Time Series Analysis in the Time Domain

CONVOLUTION
CROSS-CORRELATION
AUTOCORRELATION

O-C DIAGRAM

Time Domain Analysis: Convolution

- The convolution *h(t)* of two functions *x(t)* and *y(t)* is

$$h(t) = x(t) * y(t) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} x(\tau) y(t-\tau)\, d\tau$$

Basic properties are
commutativity: $x * y = y * x$
distributivity over addition: $x * (y+z) = x * y + x * z$

- **Convolution theorem:** The Fourier transform of the convolution is the product of the individual Fourier transforms: $F(x * y) = F(x) \cdot F(y)$

$F(x \cdot y) = F(x) * F(y)$

e.g.: $x(t) * y(t) \Leftrightarrow X(\nu) \cdot Y(\nu)$

Time Domain Analysis: Cross-correlation

- **Correlation (Cross-correlation)** is quite a similar operation to convolution. The correlation *Corr(x,y)* of two functions *x(t)* and *y(t)* is

$$Corr(x,y) = (x \star y)(\tau) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} x(t+\tau)y(t)\,dt$$

  The correlation is a function of τ, which is called *the lag*.

  Unlike for convolution, $x \star y \neq y \star x$

- **The cross-correlation theorem:** The Fourier transform of the cross-correlation of two functions is the product of the individual Fourier transforms , where one of them has been complex conjugated: $x(t) \star y(t) \Leftrightarrow X(v) \cdot Y^*(v)$

Time Domain Analysis: Autocorrelation

- The correlation of a function with itself is called its **autocorrelation**.
- The related autocorrelation theorem is also known as the ***Wiener-Khinchin theorem*** and states

$$x(t) \star x(t) \Leftrightarrow X(\nu)\, X^{*}(\nu) = |X|^{2}$$

- **The Fourier transform of an autocorrelation function is the power spectrum, or equivalently, the autocorrelation is the inverse Fourier transform of the power spectrum.**

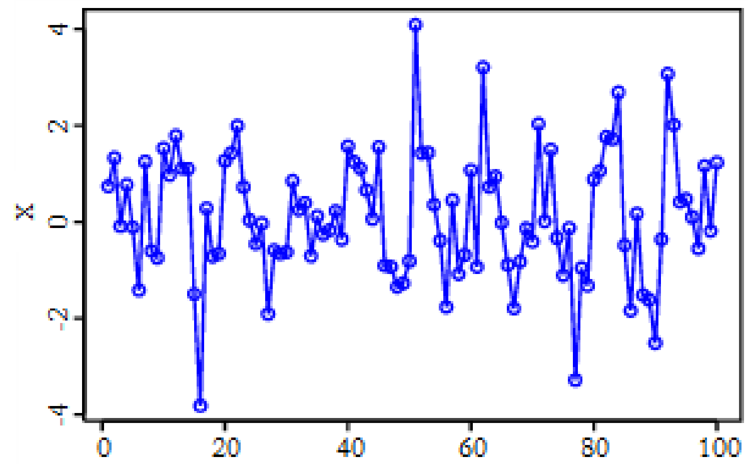
Time Series ⇔ DFT ⇔ Autocorr. ⇔ PSD

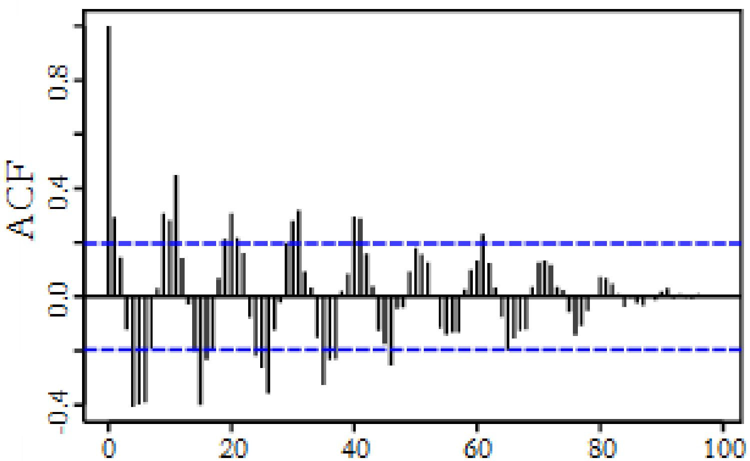
x_j (function)	$\Leftrightarrow$ DFT	X_k (transform)
$\Downarrow$		$\Downarrow$
$x_j \star x_j$ (autocorrelation)	$\Leftrightarrow$ DFT	$\lvert X_k \rvert^2$ (power spectrum)

Time Domain Analysis: Autocorrelation

- The *discrete autocorrelation* of a sampled function *x(t)* is just the discrete correlation of the function with itself.
- Obviously this is always symmetric with respect to positive and negative lags.

- 100 random numbers with a "hidden" sine function, and an autocorrelation of the series on the bottom.





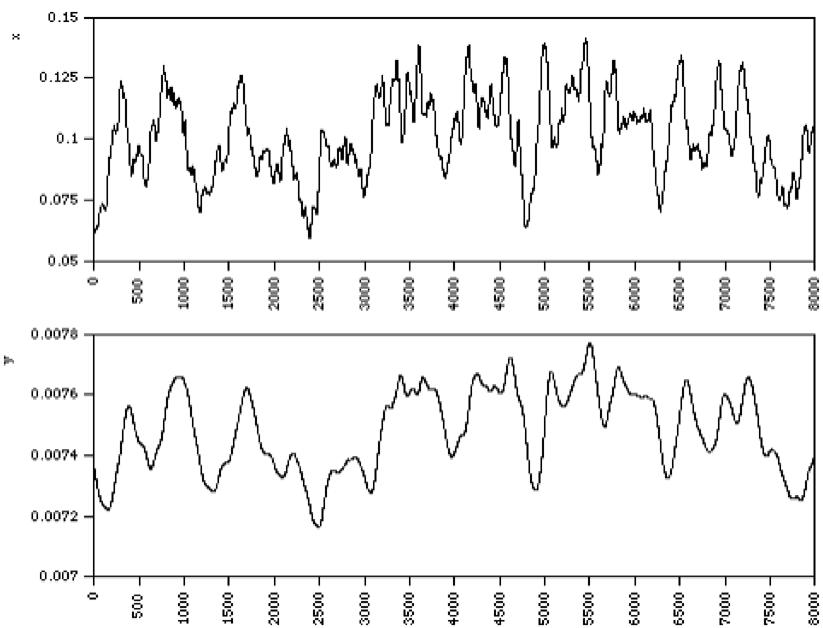
Cross Correlation

- Cross correlation is a standard method of estimating the degree to which two series are correlated.
- Consider two series $x(i)$ and $y(i)$ where i=0,1,2...N-1, with mx and my are the means of the corresponding series. The discrete cross correlation r at the lag d is defined as

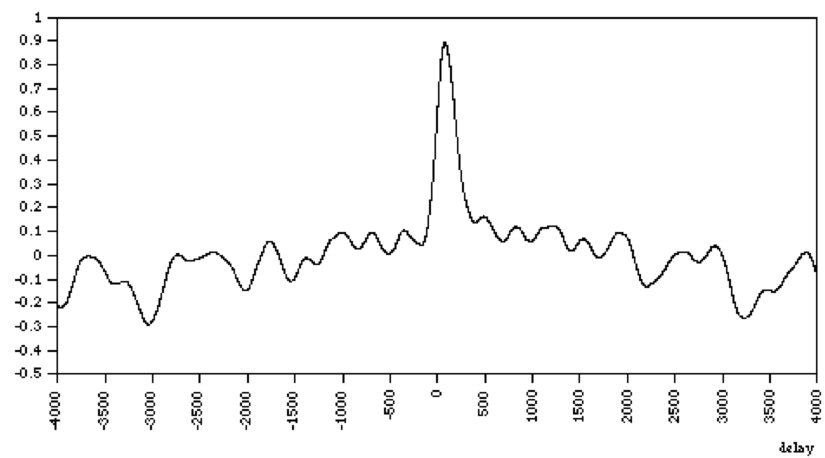
$$r(d) = \frac{\sum_i [(x(i) - mx) * (y(i-d) - my)]}{\sqrt{\sum_i (x(i) - mx)^2} \sqrt{\sum_i (y(i-d) - my)^2}}$$

Cross Correlation

Example: two time series x,y.



The cross correlation with a maximum delay of 4000.



There is a strong correlation at a delay of about 40.

O–C diagram [1]

- Basic period analysis consists of finding a reliable ephemeris of the main periodic variation and modelling of the first order effects.
- The model is usually a linear ephemeris, i.e. a prediction of eclipse times that assumes a constant period.
- ***O-C*** diagram is a powerful diagnostic tool, which compares the actual timing of an event (e.g. the mid-point of an eclipse or a pulsation cycle peak) to the moment we expect this event is occurred in a case of constant periodicity.
- ***O-C*** stands for **O**[bserved] minus **C**[alculated]

O–C diagram [2]

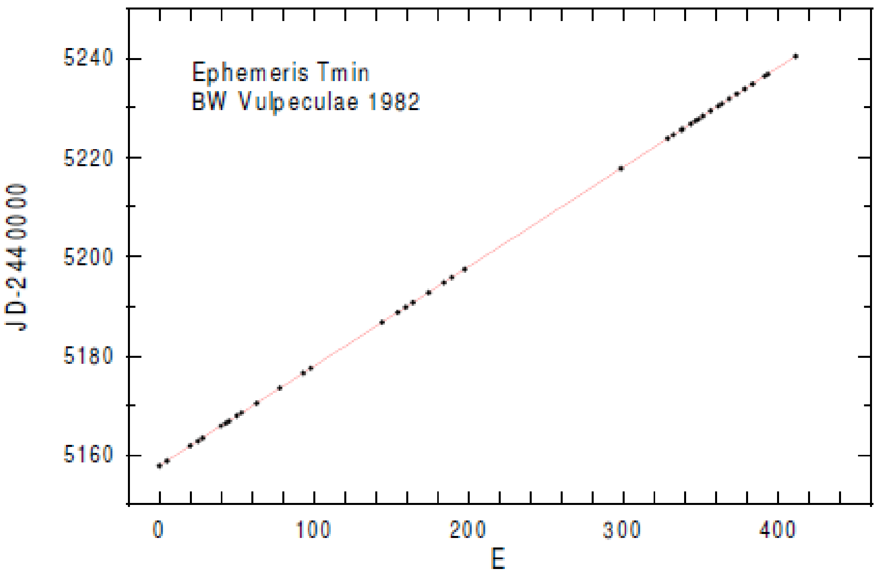
- It might appear that a period is incorrect OR variable.
- The period variations are usually delicate. By building ***O-C*** diagrams one can measure very subtle changes in the period happening with the star.
- The horizontal axis of the an **O-C** diagram most often represent time, usually expressed in days or cycles. The vertical axis is the "*O-C* " part which can expressed in days or a fraction of the period.
- Different phenomena, such as a constant but incorrect period, period increasing or decreasing at a constant rate, or sudden period changes but constant period thereafter, have distinct patterns on the ***O-C*** diagrams.

O–C diagram: constant periods [1]

If the period is constant and if its value is known, then

$$T_m = T_o + P\,E$$

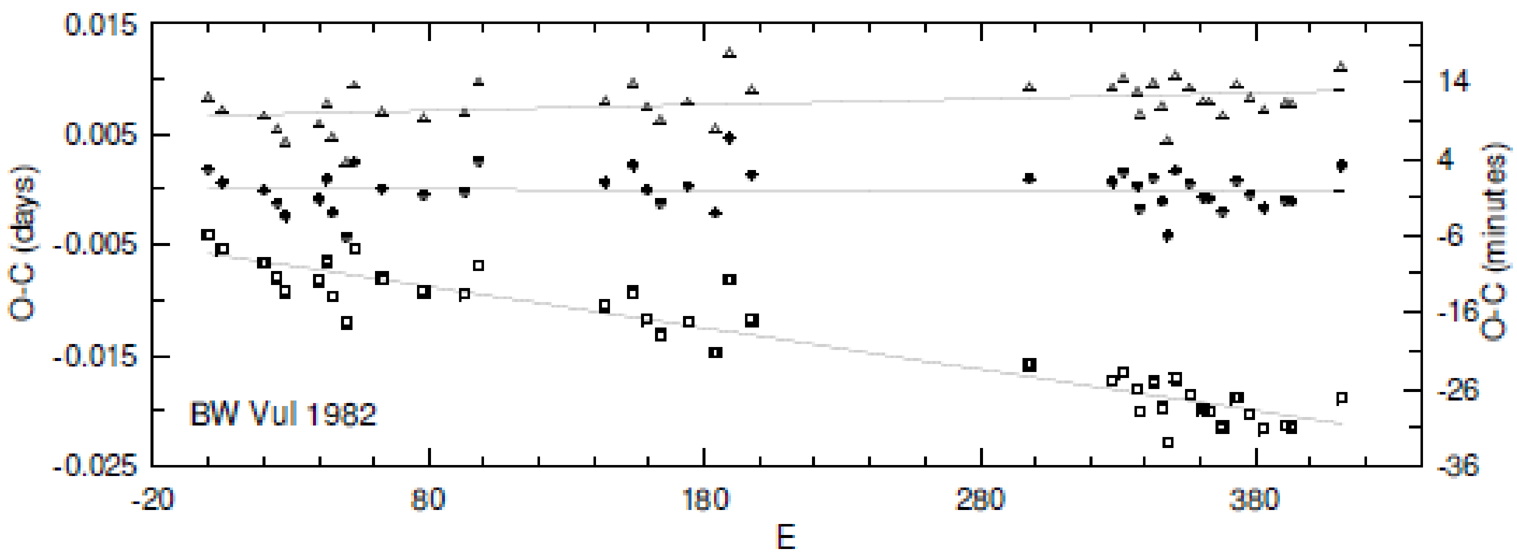
where T_m is the time of maximum or minimum light, T_o is the zero epoch and E is the number of cycles elapsed since the zero epoch.



T_o and P are obtained through a least-squares solution (like we did in the very beginning of the course). The ephemeris calculation yields $T_{min} = 2445157.8072 + 0.201043E$

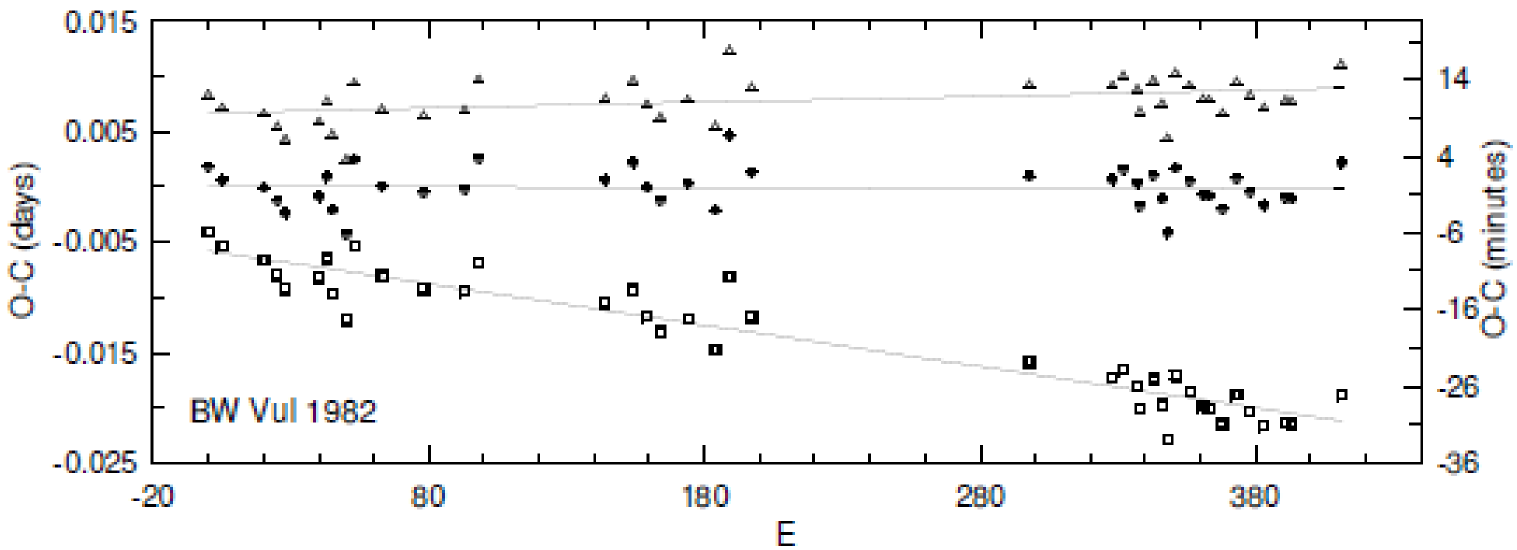
O–C diagram: constant periods [2]

- The ephemeris: $T_{\min} = 2445157.8072 + 0.201043E$



  - Middle set: based on the period P=0.201043;
  - Lower graph: slightly longer period P=0.201080;
  - Top: using slightly shorter period P=0.201037.

O–C diagram: constant periods [3]



- ***O-C*** diagram showing a positive slope indicates that the real period is longer than the period used to construct the diagram;
- A negative slope points to a real period that is shorter than the assumed one.

O–C diagram: changing periods

- **Changes** of period could be described by any mathematical formula expressing ***P*** as a function of time.

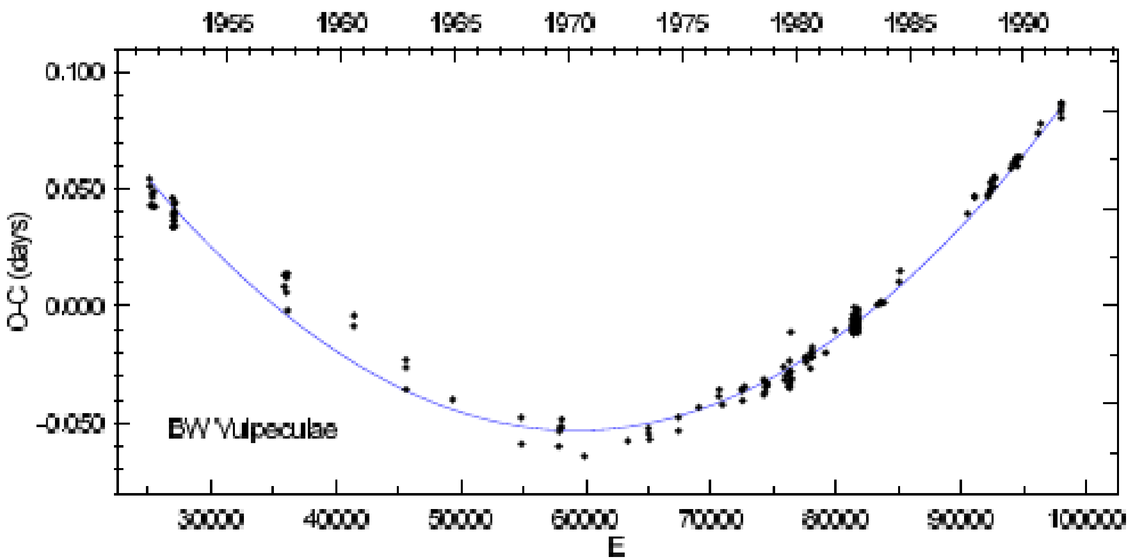
$$T_m = T_0 + \int P(t)dt \qquad \text{or} \qquad T_m = T_0 + \int P(E)dE$$

- In most cases this relation is restricted to linear variations, cyclic variations, or a combination of both.

O–C diagram: changes linear with time

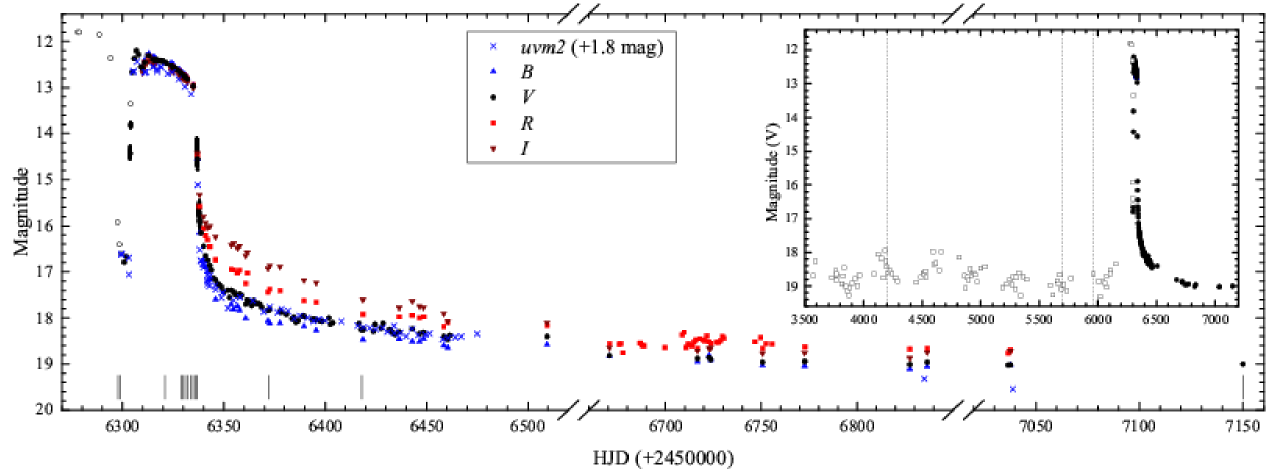
- If $\bar{P}$ is the average period over the time interval, then (**show it!**)

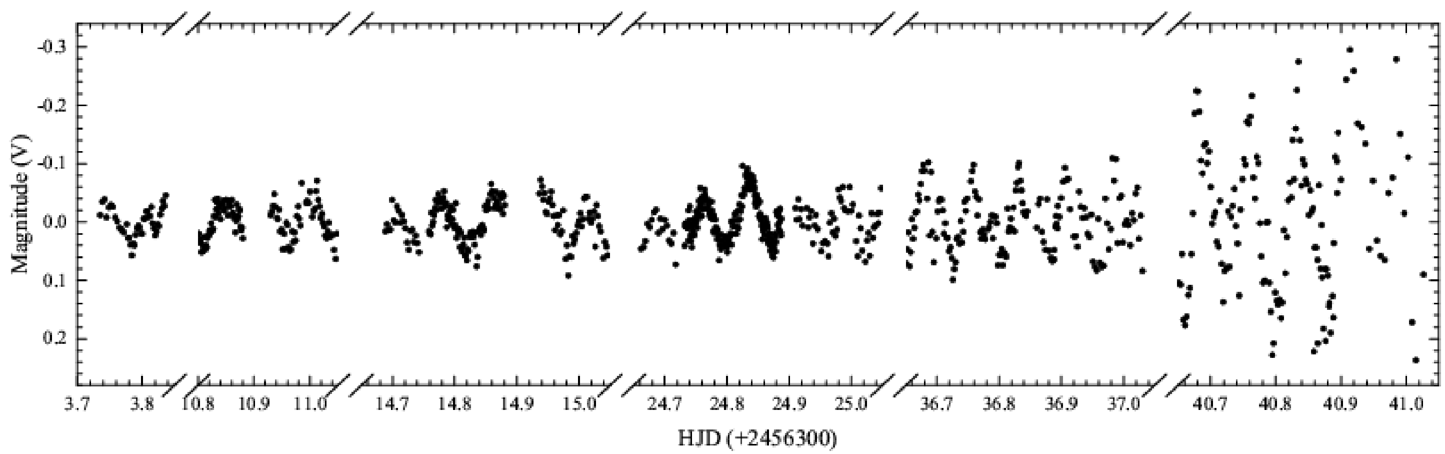
$$O - C = \frac{1}{2}\frac{dP}{dt}\bar{P}E^2$$



CVs: Superoutbursts and superhumps

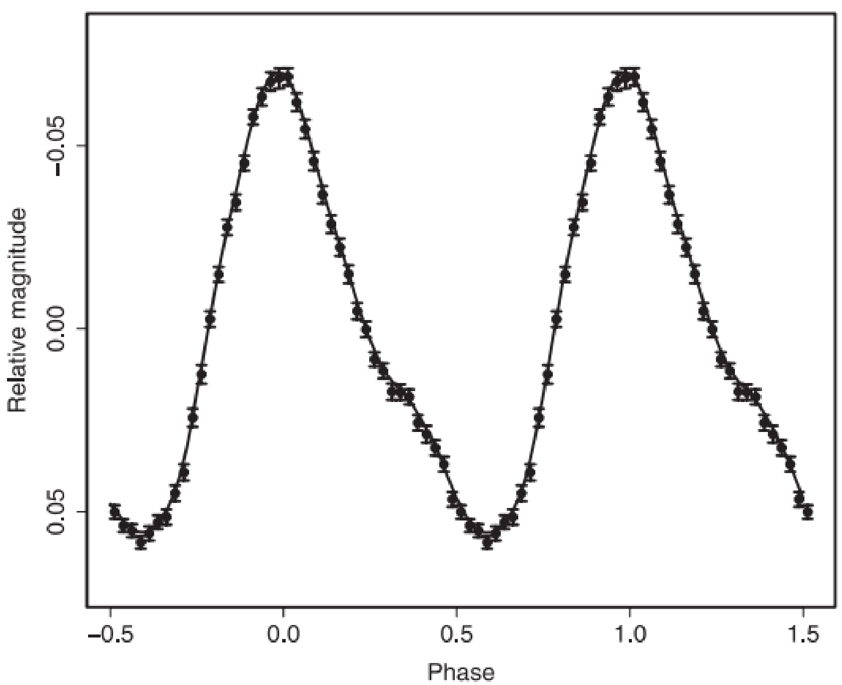
SSS J122221.7-311525



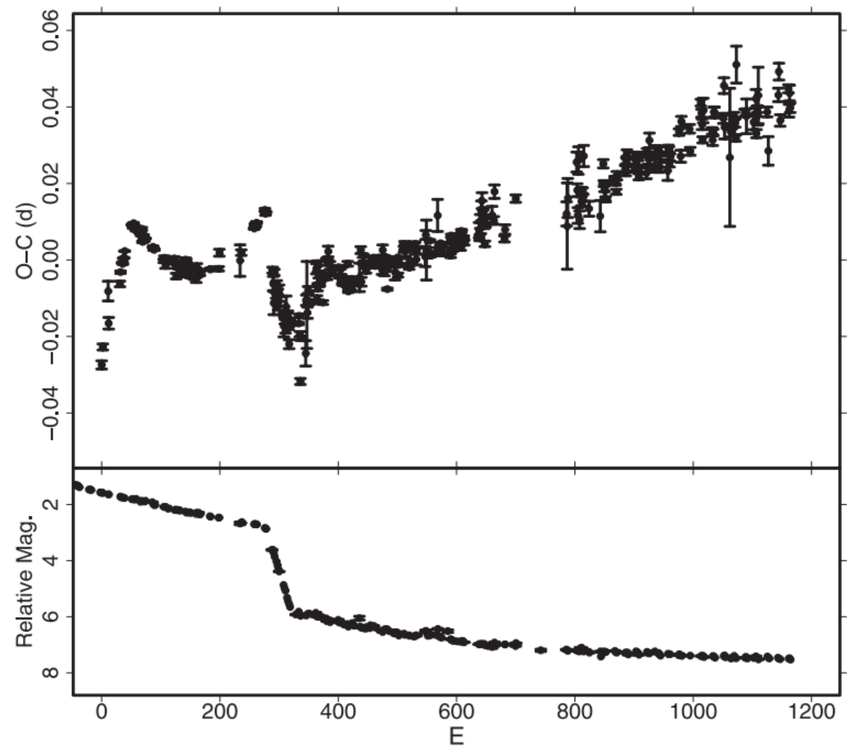


Superhumps: GW Lib

Superhump Light Curve



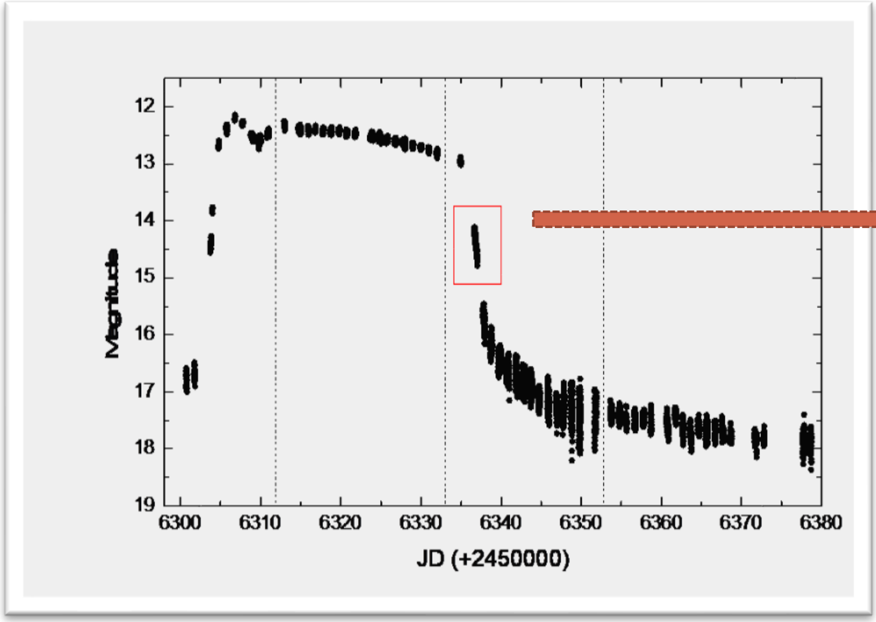
O-C diagram



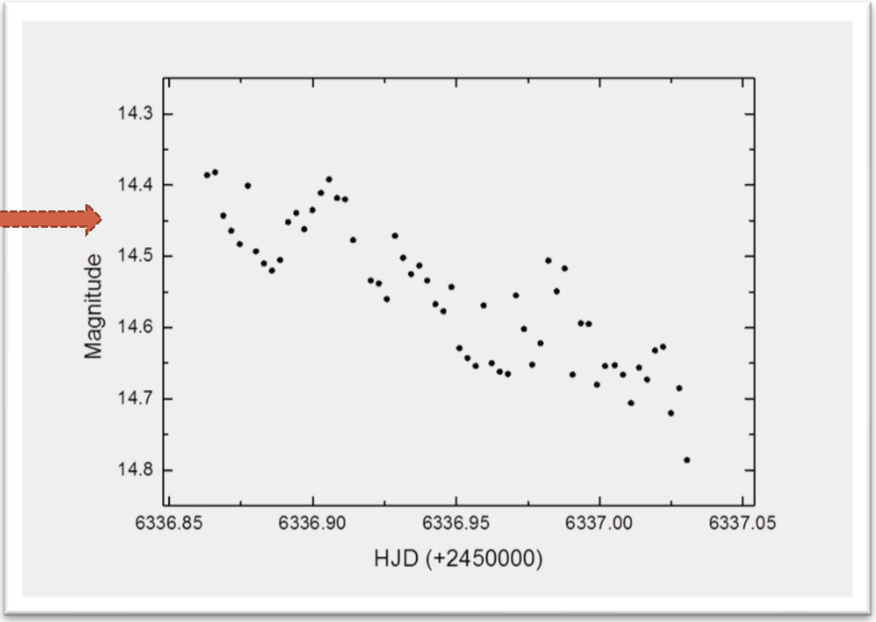
Detrending

- *Trend* in a time series is a slow, gradual change in some property of the series over the whole interval under investigation.
- Trend is sometimes defined as a long term change in the mean, but can also refer to change in other statistical properties.
- *Detrending* is the statistical or mathematical operation of removing trend from the series.

Detrending



Magnitude
12
13
14
15
16
17
18
19
6300
6310
6320
6330
6340
6350
6360
6370
6380
JD (+2450000)



Magnitude
14.3
14.4
14.5
14.6
14.7
14.8
6336.85
6336.90
6336.95
6337.00
6337.05
HJD (+2450000)

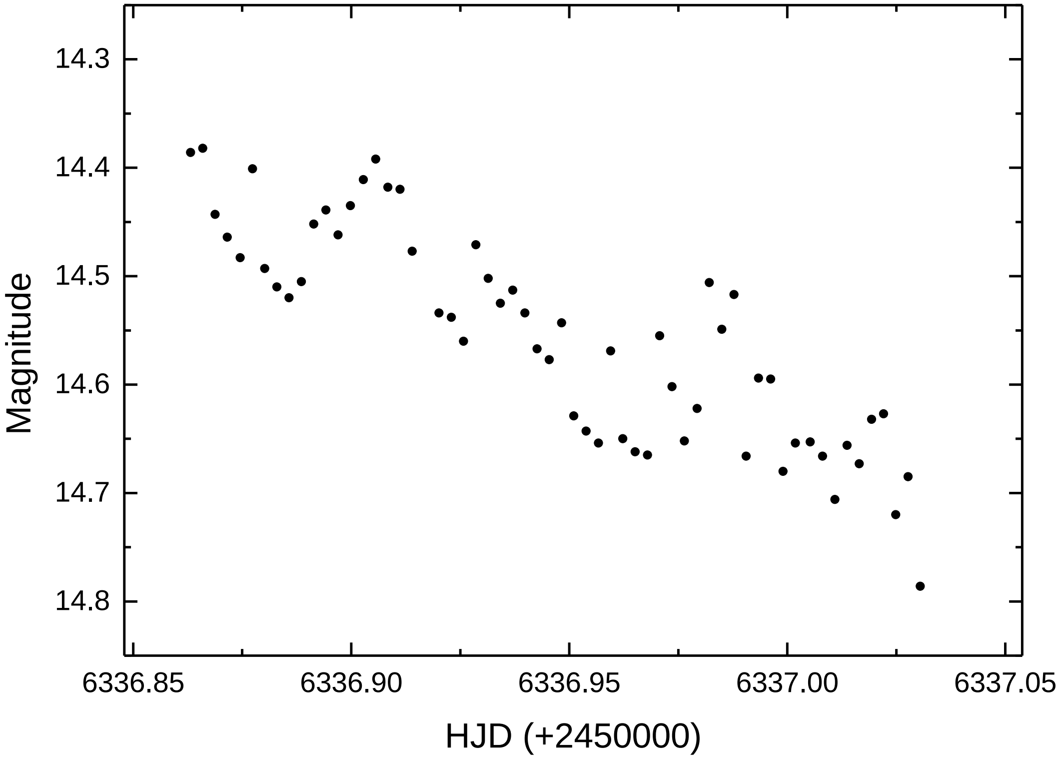
Detrending

- In studying and removing trend, it is important to understand the effect of detrending on the spectral properties of the time series. This effect can be summarized by the *frequency response* of the detrending function.

- Many alternative methods are available for detrending:
  - A simple and widely used function of time is the least-squares-fit straight line, which assumes linear trend.
  - Other functions of time (e.g., quadratic) might be better depending on the type of data.
  - An alternative to fitting a curve to the entire time series (curve fitting) is to fit polynomials of time to different parts of the time series.
  - Sometimes the mathematical form of the trend function has physical basis.

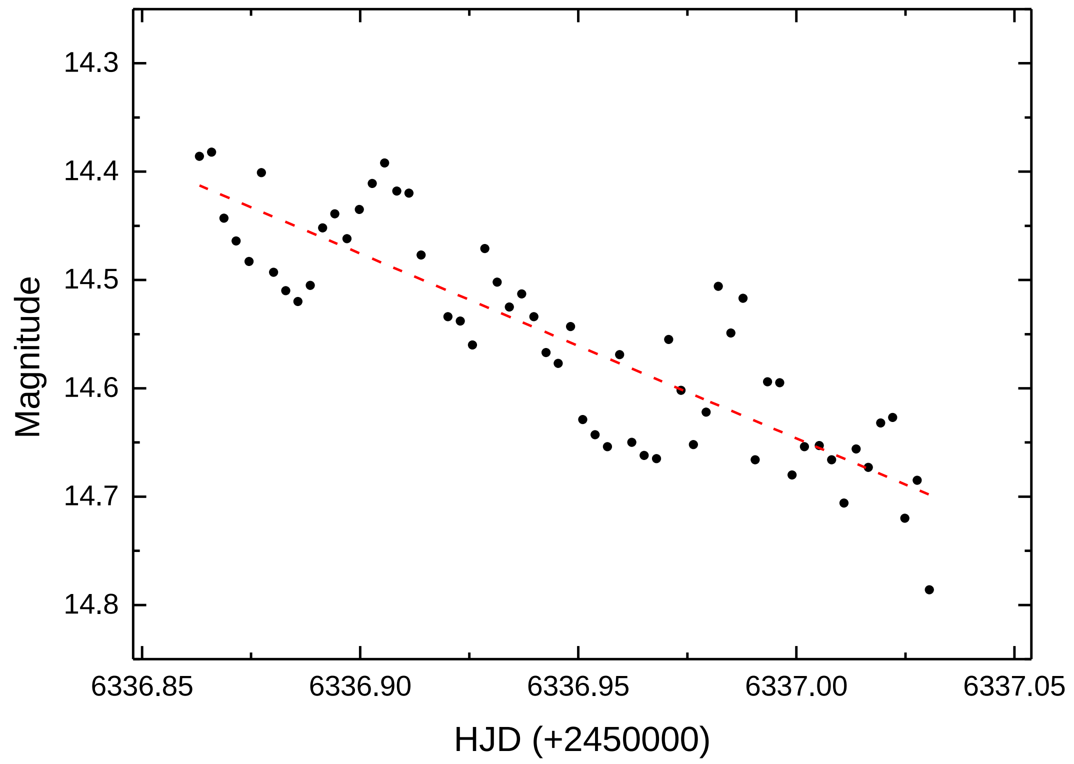
Detrending

Original light curve



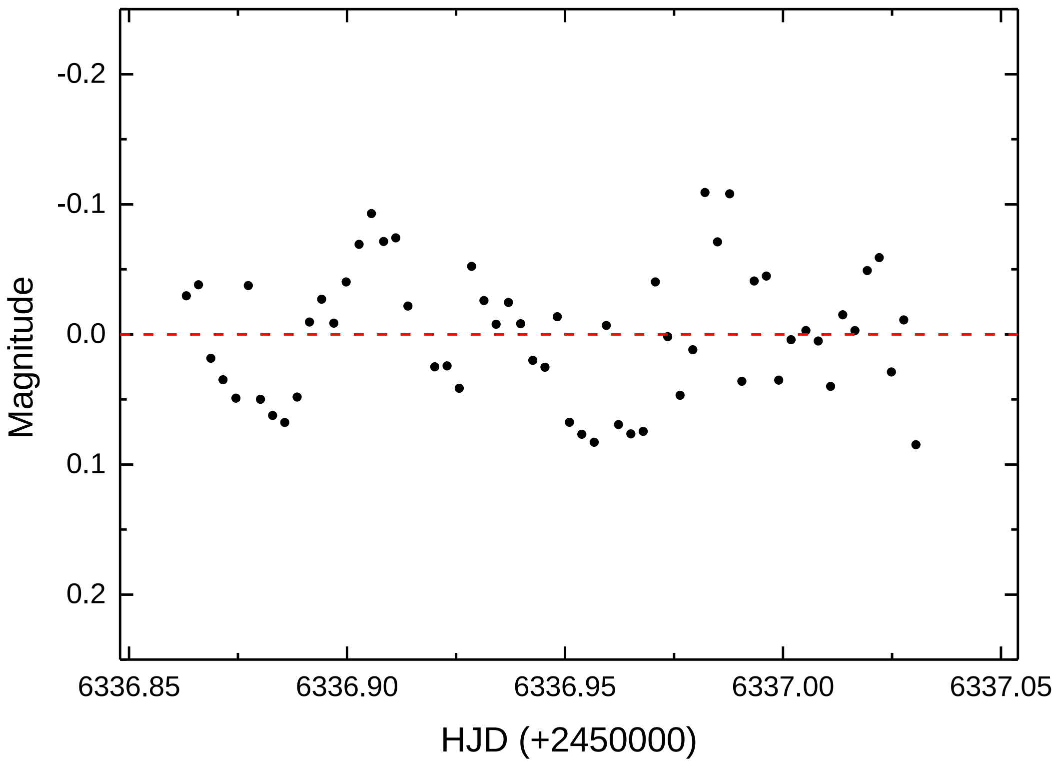
Detrending

The least-squares-fit straight line to the light curve



Detrending

Detrended light curve



Detrending

- Identification of trend in a time series is subjective because trend cannot be unequivocally distinguished from low frequency fluctuations. What looks like trend in a short segment of a time series segment often proves to be a low-frequency fluctuation – perhaps part of a cycle – in the longer series.

- We can view the entire observed time series as a segment of an unknown infinitely long series, and cannot be sure that an observed change in mean over that segment is not part of some low-frequency fluctuation imparted by a stationary process.