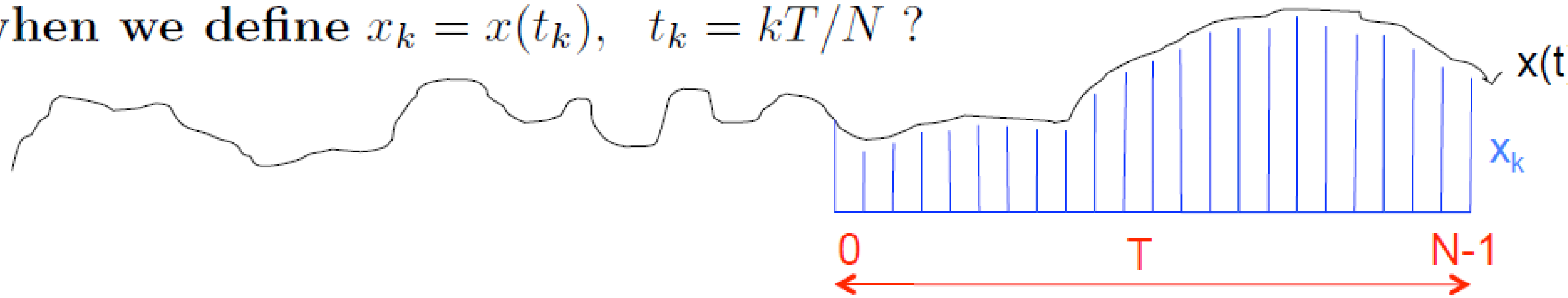
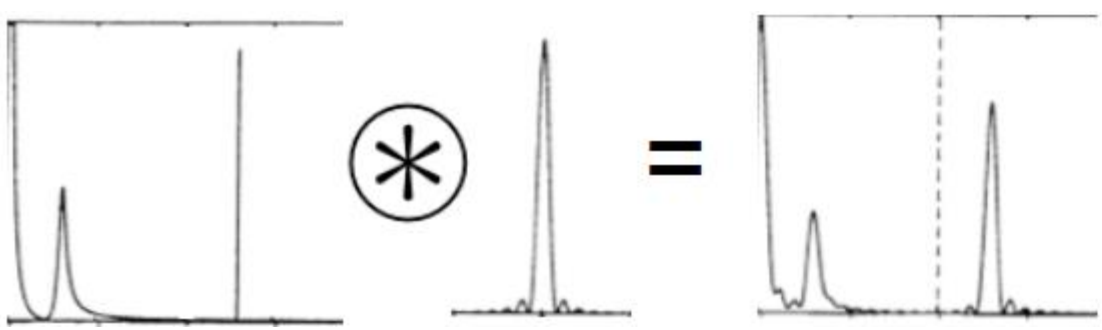
Continuous FT ⇔ Discrete FT

- How can we connect continuous and discrete FT?

What is the relation of this 'ideal case' with the discrete Fourier transform **when we define** $x_k = x(t_k), \quad t_k = kT/N$?



- We use the convolution theorem: " the transform of the product is the convolution of the transforms".



Continuous FT ⇔ Discrete FT

- We have:
  - Time-series *x(t)*
  - Window function *w(t)*:

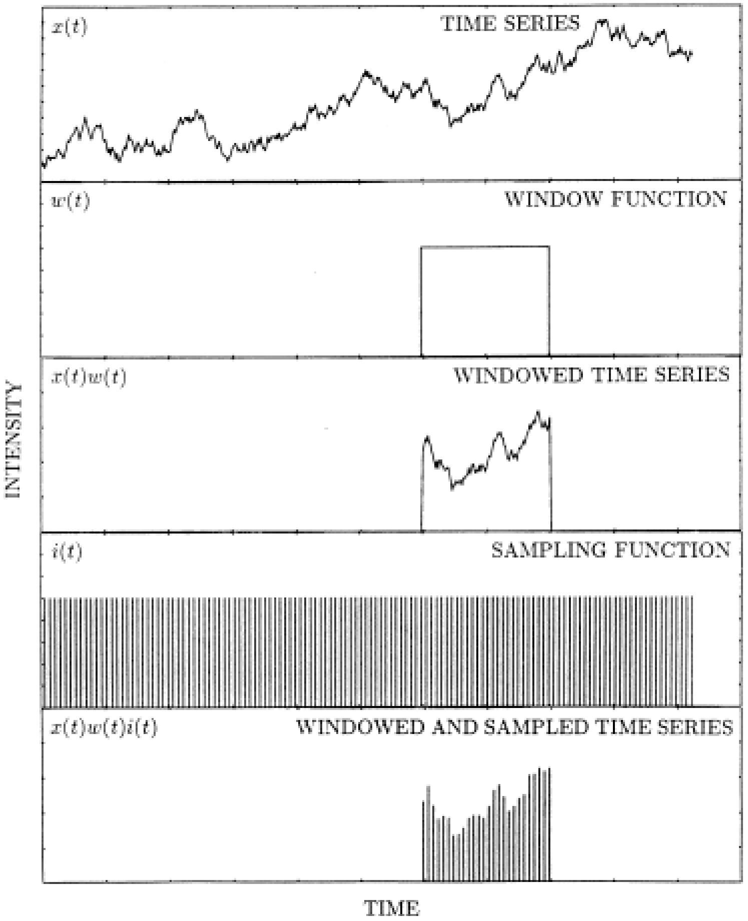
$$w(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases}$$

  - Sampling function *i(t)*:

$$i(t) = \sum_{-\infty}^{+\infty} \delta\left(t - \frac{kT}{N}\right)$$

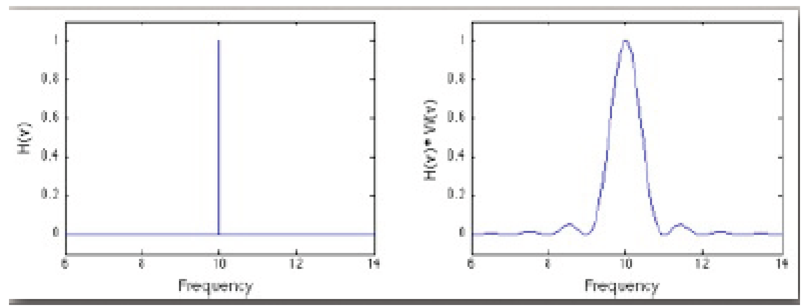
- Now we multiply:

$x_k = x(t)\, w(t)\, i(t)$

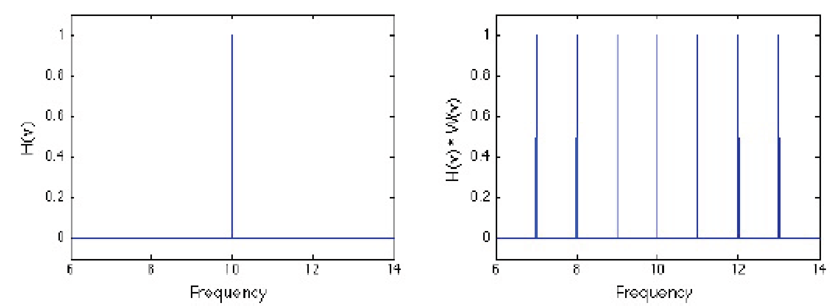


Continuous FT ⇔ Discrete FT

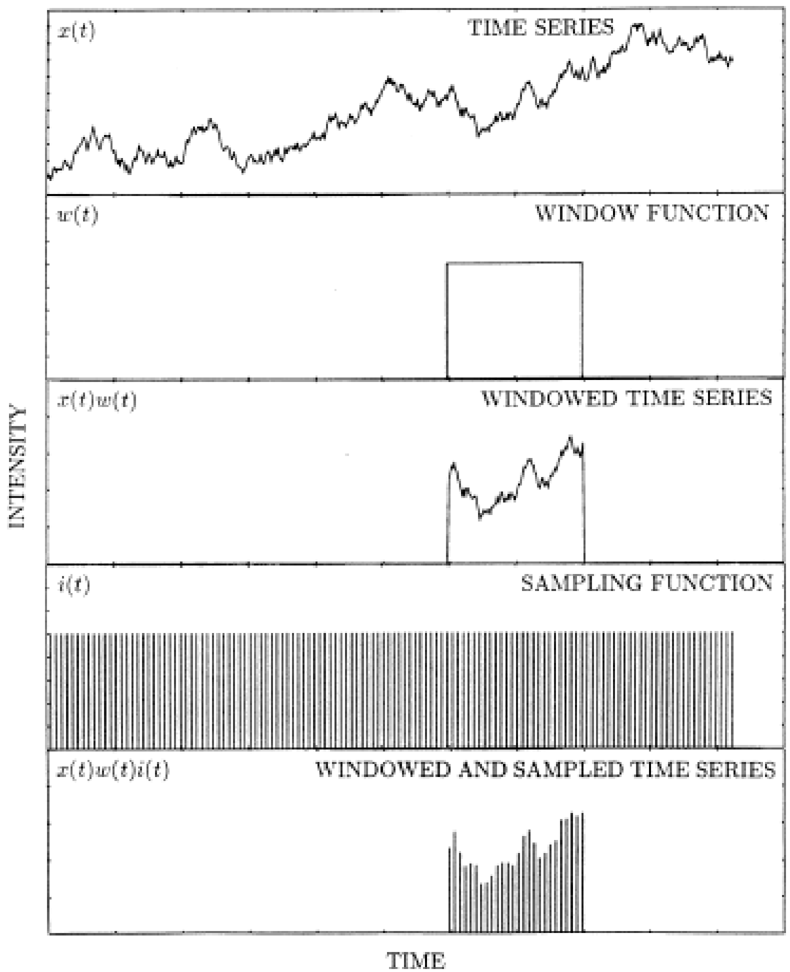
- The power spectrum of $W(\nu)$: $$|W(\nu)|^2 \equiv \left| \int_{-\infty}^{\infty} w(t) e^{2\pi\nu i t}\, dt \right|^2 = \left| \frac{\sin \pi\nu T}{\pi\nu} \right|^2$$
    - Broadening of peaks

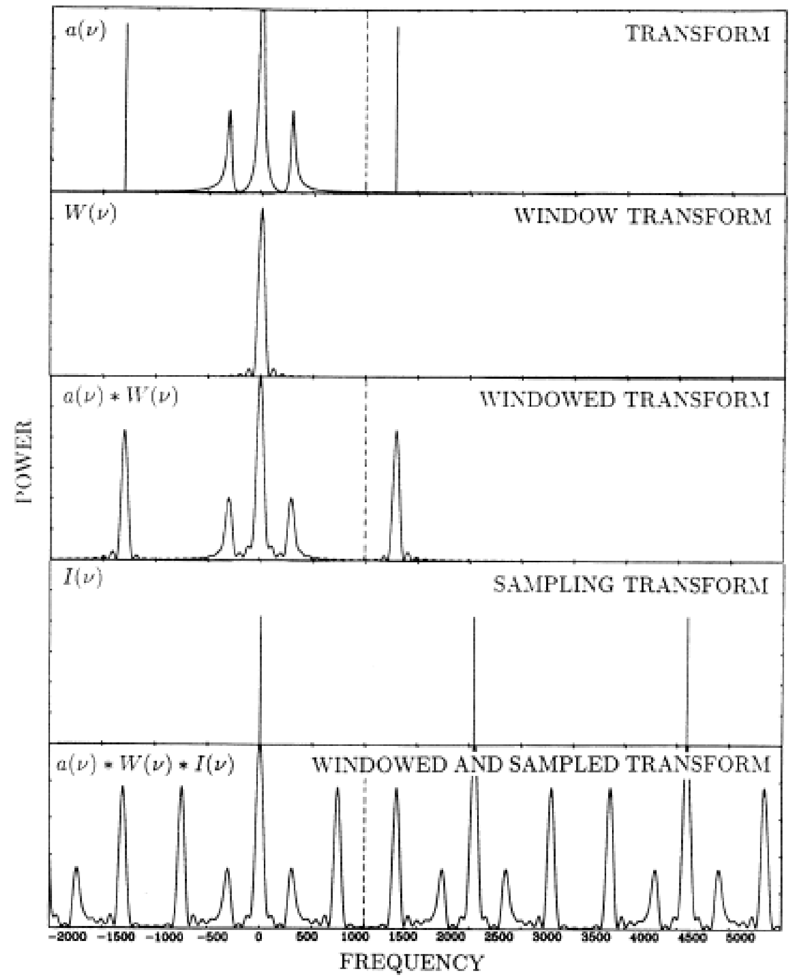


- The Fourier transform of $i(t)$: $$I(\nu) \equiv \int_{-\infty}^{\infty} i(t) e^{2\pi\nu i t}\, dt = \frac{N}{T} \sum_{\ell=-\infty}^{\infty} \delta\left(\nu - \ell \frac{N}{T}\right)$$
    - Infinite series of δ functions, with spacing $N/T =$ ***2*** $\boldsymbol{\nu_{Ny}}$



Continuous FT ⇔ Discrete FT





Summary of discrete FT effects

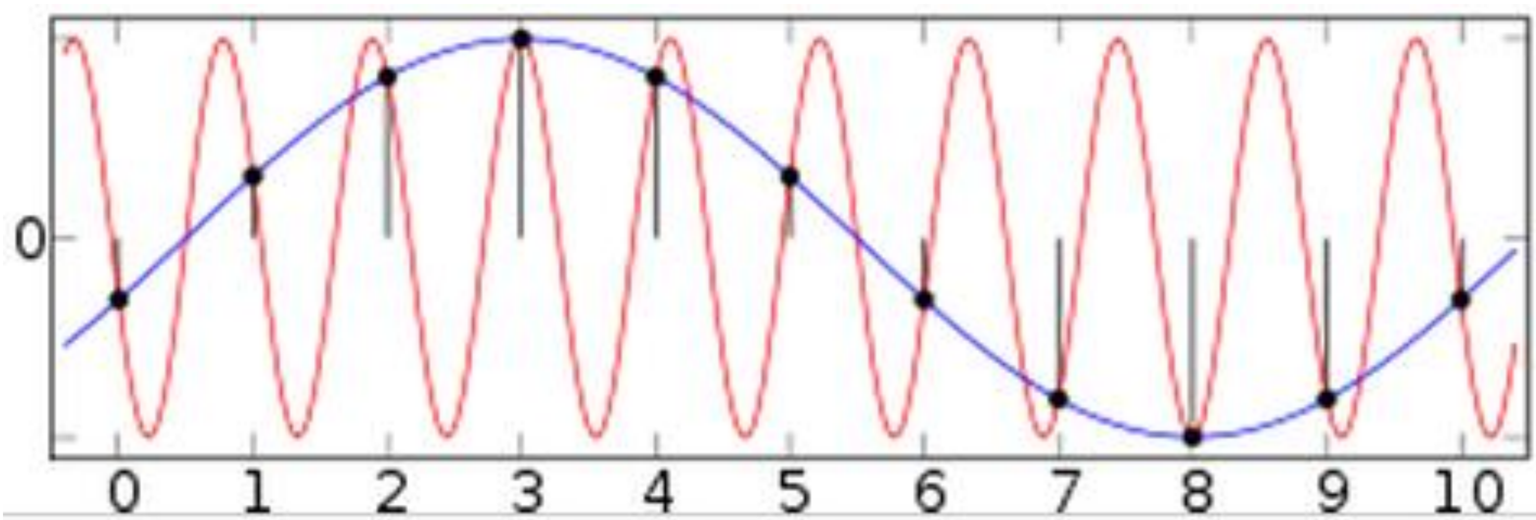
- WINDOW: broadening & sidebands
- SAMPLING: aliasing

Aliasing

- FT is symmetric in frequency for a real signal;
- Alias repeats it every $2\nu_{Ny}$:
- The power spectrum of the convolved function $|a(\nu) * I(\nu)|^2$ is reflected around the Nyquist frequency ν_{Ny}.
- This causes features with a frequency exceeding ν_{Ny} by ν_x (so, located at $\nu = \nu_{Ny} + \nu_x$) to also appear at a frequency (ν_{Ny} - ν_x) — a phenomenon known as aliasing; the reflected feature is called the alias of the original one.

Aliasing

- Two different sinusoids fit the same set of samples:



Is aliasing a problem?

- Not such a big problem for high-energy astronomy;
- In practice, we do not really discretely sample the data, but rather bin the data up;
- That means that before discrete sampling we convolve the *x(t)* with the bin width.
- So, in the frequency domain, we multiply $a(\nu)$ with

$$B(\nu) = \frac{\sin(\frac{\pi\nu N}{T})}{\pi\nu N/T}$$

- Suppression of high frequencies.

Is windowing a problem?

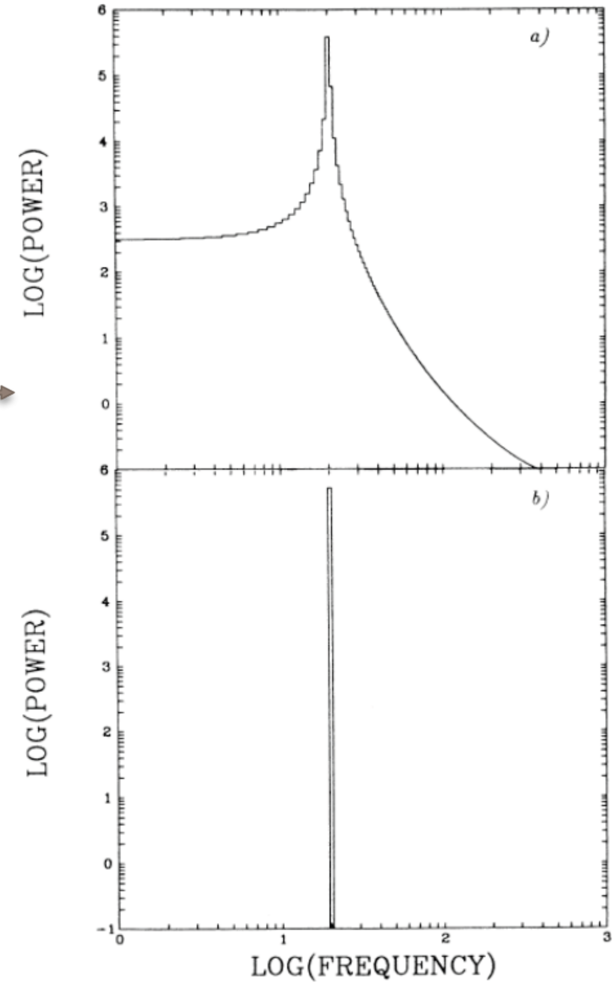
- Yes:
  - It broadens delta peaks;
  - It flattens the slopes of noise components (sidelobes);
  - For steep spectra the ”leakage” can be severe.

- Solution: The longer the observation, the better.

Discrete FT of a sinusoid

- The continuous FT of a sinusoidal signal is a δ-function.

- However, if you calculate the **D**FT of some arbitrary sinusoid, the result will be like this



- a) The discrete power spectrum of a sine-wave with an arbitrary frequency.
- b) Same, but with the frequency equal to a Fourier frequency.

Discrete FT of a sinusoid

The power spectrum of a sinusoid ***A sin*** $(2\pi\nu_{sine}t_k+\varphi)$:

$$|a_j|^2 = \frac{1}{4}A^2N^2\left(\frac{\sin\pi x}{\pi x}\right)^2\left[\left(\frac{\pi x/N}{\sin\pi x/N}\right)^2+\left(\frac{\pi x/N}{\sin[\pi(2j+x)/N]}\right)^2+\right.$$

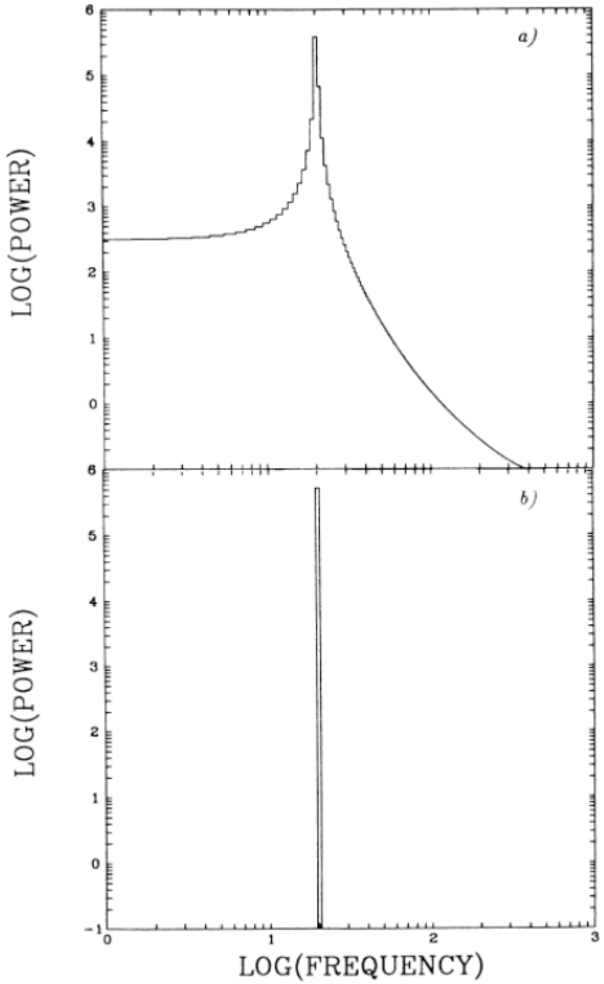
$$\left.+2\left(\frac{\pi x/N}{\sin\pi x/N}\right)\left(\frac{\pi x/N}{\sin[\pi(2j+x)/N]}\right)\cos[(N-1)(2\pi(j+x)/N)+2\phi]\right]$$

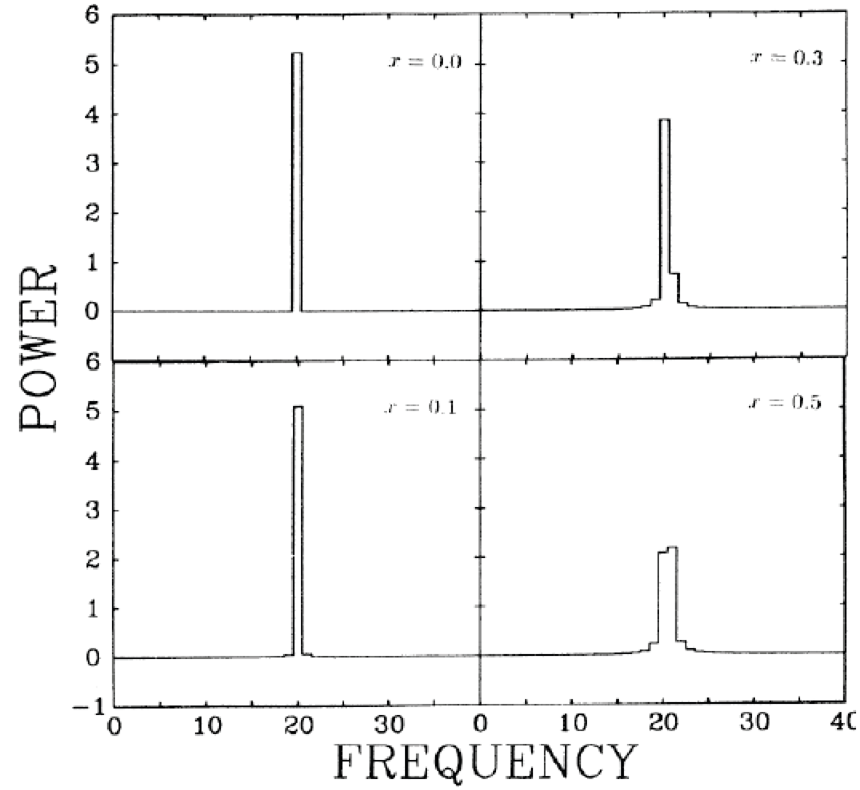
$$x = (\nu_{\mathrm{sine}} - \nu_j)T$$

$$\approx \frac{1}{4}A^2N^2\left(\frac{\sin\pi x}{\pi x}\right)^2$$

$x/N \ll 1$ and $0 \ll j/N \ll \frac{1}{2}$

Discrete FT of a sinusoid





Periodogram & Power Spectrum

- The periodogram is an estimate of the spectral density of a signal. The term was coined by Arthur Schuster in 1898 (*the Schuster Periodogram*).
- A Power Density Spectrum is computed as the squared Fourier amplitudes with **some normalization**:

$$a_j = \sum_{k=0}^{N-1} x_k \, e^{2\pi i jk/N} \qquad j = -\frac{N}{2}+1, \dots, \frac{N}{2}$$

$$P_j = (Normalization)|a_j|^2$$