Noise Power Distribution

- Flux measurements are **always** accompanied by noise.
- The light curve can be divided into its independent components: the deterministic signal ***S*** and the noise ***N***. For an individual time bin, the total number of counts is composed of the sum of the signal and the noise, i.e., $x_k = s_k + n_k$.
- Noise powers follow a chi-squared distribution with 2 dof:
  - $P_j \propto A_j^2 + B_j^2$, where $A_j = \sum_k x_k \cos \omega_j t_k$ and $B_j = \sum_k x_k \sin \omega_j t_k$; $\quad k = 0, \ldots, N-1$
  - So, each A_j and each B_j is a linear combination of the x_k. Hence if the x_k are normally distributed then A_j and B_j are as well → $P_j \propto \chi^2$ with 2 dof by definition.
  - If x_k follow some other distribution (e.g. Poisson) then the central limit theorem ensures that A_j and B_j are still approximately normal (for large N) → P_j are still approximately χ^2 with 2 dof.
  - Exact expressions depend on the normalization of the P_j.

Power Spectrum – Leahy Normalization

- We will adopt the **Leahy** et al. (1983) normalization:

$$P_j \equiv \frac{2}{N_{tot}} |a_j|^2 \quad j = 0, \dots, \frac{N}{2}; \;\; where\; N_{tot} = N_{ph} = \sum_k x_k = a_0$$

- The Leahy normalization is chosen such that if the x_k are Poisson distributed, then the P_j exactly follow the chi-squared distribution with 2 dof, χ^2.
- For the Poisson process, the variance (square of the standard deviation) is equal to the number of counts.

Properties of Leahy normalized PDS

- Variance in the real time series x_k:

$$Var(x_k) \equiv \sum_k (x_k - \bar{x})^2 = \sum_k {x_k}^2 - \frac{1}{N}\left(\sum_k x_k\right)^2 =$$

$$= \frac{1}{N}\sum_j |a_j|^2 - \frac{1}{N}{a_0}^2 = \frac{1}{N}\sum_{\substack{j=-N/2 \\ j\neq 0}}^{N/2-1} |a_j|^2$$

$$\sum_{k=0}^{N-1} {x_k}^2 = \frac{1}{N}\sum_{j=-N/2}^{N/2-1} |a_j|^2$$

Parseval's theorem

Leahy normalization

$$P_j \equiv \frac{2}{N_{tot}}|a_j|^2 \quad j = 0, \ldots, N/2$$

$$Var(x_k) = \frac{N_{tot}}{N}\left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2}P_{N/2}\right)$$

variance is sum of powers!

The dimension of P_j is the same as x_k and a_j: $[P_j] = [a_j] = [x_k]$

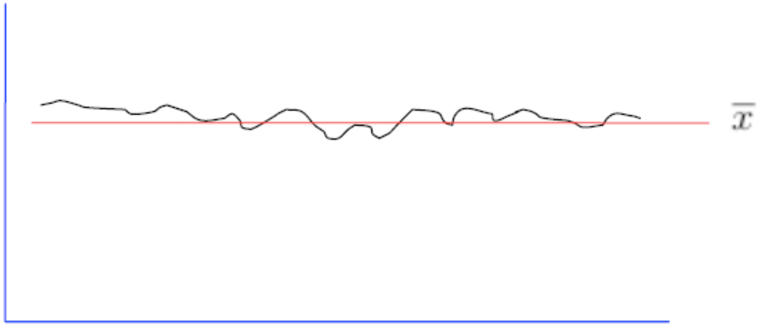
$P_j \propto |a_j|^2/a_0$

Properties of Leahy normalized PDS

Often, the variance is expressed in terms of
Fractional root-mean-square (rms) amplitude of a signal in a time series x_k:

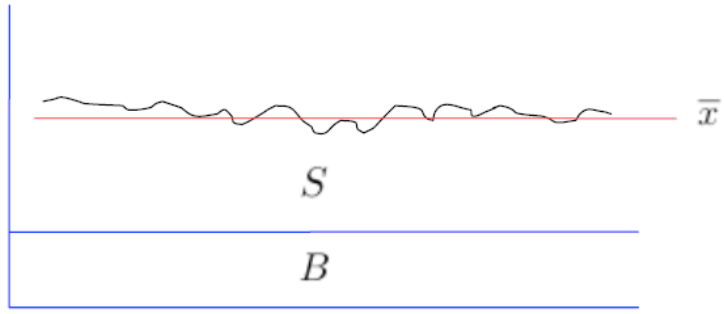
$$r \equiv \frac{\sqrt{\frac{1}{N} Var(x_k)}}{\bar{x}} = \frac{N}{N_{tot}} \sqrt{\frac{N_{tot}}{N^2} \left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2} P_{N/2} \right)} = \sqrt{\frac{1}{N_{tot}} \left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2} P_{N/2} \right)}$$

r is dimensionless and often expressed in % (percentage rms variation).



Properties of Leahy normalized PDS

- "rms normalized" power density: $q(\nu_j) \equiv TP_j/N_{ph}$
  physical unit of $q(\nu_j)$ is (rms/mean)²/Hz
- "Source" fractional rms amplitude: If the x_k are the sum of source and background: $x_k = b_k + s_k$, then the rms amplitude as a fraction of just the s_k:
  $r_s = r\frac{B+S}{S}$,
  where B and S are sums of the b_k and s_k, so $B+S = \sum_k x_k = N_{ph}$



- "Source rms normalized" power density ("Miyamoto" normalization):
  $q_S \equiv q\left(\frac{B+S}{S}\right)^2 = TP_j\frac{B+S}{S^2}$
  the same unit as q: (rms/mean)²/Hz

Requires a model or a measurement of the background count rate

Leahy normalization of the PDS of a sinusoid

The power spectrum of a sinusoid ***A sin (2πν$_{sine}$t$_k$+φ)***:
Slide 80

$$|a_j|^2 = \frac{1}{4}A^2N^2\left(\frac{\sin\pi x}{\pi x}\right)^2\left[\left(\frac{\pi x/N}{\sin\pi x/N}\right)^2 + \left(\frac{\pi x/N}{\sin[\pi(2j+x)/N]}\right)^2 + \right.$$

$$\left. + 2\left(\frac{\pi x/N}{\sin\pi x/N}\right)\left(\frac{\pi x/N}{\sin[\pi(2j+x)/N]}\right)\cos[(N-1)(2\pi(j+x)/N)+2\phi]\right]$$

$$x = (\nu_{\rm sine} - \nu_j)T$$

$$\approx \frac{1}{4}A^2N^2\left(\frac{\sin\pi x}{\pi x}\right)^2 \qquad x/N \ll 1 \text{ and } 0 \ll j/N \ll \frac{1}{2}$$

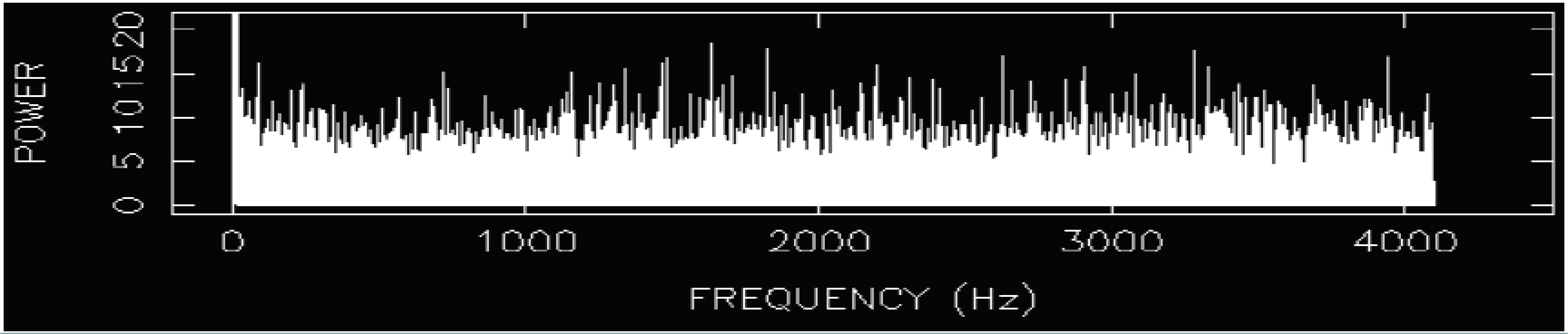
Leahy normalization

$$P_j \equiv \frac{2}{N_{tot}}|a_j|^2 \qquad j = 0, \ldots, N/2$$

Then, for x=0, $P_{j,\,sine} = \frac{1}{2}\frac{N^2}{N_{tot}}A^2$

Properties of Leahy normalized PDS

- The Leahy normalization is chosen such that if the x_k are Poisson distributed, then the P_j exactly follow the chi-squared distribution with 2 dof, χ^2.
- Properties of this distribution:
  - The mean power is 2;
  - the standard deviation is 2!
- So, the power spectrum is very noisy. This does not improve with:
  - longer observation — you just get more powers
  - broader time bins — you just get a lower ν_{Ny}



Statistics of Power Spectra

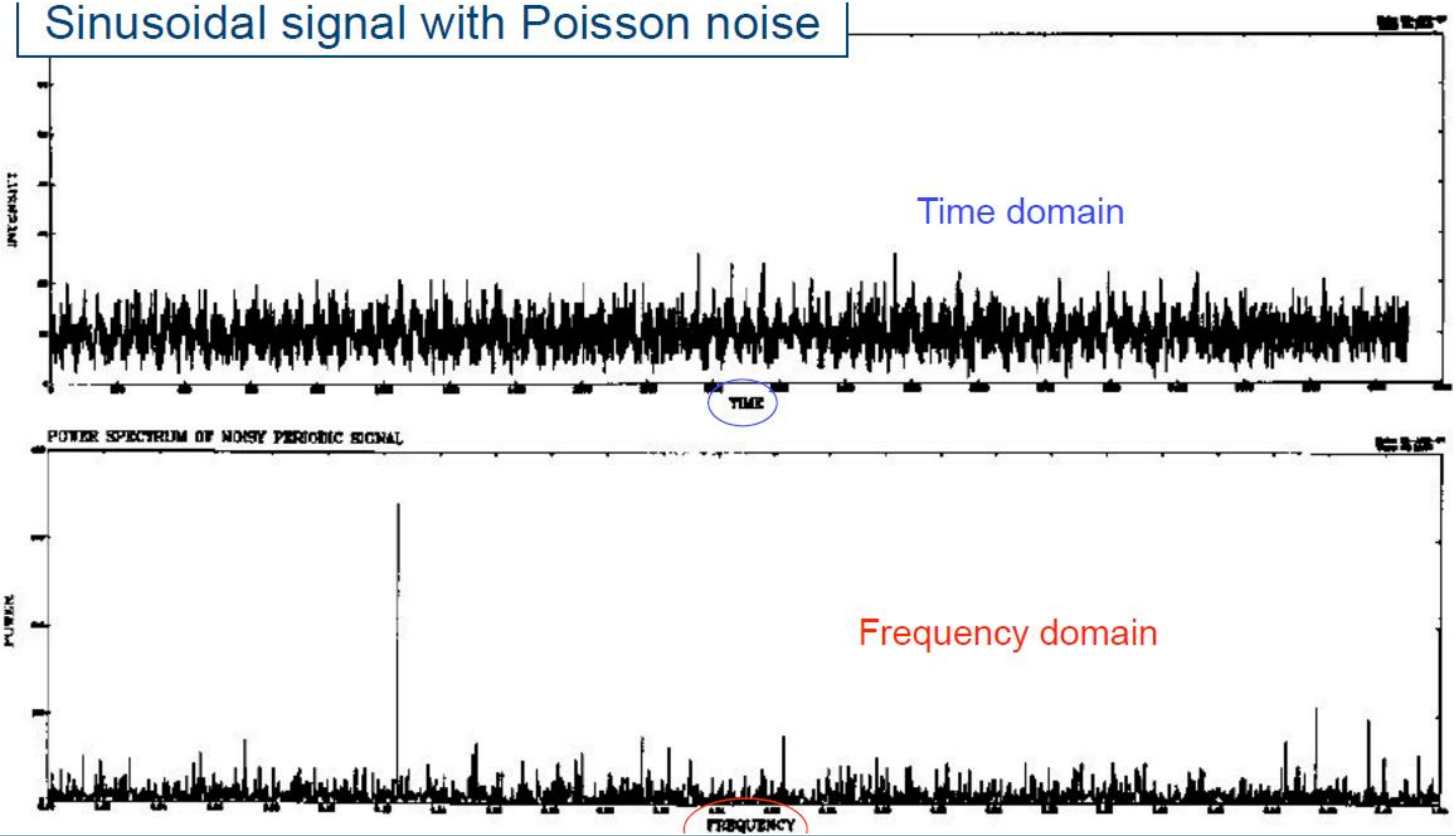
- Flux measurements are always accompanied by noise.
- The light curve can be divided into its independent components: the deterministic signal S and the noise N. For an individual time bin, the total number of counts is composed of the sum of the **signal** and the noise, i.e., $x_k = s_k + n_k$.
- Examples of **deterministic signals**:
  - a non-periodic deterministic variation, such as a nova light curve;
  - A periodic variation, such as an eclipsing binary or a RR Lyr light curve;
  - a multiply periodic variation, such as a spectroscopic triple system;
  - a modulated periodic variation where either the amplitude, frequency, or phase may vary with time - for example a pulsating system in a binary orbit.

Statistics of Power Spectra

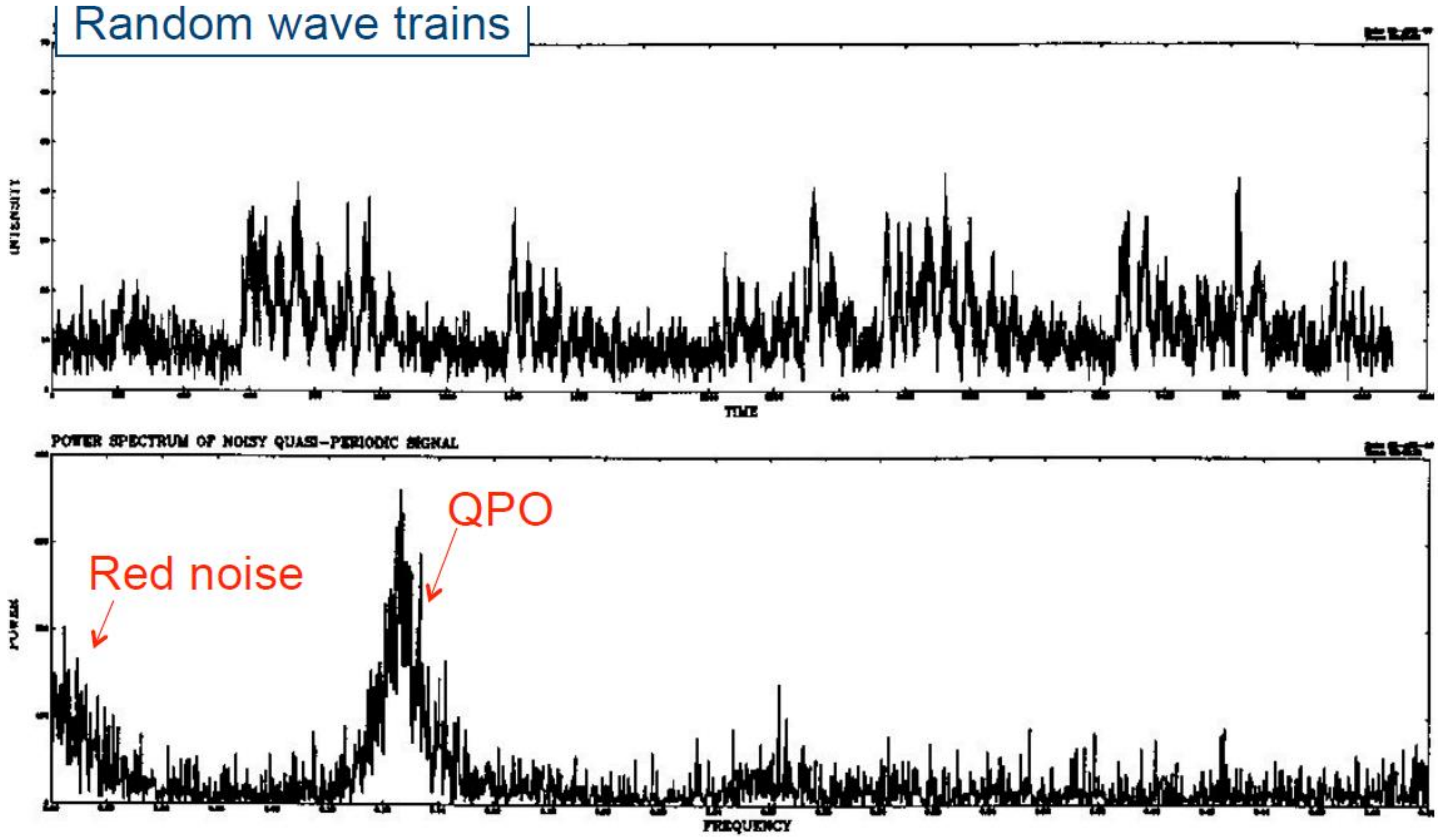
- 'Noise' (= random aka **stochastic processes**) in the light curve produces peaks and broad components in the power spectrum.
- Examples of noise:
  - Counting statistics noise (Poisson noise) -> white noise;
  - Poisson noise modified by instrumental effects (e.g. dead-time) and other instrumental noise;
  - Noise that is (stochastic) intrinsic source variability: QPO, band limited noise, red noise, etc.
- All these can occur at the same time, possibly together with deterministic signals.
- They can be the **background** against which you are trying to detect something else
- Or they can be the **signal** you are trying to detect.

Examples of power spectra: Periodic signal

Sinusoidal signal with Poisson noise

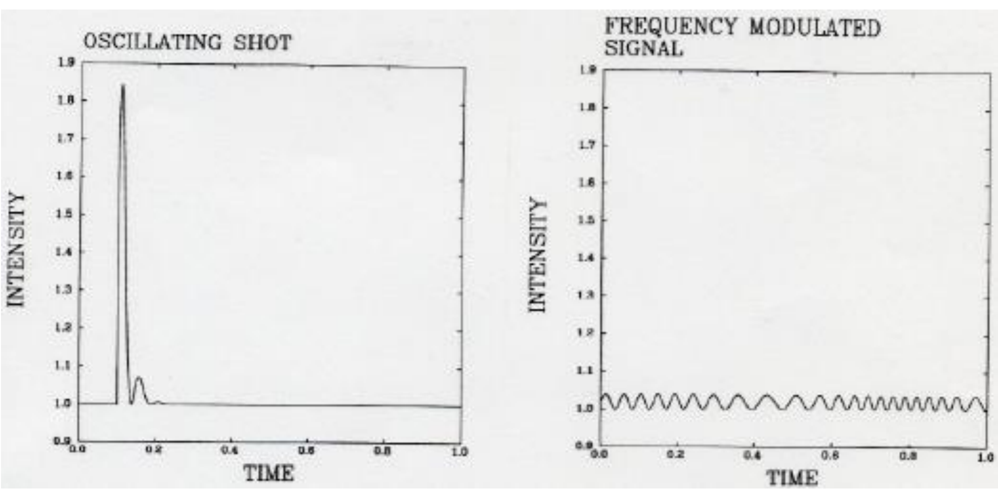


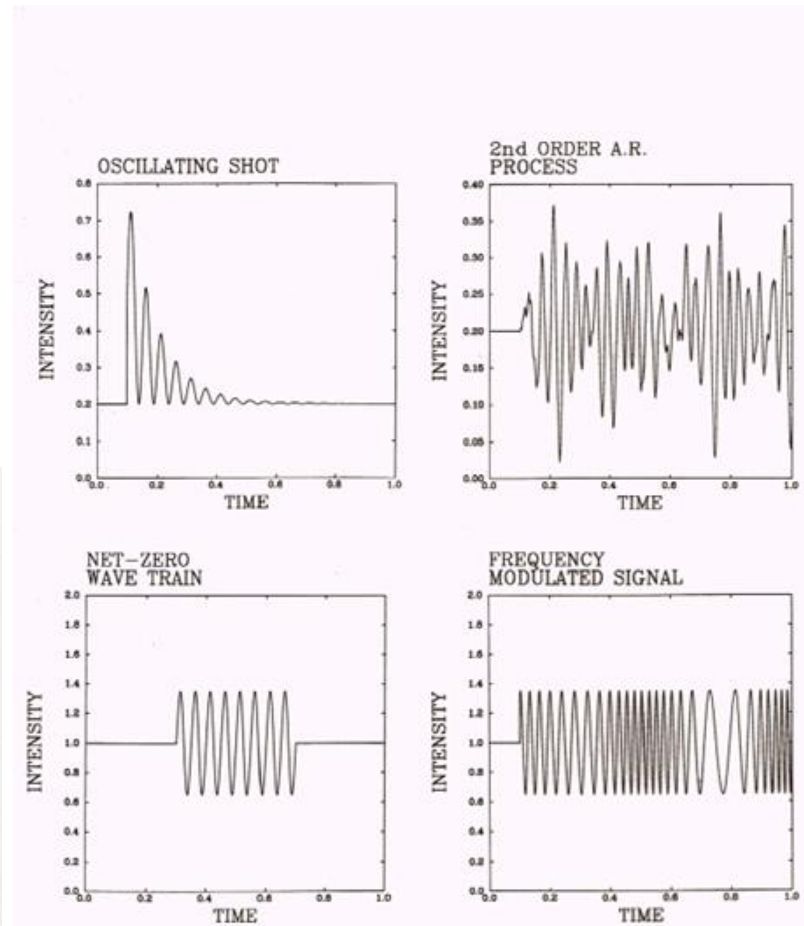
Examples of power spectra: QPO and red noise



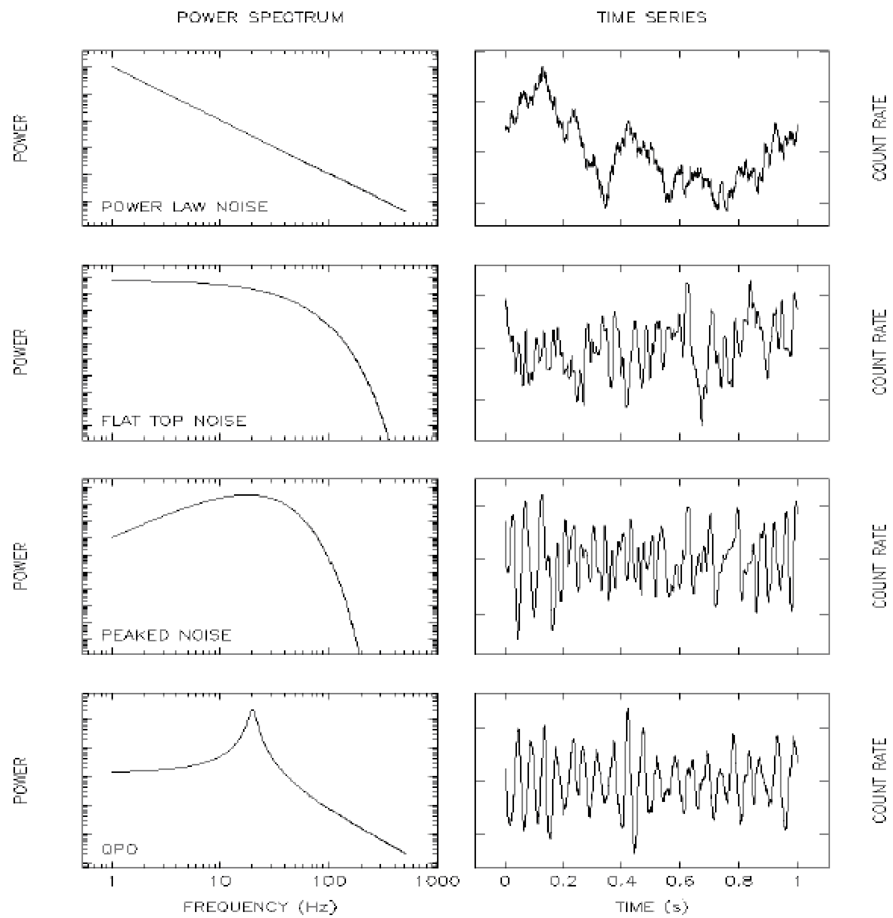
Various possible QPO signals

Various possible time-domain signals can underlay the QPO peak we see in frequency domain



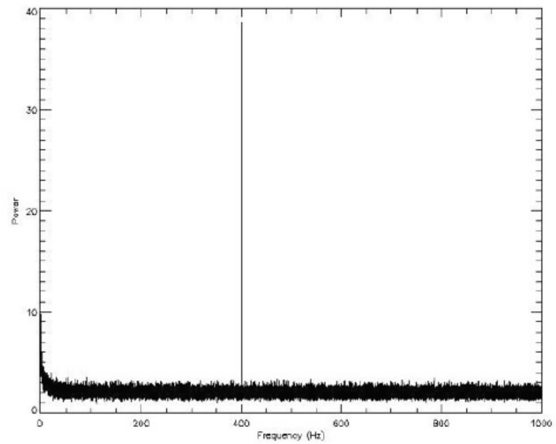


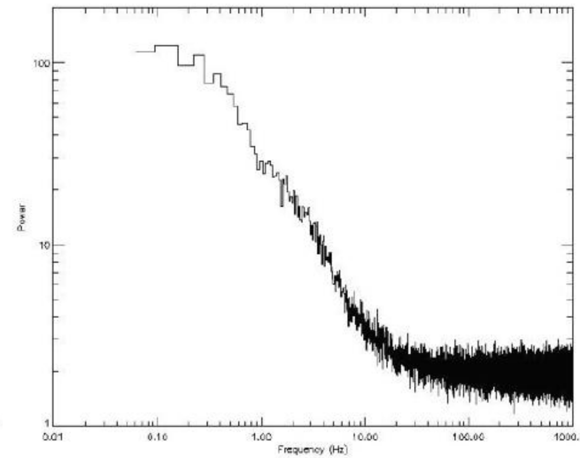
Statistics of Power Spectra

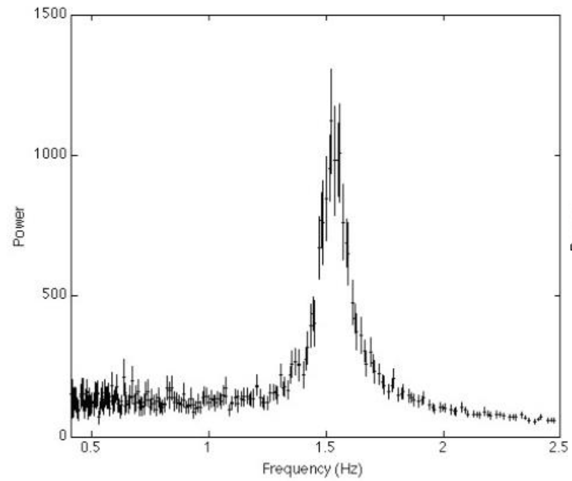


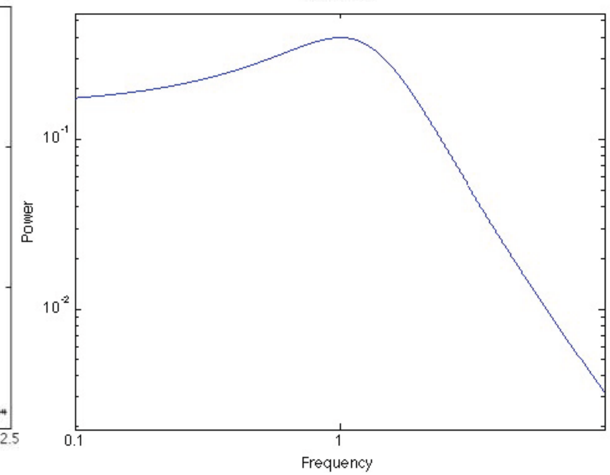
Main types of signals

- Coherent pulsation
- Broad-band noise
- Broad peak (QPO)
- Peaked-noise









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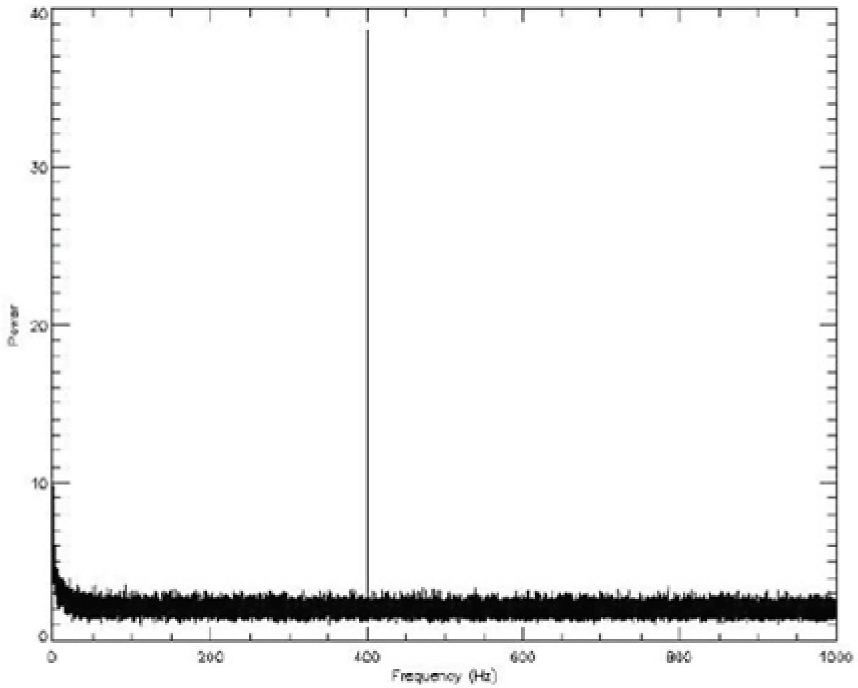

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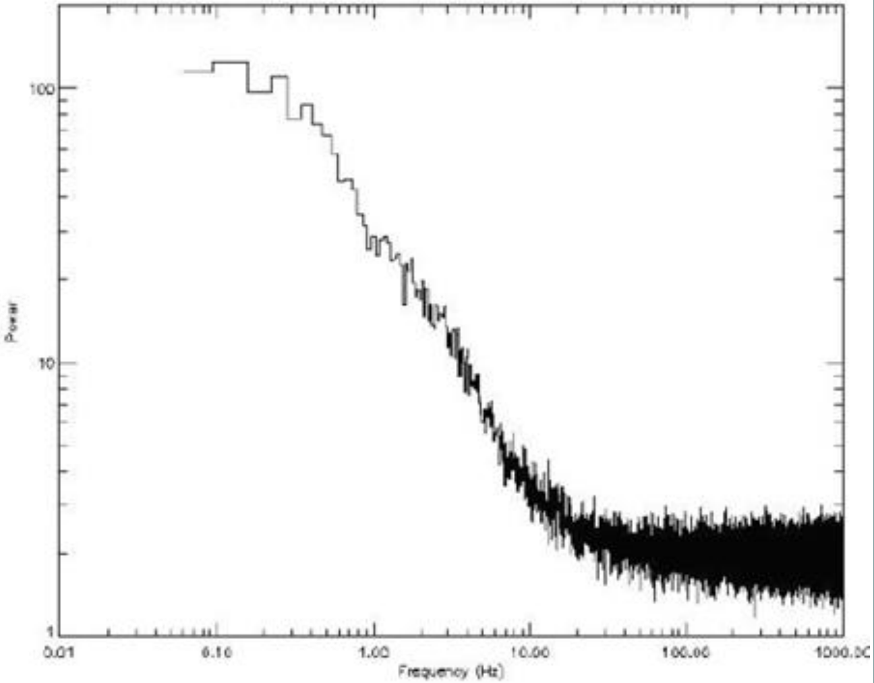

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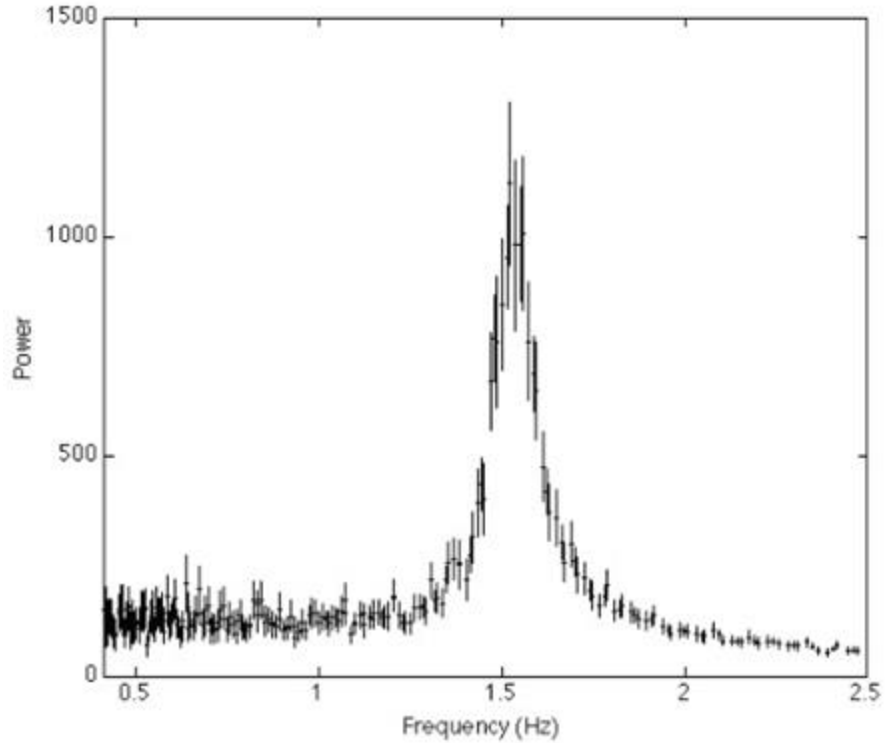
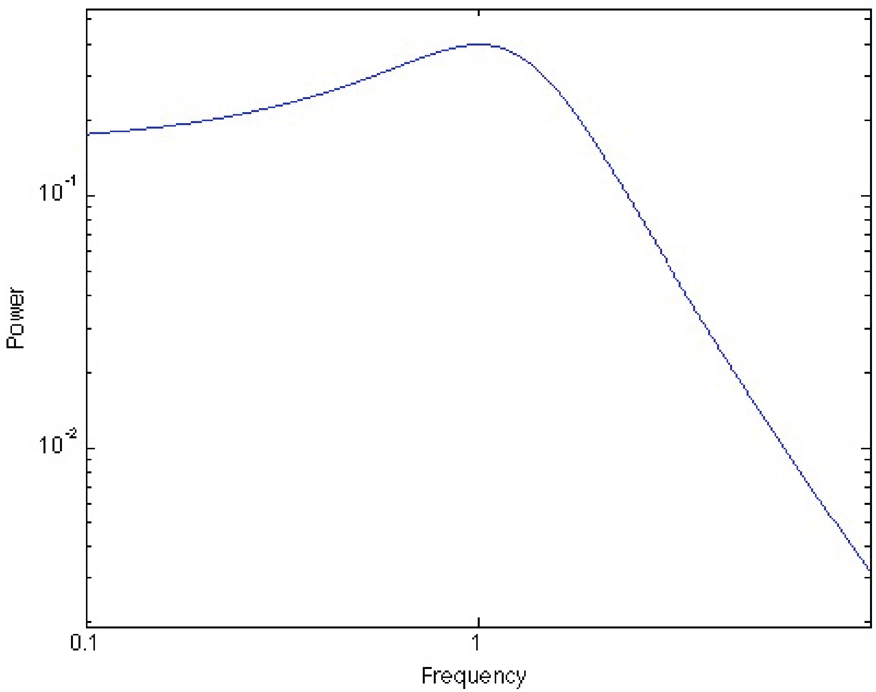


Main types of signals

- Coherent pulsation
- Broad-band noise
- Broad peak (QPO)
- Peaked-noise

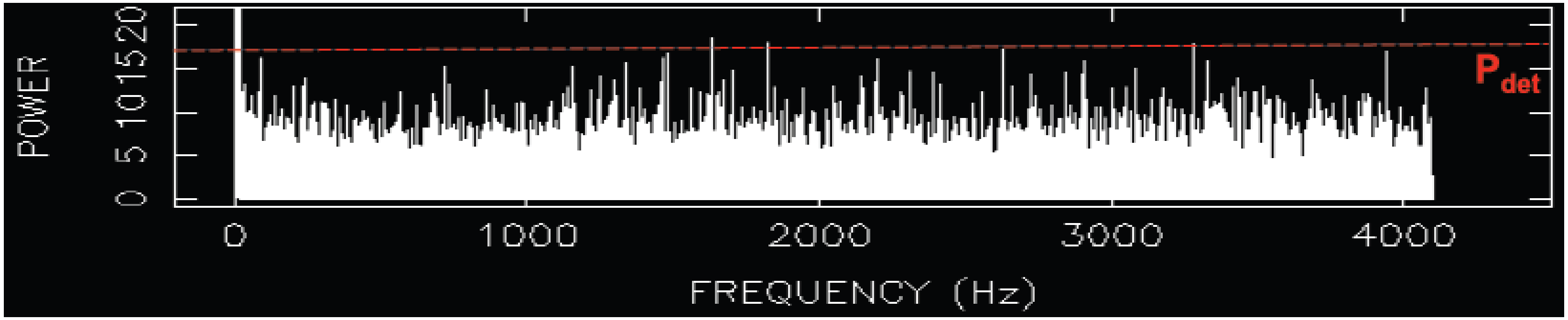


Coherent Signals

- Much analysis involves "coherent" signals, i.e. periodic signals whose phase is constant over the relevant duration
  - $Q = \nu/\Delta\nu >> 1000$
- Examples:
  - Pulses from rotating pulsars;
  - Orbital modulation or eclipses;
  - Precession periods.

Statistics of Power Spectra

- How to determine the significance of peaks found in power spectra? How big must a power be to constitute a significant excess over the noise?
- Let's define ε as the probability that a noise fluctuation exceeds P_{det}. The $(1-\varepsilon)$ confidence detection level P_{det} is a level that has a false alarm probability of ε. If there is just noise, $\mathrm{Prob}(P_j > P_{det}) = \varepsilon$. We want ε to be small, e.g., ε =1% for 99% confidence.
- If $P_j > P_{det}$ then with 99% confidence there is something else than just noise, a source signal.



Statistics of Power Spectra

- To determine P_{det}, we need to know **the noise power distribution**.

- **Warning:** Because in High-Energy Astrophysics we are counting individual photons, the relevant statistics are ***Poisson***, not ***Gaussian***.
- The Leahy normalization is chosen such that if the x_k are Poisson distributed, then the P_j exactly follow the chi-squared distribution with 2 degrees of freedom, χ^2. This is actually an exponential distribution:

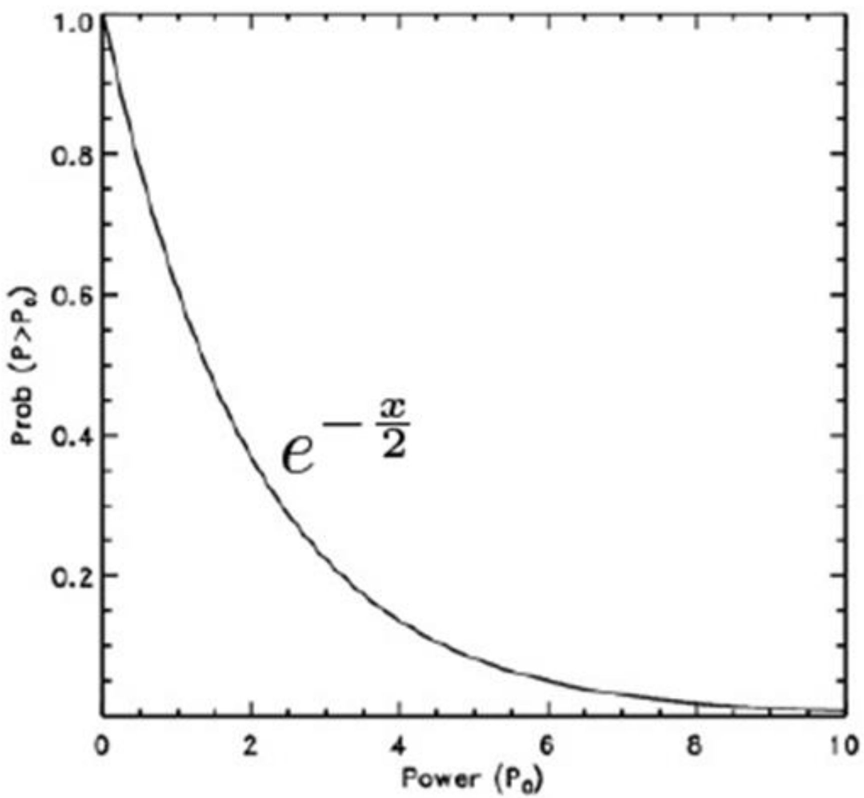
$$\varepsilon = Prob_{single}(P_j > P_{det}) = e^{-P_{det}/2} \longrightarrow P_{det} = -2 \ln \varepsilon$$

- Properties of this distribution: $\langle P_{noise} \rangle = 2$; $\mathrm{Var}(P_{noise}) = 4$

Statistics of Power Spectra

- Examples:
  - ε=1% corresponds to P_{det}=9.2;
  - a power of 40 has a probability of $e^{-40/2}=2\times10^{-9}$ of being noise.
- Since a large number of independent frequencies N_{trial} are examined, the detection threshold has to be defined as that power that has an ε (small) probability to be exceeded in one frequency bin out of the N_{trial} examined.
  - One should divide ε by the number of trials.

$$\varepsilon = N_{trial}\, e^{-P_{det}/2}$$



Statistics of Power Spectra

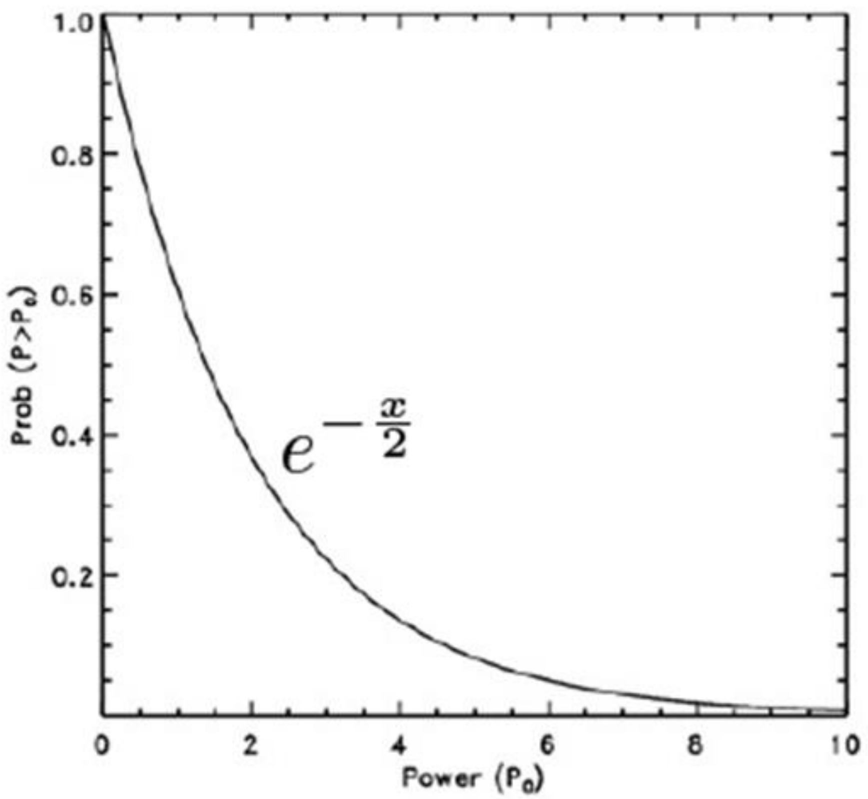
Important! The number of trial powers N_{trial} over which the search has been carried out:

N_{trial} = to the powers in the PSD if all the Fourier frequencies are considered;

N_{trial} < than the powers in the PSD if a smaller range of frequencies has been considered.

- Examples (cont.): N_{trial}=10 000
  - ε=1% corresponds to P_{det}=27.6;
  - a power of 40 has a probability of $e^{-40/2}=2\times10^{-5}$ of being noise.

Still significant!!



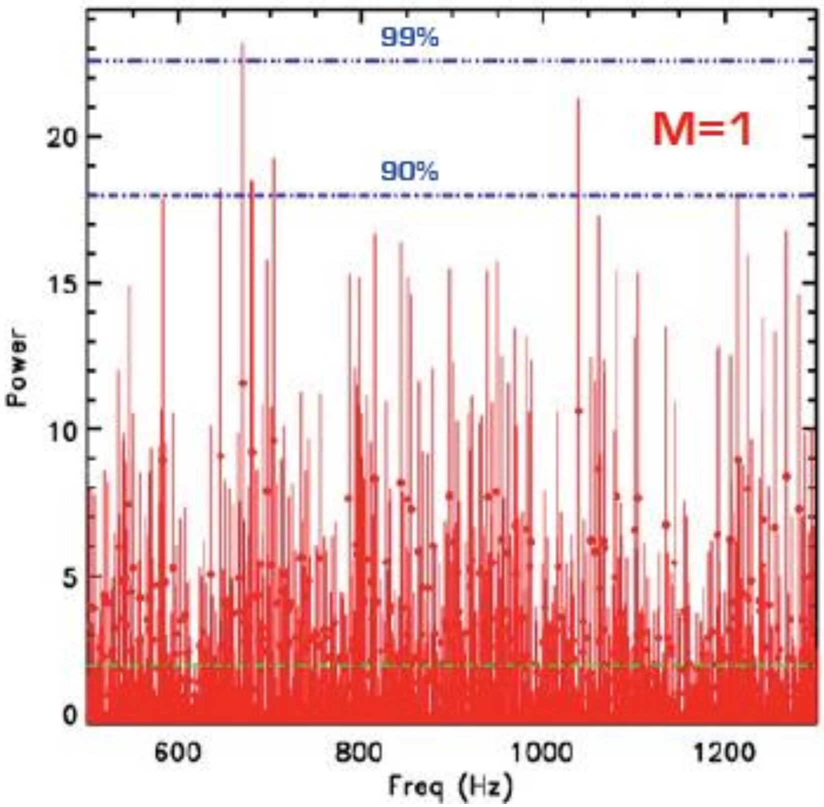
Rebinning and Averaging

- The power spectrum is very noisy. Smoothing methods:
  - Average several power spectra of subsegments of the time series;
  - Average adjacent bins in a power spectrum: rebinning;
  - Windowing is also possible.
- Averaged power distribution:
  - Individual P_j follow the chi-squared distribution with 2 dof.
  - Additive property of χ^2 distribution: sum of M powers is distributed as χ^2_{2M}
- M – the number of the time series, W – Frequency rebinning factor:

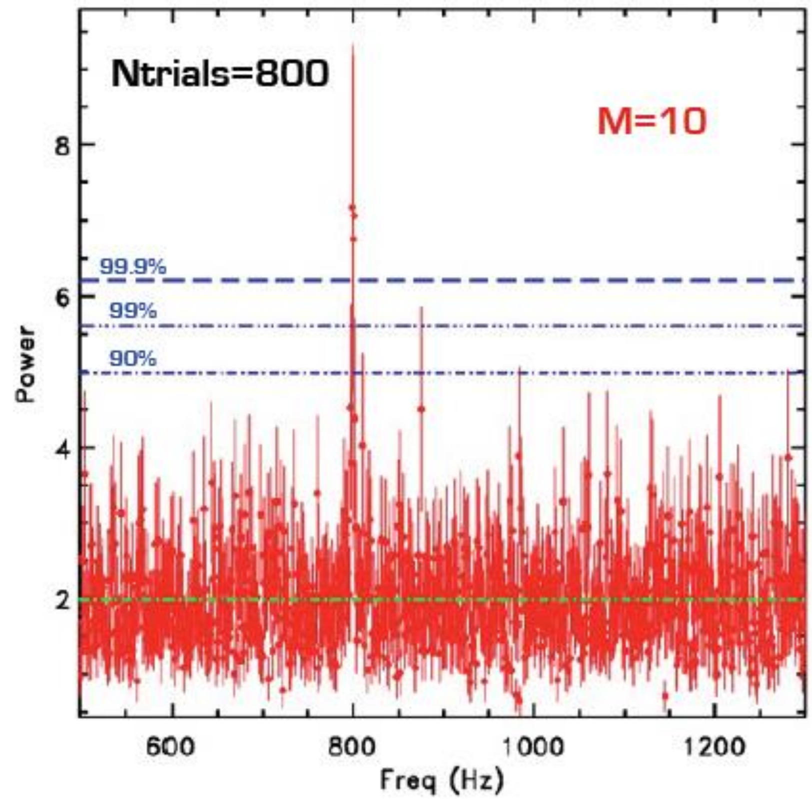
  $<P_{noise}>= 2$; $Var(P_{noise})=4/MW$ (the number of trials decreases)
- **Central limit theorem:**
  for large MW the distribution of $\overline{P_{WM}}$ tends to normal (Gaussian), with mean 2 and standard deviation $2/\sqrt{MW}$

Signal Detection

M=1,
Noisy PDS



M=10,
A signal is clearly detected



A note about rebinning

- **Coherent peak:** narrow power distribution – the longer the observation span, the better. The signal power to decrease by 1/MW.
  Is it worth to average or rebin? No.
  - The signal power decreases faster than the threshold power when averaging/rebinning;
  - If the frequency varies (orbital motion) is even worse as you average signal with noise.

- **Broad peak:** broad power distribution - length of observation not crucial - rebinning helps.

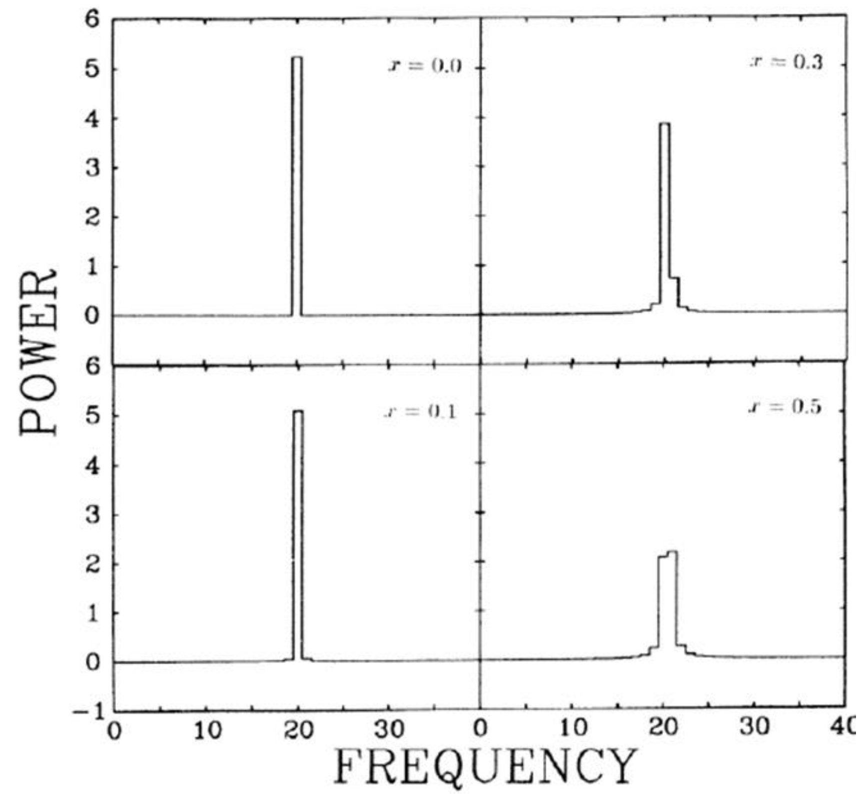
Signal detection optimization

- The power spectrum of a sinusoidal signal $x_k = A\cos(2\pi\nu_{sine} t_k + \varphi)$:

$$|a_j|^2 \approx \frac{1}{4} A^2 N^2 \left(\frac{\sin \pi x}{\pi x}\right)^2$$

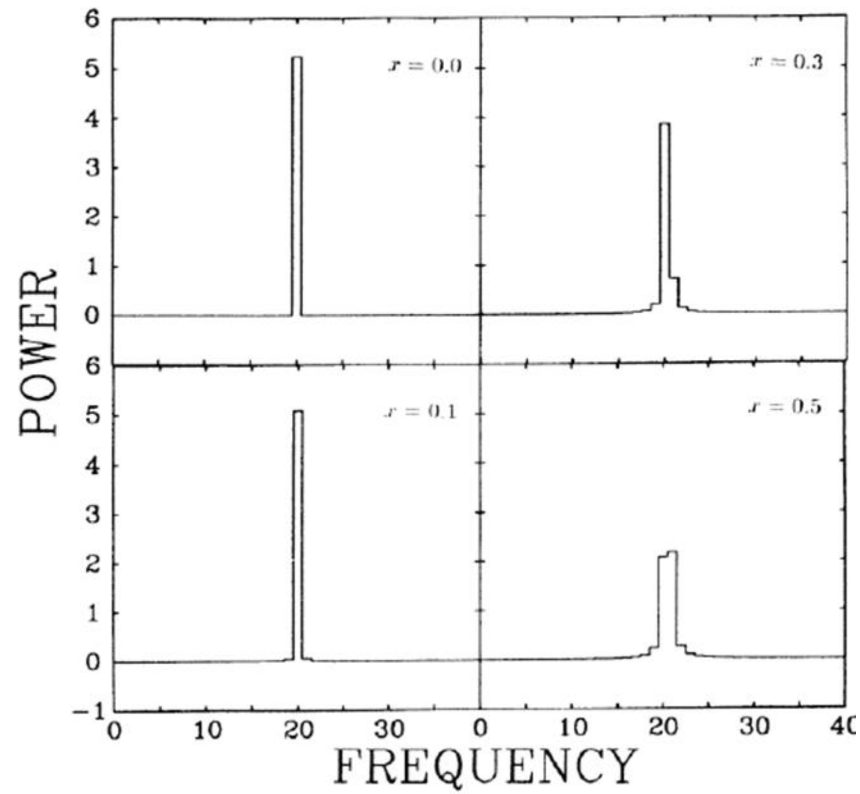
where $x = (\nu_{sine} - \nu_j)T$

- The highest power in the signal power spectrum will be obtained at the Fourier frequency ν_j closest to ν_{sine}. Normalized to a power of 1 for $\nu_{sine} = \nu_j$ (x = 0), this power varies between 0.405 and 1, with an average value of 0.773



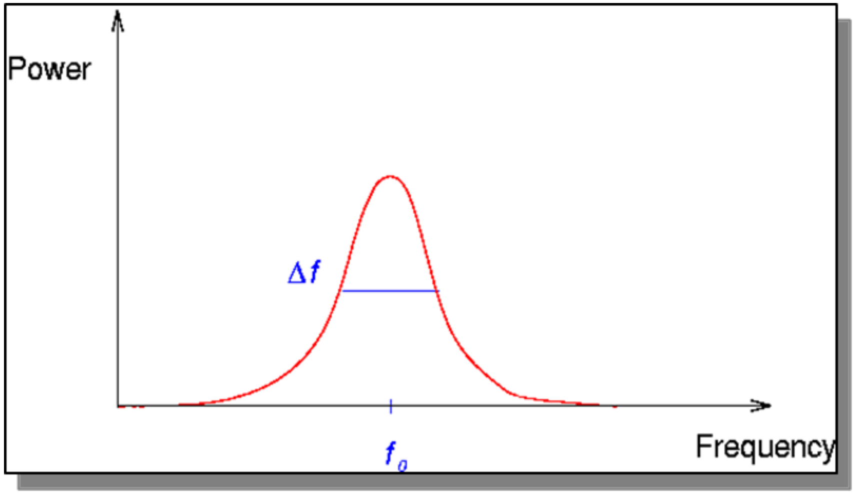
Signal detection optimization

- **Implications:** When searching for strictly coherent signals it is important to rely upon the original/maximum Fourier resolution (1/T).



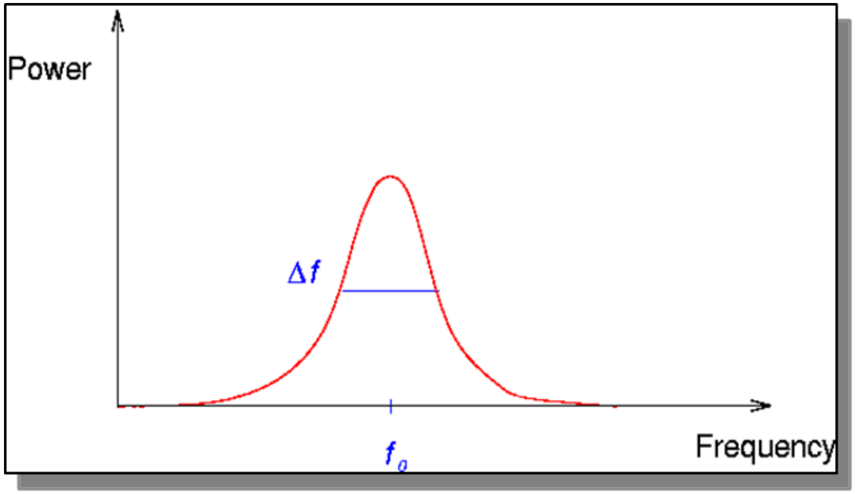
Signal detection optimization

- Similar reasoning shows that the signal power for a feature with finite width $\Delta\nu$ drops proportionally to $1/MW$ when degrading the Fourier resolution. However, as long as feature width exceeds the frequency resolution, $\Delta\nu > MW/T$, the signal power in each Fourier frequency within the feature remains approx. constant. When $\Delta\nu < MW/T$ the signal power begins to drop.



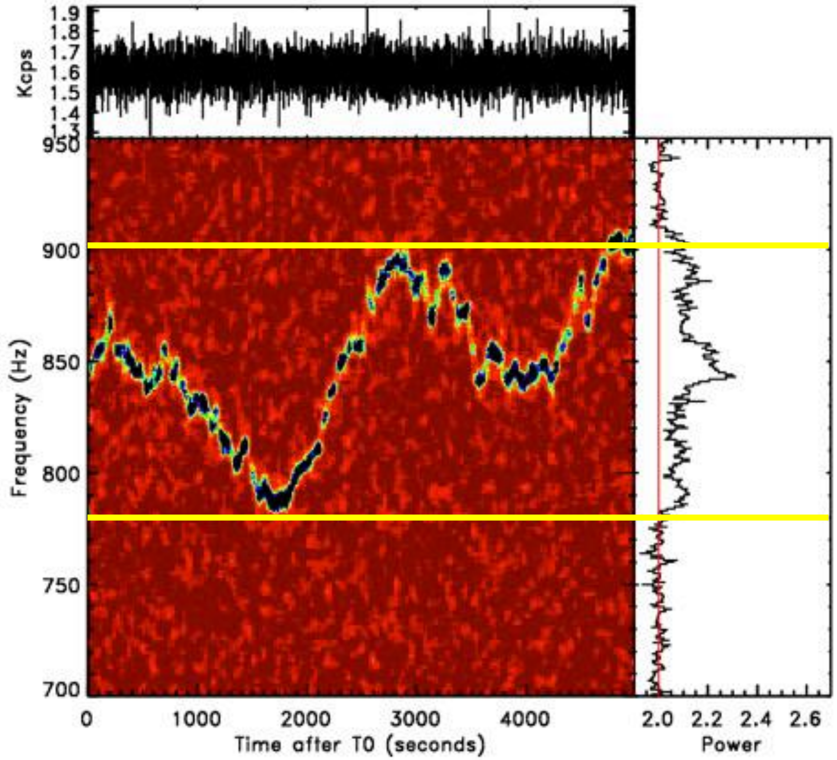
Signal detection optimization

- **Implications:** The search for QPOs is a three step interactive process.
- Firstly, estimate (roughly) the feature width.
- Secondly, run again a PSD by setting the optimal value of MW equal to ~Δν T. Two or three iterations are likely needed.
- Finally, use χ^2 hypothesis testing to derive significance of the feature, its centroid and r.m.s.



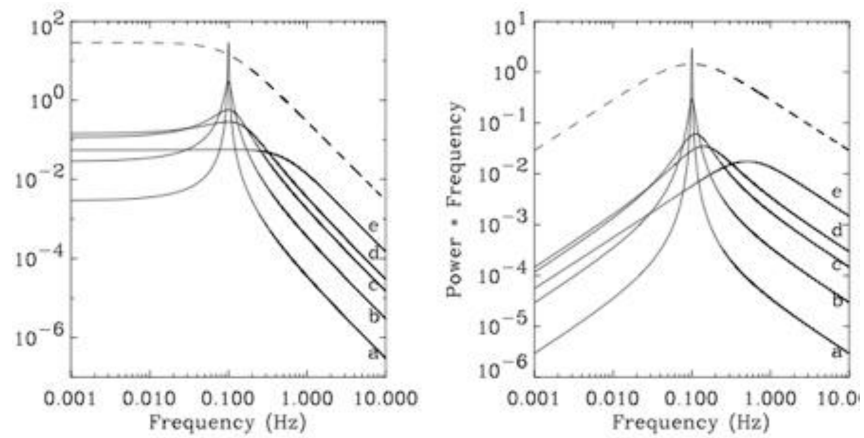
Measuring narrow features in PSD

- **The QPO frequency varies with time (on short timescales).**
- To minimize the pollution of the frequency drift to the measured QPO parameters, PDS must be integrated on the shortest possible timescales
- **Useful tip:** Produce a dynamical PSD
  - Smooth it in time and frequency
  - Restrict the frequency range to where you see the QPO



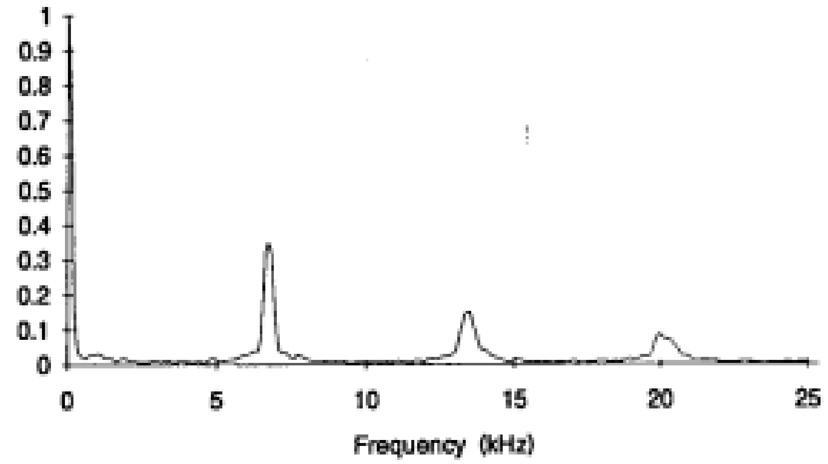
Power spectrum plots

- Multiply the power spectrum by the frequency
- Obtain a νP_ν representation
- Useful to see where the power per decade peaks
- Characteristic frequencies are peaks in νP_ν



Periodic Non-sinusoidal Signals

- Power for Periodic Nonsinusoidal Signals is spread over harmonics of the modulation frequency: Confidence lower.



Summary: Detecting something in a power spectrum

The process of detecting something in a power spectrum against the background of noise has several steps:

- knowledge of the probability distribution of the noise powers;
- knowledge of the interaction between the noise and the signal powers (determination of the signal upper limit);
- The detection level: Number of trials (frequencies and/or sample);
- Specific issues related to the intrinsic source variability (non Poissonian noise);
- Specific issues related to a given instrument/satellite (spurious signals – spacecraft orbit, wobble motion, large data gaps, etc.).