Harmonic Analysis

CONTINUOUS AND DISCRETE FOURIER TRANSFORM

POWER SPECTRUM

Time and Frequency domains

- A physical process can be described either in the time domain, by the values of some quantity x as a function of time t, e.g., $x(t)$, or else in the frequency domain, where the process is specified by giving its amplitude X (generally a complex number indicating phase also) as a function of frequency ν, that is $X(\nu)$, with $-\infty < \nu < +\infty$. For many purposes it is useful to think of $x(t)$ and $X(\nu)$ as being two different representations of the same function.
- One goes back and forth between these two representations by means of the **Fourier transform equations**.

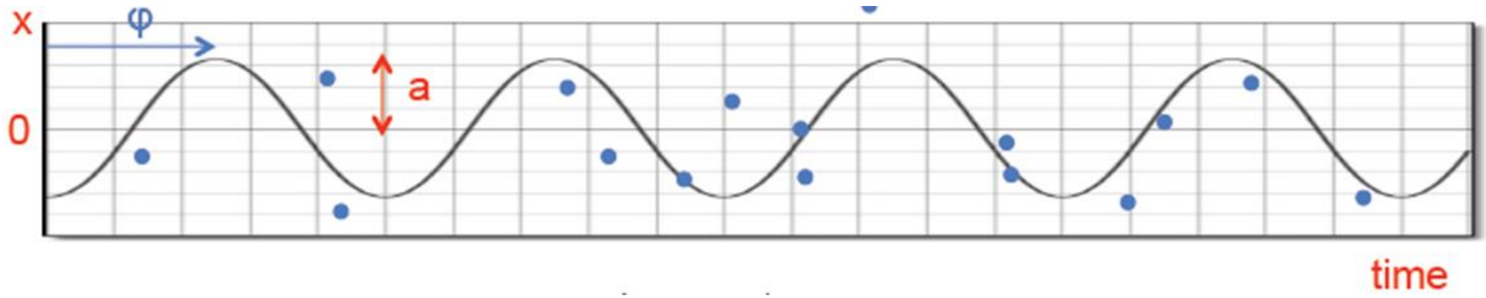
FOURIER TRANSFORM

Joseph Fourier (1768-1830)

- A Workhorse of the Timing World (or a part of this).
- The Additive Model for a Time Series:
  data are realizations of random variables Y_t that are themselves sums of different components (for example, signal and noise).

FOURIER TRANSFORM

- A Fourier transform is a decomposition of the signal into sine waves



- At ω, best-fit sinusoid is:

$$x(t) = a\cos(\omega t - \varphi) = A\cos\omega t + B\sin\omega t$$

$$a = \sqrt{A^2 + B^2} \text{ and } \tan\varphi = -B/A$$

FOURIER TRANSFORM

- Do this at many frequencies ω_j, then

$$x(t) = \frac{1}{N}\sum_j a_j \cos\left(\omega_j t - \varphi_j\right) = \frac{1}{N}\sum_j (A_j \cos \omega_j t + B_j \sin \omega_j t)$$

- The Fourier coefficients A_j and B_j can be straightforwardly computed as:

$$A_j = \sum_k x_k \cos \omega_j t_k$$

$$B_j = \sum_k x_k \sin \omega_j t_k$$

where $x_k = x(t_k)$

FOURIER TRANSFORM

- A_j and B_j are simply the correlation of the signal x_k with a sine or cosine wave of frequency ω_j;
- If there is a good correlation then the corresponding Fourier coefficient is large and gives a large contribution to the sum;
- So, good correlation:
  large A, B — bad correlation: small A, B.

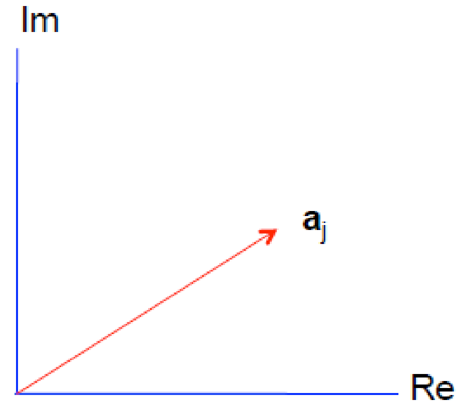
Complex representation

- At each ω we get two numbers: (A, B) or (a, φ). For easier handling, it is possible to represent the Fourier transform in terms of complex numbers:

$$a_j = \sum_k x_k \, e^{i\omega_j t_k}$$

$$x_k = \sum_j a_j \, e^{-i\omega_j t_k}$$

$i^2 = -1$



The complex numbers a_j – complex Fourier amplitudes:

$$a_j = |a_j| \; e^{i\varphi_j} = |a_j| \, (\cos\varphi_j + i \sin\varphi_j)$$

Complex representation

The Euler relation:

$$e^{ix} = \cos x + i \sin x$$

and its inverse:

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$
$$\sin x = \frac{1}{2i}\left(e^{ix} + e^{-ix}\right)$$

Do not worry! Observed data are strictly real-valued.
We consider both positive and negative frequencies, $\omega_{-j} = -\omega_j$
Imaginary terms at $+j$ and $-j$ cancel out in Σ_j to produce strictly **real** terms $2|a_j| \cos(\omega_j t_k - \varphi_j)$

Complex representation

Good explanation:

http://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/

Continuous Fourier transform

- Decomposes a function into an infinite number of sinusoidal waves.
- Signal $x(t)$: $-\infty < t < +\infty$
- Transform $a(v)$: $-\infty < v < +\infty$

$$a(\nu) = \int_{-\infty}^{\infty} x(t) e^{2\pi \nu i t}\, dt \qquad -\infty < \nu < \infty$$

$$x(t) = \int_{-\infty}^{\infty} a(\nu) e^{-2\pi \nu i t}\, d\nu \qquad -\infty < t < \infty$$

Basic properties of Fourier transform

Let's consider transform pairs: $x(t) \Leftrightarrow a(v)$

- **Linearity:** The transform of the sum of two functions is equal to the sum of the transforms:

$$x(t) + y(t) \Leftrightarrow a(v) + b(v)$$

  We can analyse complex optical systems by looking at different frequencies separately.

- **Time scaling:** The transform of a constant times a function is that same constant times the transform of the function: $x(ct) \Leftrightarrow \frac{1}{|c|} a\left(\frac{v}{c}\right)$

Basic properties of Fourier transform

- **Convolution:**
  The convolution of the two functions *x(t)* and *y(t)* is

$$x(t) * y(t) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} x(\tau) y(t-\tau)\, d\tau$$

  Basic properties are
  commutativity: $x * y = y * x$
  distributivity over addition: $x * (y+z) = x * y + x * z$
- **Convolution theorem:** The Fourier transform of the convolution is the product of the individual Fourier transforms: $x(t) * y(t) \Leftrightarrow a(v)\, b(v)$

Basic properties of Fourier transform

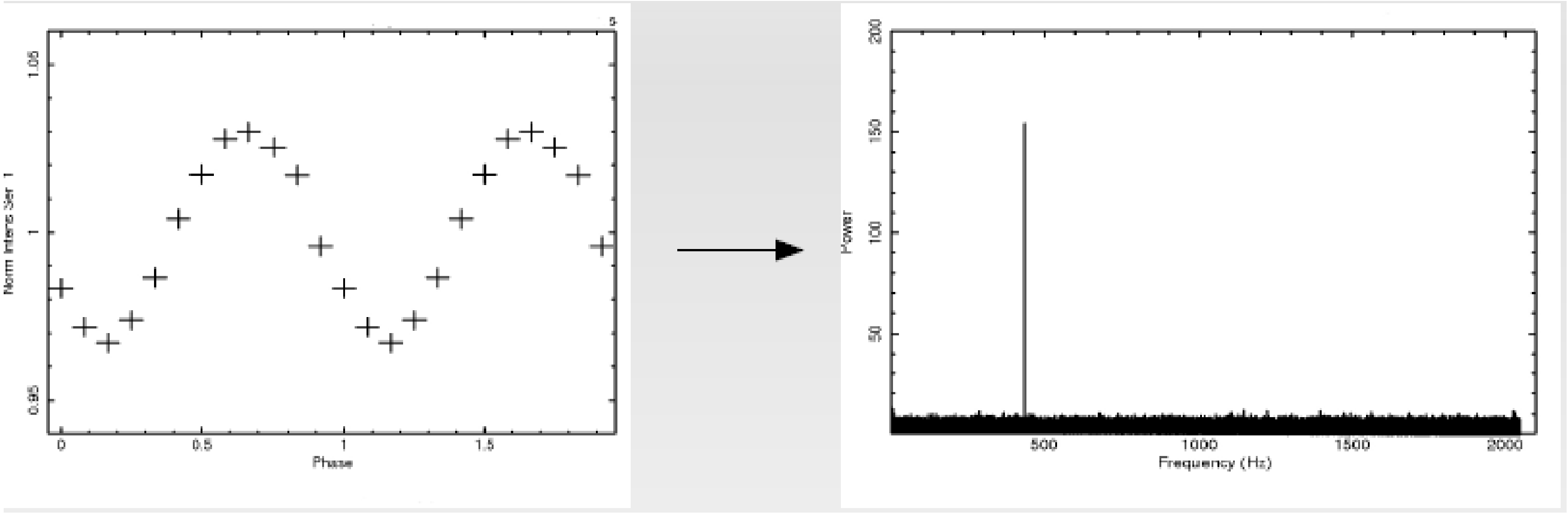
- Dirac delta function (δ-function):

$$\delta(x) = 0, \quad x \neq 0$$
$$\delta(x) = \infty, \quad x = 0$$

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

Continuous Fourier transform

- The continuous Fourier transform of an infinitely extended sine (or cosine) wave is a delta function (this is not in general true for the discrete Fourier transform → see later).



Continuous Fourier transform

- The continuous Fourier transform has a number of pleasing properties.
- Therefore, theoretical predictions of the shape of the Fourier transform of a signal are usually in terms of the continuous Fourier transform.
- … but we don’t have either continuous or infinite signals.

- **Fourier theorem:** the **discrete** Fourier transform gives a complete description of the discrete signal.

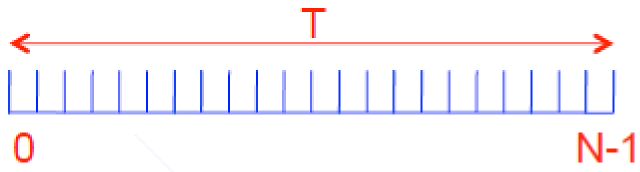
Discrete Fourier transform of real time series

- Time series, x_k, $k=0, \dots, N-1$
- The discrete Fourier transform decomposes this signal into N sine waves, a_j, $j= -N/2+1, \dots, N/2$

$$a_j = \sum_{k=0}^{N-1} x_k \, e^{2\pi i j k/N} \qquad j = -\frac{N}{2}+1, \dots, \frac{N}{2}$$

$$x_k = \frac{1}{N} \sum_{j=-\frac{N}{2}+1}^{N/2} a_j \, e^{-2\pi i j k/N} \qquad k = 0, \dots, N-1$$

Discrete Fourier transform of real time series



- Time step, $\delta t = T/N$
- Frequency step, $\delta\nu = 1/T$
- x_k refers to time $t_k = kT/N$
- a_j refers to frequency $\omega_j = 2\pi\nu_j = 2\pi j/T$
- *So, for* $e^{i\omega_j t_k}$ we have $e^{2\pi ijk/N}$

- Note that the number (N) of input values x_k equals the number of output values a_j.
- If x_k are uncorrelated, then a_j are uncorrelated as well.

Discrete Fourier transform of real time series

- The highest frequency you need for a complete description of the discrete signal is the **Nyquist frequency**

  $\boldsymbol{\nu_{Ny}=\nu_{N/2}=\frac{N}{2T}}$ – half the "sampling" frequency
- Lowest frequency (>0) = frequency of the first frequency step = $1/T$ = frequency of sinusoid that fits exactly once on T
- At zero frequency you get

  $\boldsymbol{a_0 = \sum_k x_k = N_{ph}}$ – the total number of counts (e.g., photons detected), always real for real x_k

Fast Fourier Transform (FFT)

- FFT is an efficient algorithm to compute the discrete Fourier transform and its inverse;
- The FFT has been called the most important numerical algorithm of our lifetime;
- The computation time can be reduced by several orders of magnitude (especially for long data);
- **Number of points $N_{bin} = 2^n$; n – integer.**

Power spectrum

- **Parseval's theorem:**
The total power in a signal is the same in the time domain and in the frequency domain:

$$Total\ Power \equiv \int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |a(\nu)|^2\, d\nu$$

- For a given signal, the **power spectrum** gives a plot of the portion of a signal's power (energy per unit time) falling within given frequency bins.

Power Spectral Density (PSD)

- How much ***power*** is contained in the frequency interval bentween *v* and *v+dv* ?

  Power spectral density (PSD):

  $$P(v) \equiv |a(v)|^2, \; -\infty < v < \infty$$

  One-sided PSD:

  $$P(v) \equiv |a(v)|^2 + |a(-v)|^2, 0 \leq v < \infty$$

  We have real *x(t)*, then:

  $$P(v) \equiv 2|a(v)|^2$$

Power Density Spectrum

- **Parseval's theorem**:

$$\sum_{k=0}^{N-1} {x_k}^2 = \frac{1}{N} \sum_{j=-\frac{N}{2}+1}^{N/2} |a_j|^2$$

- A Power Density Spectrum is computed as the squared Fourier amplitudes with **some normalization**.
- A light curve of N bins (x_k) is then translated into a PDS of N/2+1 independent amplitudes.

Power Density Spectrum

- **Non-linear transformation:**
  Suppose that the data x_k are the sum of y_k and z_k.

$$\sum_k x_k\, e^{i\omega_j t_k} = \sum_k (y_k + z_k)\, e^{i\omega_j t_k} = \sum_k y_k\, e^{i\omega_j t_k} + \sum_k z_k\, e^{i\omega_j t_k}$$

- **But!** $x_k = y_k + z_k \;\Rightarrow\;$

$$a_j = b_j + c_j$$

$$|a_j|^2 = |b_j|^2 + |c_j|^2 + crossterms$$

- If independent (random noise added), cross terms average out to zero