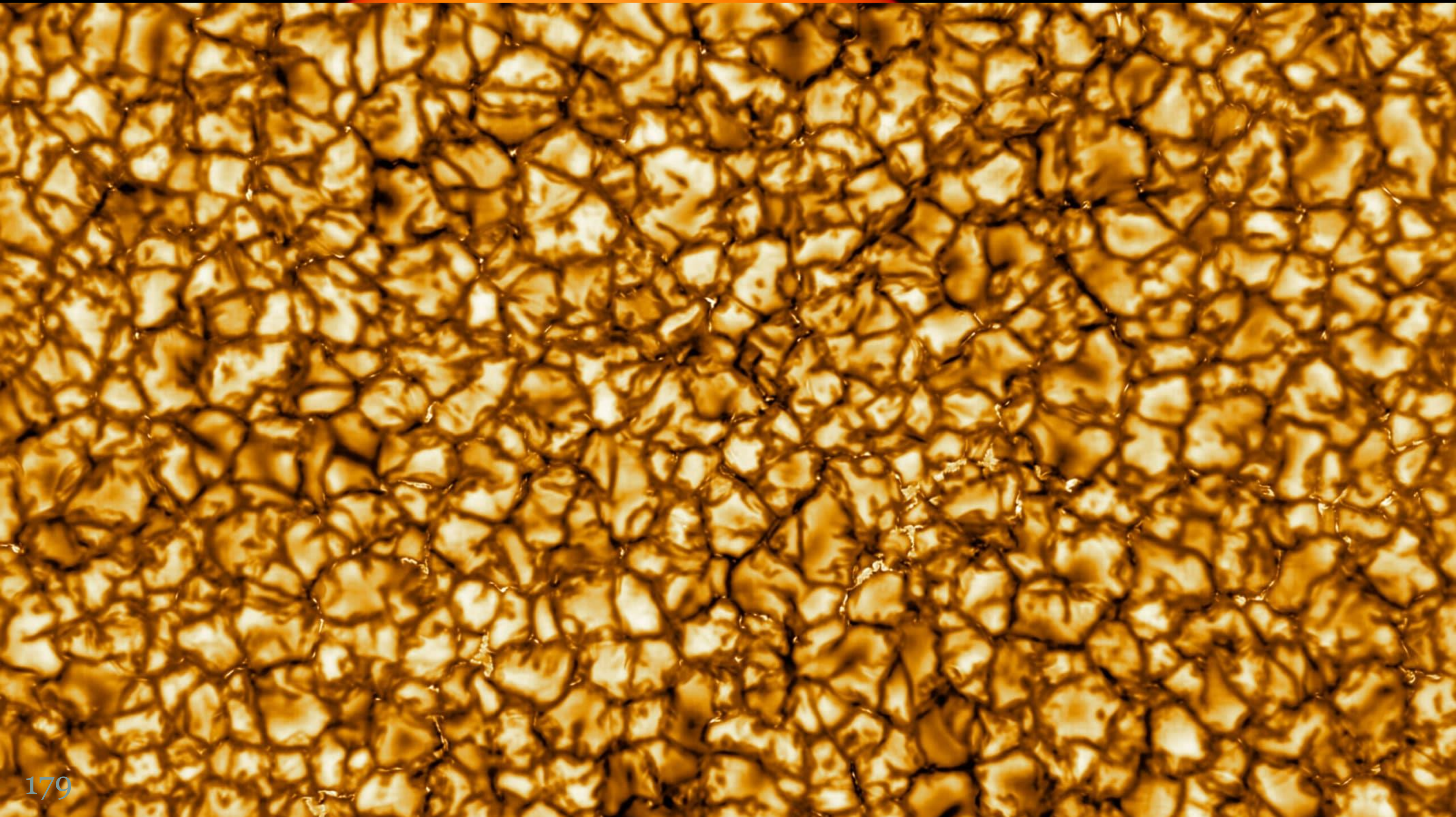
- Now let's consider how we could evolve a model

$$(C_{X,Y,Z})_{m,t_0+\delta t} = (C_{X,Y,Z})_{m,t_0} + \frac{\partial (C_{X,Y,Z})_m}{\partial t}$$

However ...

Solar surface

Granule size ~1000 km



Solar surface

Granule size ~1000 km