Simple theory of line formation

SIMPLE LINE TRANSFER
SCHUSTER-SCHWARZSCHILD MODEL
THEORY OF LINE FORMATION
CURVE OF GROWTH

Schuster-Schwarzschild model

We now turn to the solution of the transfer equation for both line and continuum radiation. We will adopt the Schuster-Schwarzschild model, which assumes that the line is formed above the continuum and that continuous opacity plays only indirect role.

The total absorption coefficient within an arbitrary line is the sum of the line (α_L) and continuum (α_C) contributions i.e. $\alpha_\lambda = \alpha_L + \alpha_C$ as is the total emission coefficient ($\varepsilon_\lambda = \varepsilon_L + \varepsilon_C$). Hence,

$$S_\lambda = (\varepsilon_L + \varepsilon_C)/(\alpha_L + \alpha_C)$$

and

$$d\tau_\lambda = -(\alpha_L + \alpha_C)\, dz \qquad \tau_\lambda = \tau_L + \tau_C$$

So, we can write the transfer equation as usual:

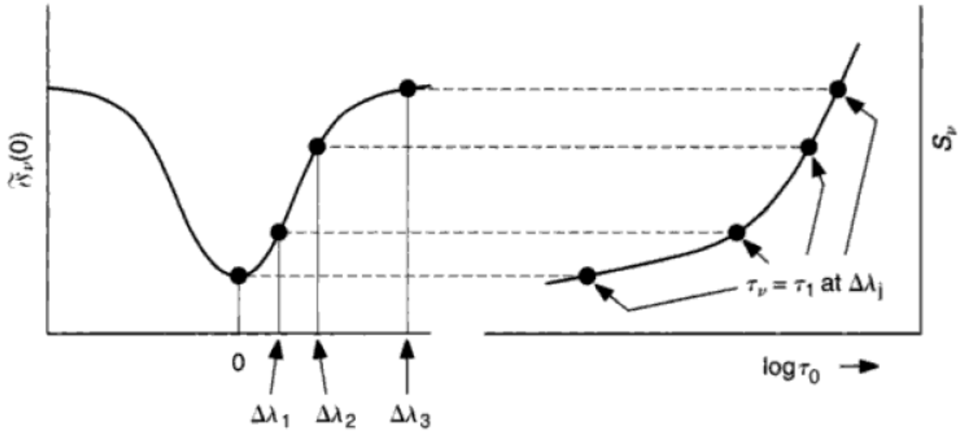
$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

Line source function

- We have seen earlier that the emergent flux from the stellar surface is π times the Source function at an optical depth of $2/3$:

$$F_\lambda(0) = \pi S_\lambda(\tau = 2/3)$$

- Across a line profile, α_λ varies, being larger towards the centre. The condition $\tau_\lambda = 2/3$ is true higher up in the atmosphere for λ near line centre and holds for progressively deeper layers for λ further into the wing.
- Assuming S_λ is a slowly varying function of λ (i.e. constant over the line width), $\pi S_\lambda(\tau_1 = 2/3) = F_\lambda(0)$ provides a mapping between F_λ as a function of λ and S_λ as a function of τ_λ



Theory of line formation

Because of larger absorption in the line, it is formed **higher** up in the atmosphere where T is lower **=>** absorption line.

$$\tau_\lambda = \tau_L + \tau_C$$

Consider **weak** lines: the layer $\tau_\lambda = 2/3$ is close to the layer with $\tau_C = 2/3$.

$$\alpha_L << \alpha_C \rightarrow \alpha_\lambda = \alpha_C (1 + \alpha_L / \alpha_C)$$

We can evaluate S_λ by a Taylor expansion around the point $\tau_C = \tau_\lambda$:

$$S_\lambda(\tau_\lambda = 2/3) \approx S_\lambda(\tau_C = 2/3) + \left.\frac{dS_\lambda}{d\tau_c}\right|_{\tau=2/3} \Delta\tau_C$$

$$\tau_\lambda / \tau_C = \alpha_\lambda / \alpha_C \rightarrow \tau_C = (\tau_L + \tau_C)\frac{\alpha_C}{\alpha_L + \alpha_C} \approx \frac{2}{3}\frac{\alpha_C}{\alpha_L + \alpha_C} \approx \frac{2}{3}\left(1 - \frac{\alpha_L}{\alpha_C}\right) \text{ for } \alpha_L << \alpha_C$$

$$\tau_C = \tau_\lambda + \Delta\tau_C = \frac{2}{3} + \Delta\tau_C \rightarrow \Delta\tau_C = -\frac{2}{3}\frac{\alpha_L}{\alpha_C}$$

Such a line is called optically thin.

Theory of line formation

$$S_\lambda(\tau_\lambda = 2/3) \approx S_\lambda(\tau_C = 2/3) - \frac{2}{3}\frac{\alpha_L}{\alpha_C}\frac{dS_\lambda}{d\tau_c}\bigg|_{\tau=2/3}$$

The line equivalent width is then (LTE: $S_\lambda = B_\lambda$)

$$W_\lambda = \int \frac{F_c - F_\lambda}{F_c} d\lambda = \int d\lambda \frac{B_\lambda(\tau_c = 2/3) - B_\lambda(\tau_\lambda = 2/3)}{B_\lambda(\tau_c = 2/3)}$$

$$W_\lambda = \int d\lambda \frac{dB_\lambda(\tau_c = 2/3)}{d\tau_c}\bigg|_{\tau_c=2/3} \left(\frac{2}{3}\frac{\alpha_L}{\alpha_C}\right) \frac{1}{B_\lambda(\tau_c = 2/3)} =$$

$$W_\lambda = \frac{2}{3}\int d\lambda \frac{d\ln B_\lambda\,(\tau_c = 2/3)}{d\tau_c}\bigg|_{\tau_c=2/3} \left(\frac{\alpha_L}{\alpha_C}\right)$$

Weakly depends on λ

$$W_\lambda = \frac{2}{3}\frac{1}{\alpha_C}\frac{d\ln B_\lambda\,(\tau_c = 2/3)}{d\tau_c}\bigg|_{\tau_c=2/3} \times \int_0^\infty \alpha_L d\lambda$$

If there is **no** temperature gradient with the temperature decreasing outwards, then there are **no** absorption lines in the spectrum.

The profile mimics the shape of α_L. Line strength can be increased by decreasing the continuous absorption α_C or by increasing the line absorption α_L.

Theory of line formation

$$W_\lambda = \frac{2}{3}\frac{1}{\alpha_C}\frac{d\ln B_\lambda\,(\tau_c = 2/3)}{d\tau_c}\bigg|_{\tau_c=2/3} \times \int_0^\infty \alpha_L d\lambda$$

$$\alpha_L = \sigma_L n, \qquad N = \int n\,dr = \frac{n}{\alpha_C}\int \alpha_C\,dr = \tau_c \frac{n}{\alpha_C} \approx \frac{2}{3}\frac{n}{\alpha_C} \rightarrow W_\lambda \propto N$$

For optically thin lines with $\alpha_L << \alpha_C$, $W_\lambda \propto N$

Strong lines

For $\alpha_L << \alpha_C$, the line is optically thin, and its strength increases proportionally with α_L / α_C. If $\alpha_L/\alpha_C>1$, the line becomes optically thick, reaching a maximum depth R_λ. For very thick lines with $\alpha_L/\alpha_C=\infty$, the intensity in the line centre is given by the source function $S_\lambda(\tau_\lambda=0)$, or $B_\lambda(\tau_\lambda= 0)$ in LTE. This is **not** zero since $T(\tau_\lambda= 0)$ is **non-zero**.

If non-LTE applies, when $S_\lambda \neq B_\lambda$, $S_\lambda(\tau_\lambda=0)$ may tend towards zero, for instance, in resonance lines (arising from transitions between the ground states and the first energy level).

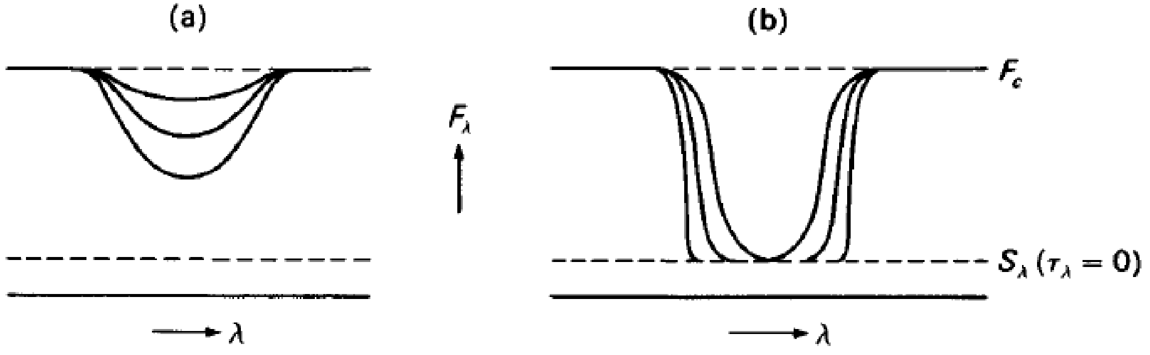


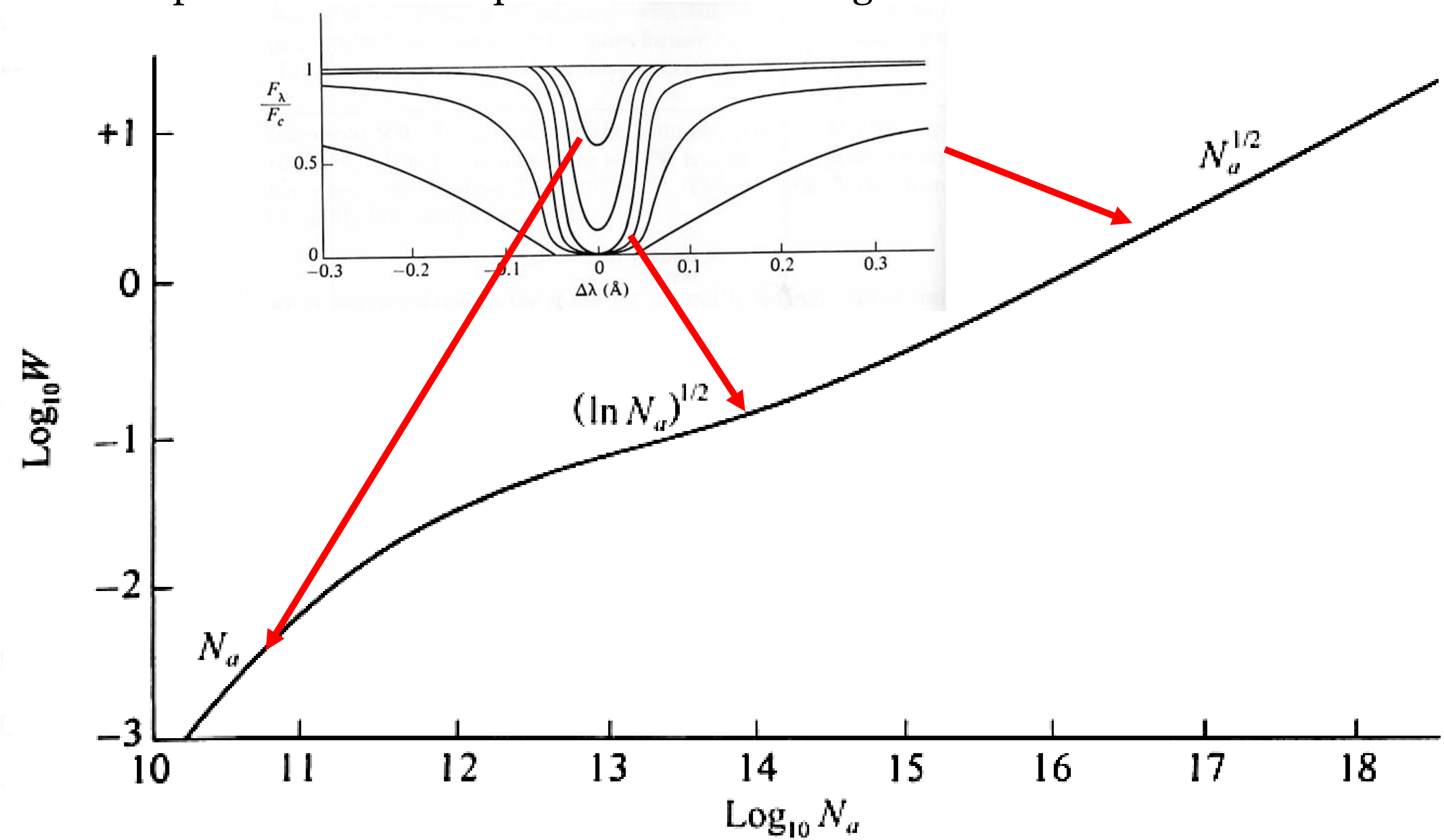
Fig. 10.12. Changes of the line profile with increasing κ_L/κ_c for (a) optically thin and (b) optically thick lines.

Curve of Growth

- The **Curve of growth** describes how the equivalent width (line strength) W_λ depends on the number of absorbing atoms or ions.
- For weak, optically thin lines, as the abundance doubles, the line equivalent width also doubles in strength:
  $W_\lambda \sim N$ – this is the LINEAR part of the curve of growth.
- As the abundance continues to increase, the Doppler core of the line becomes optically thick and saturates. The wings of the line, which are still optically thin, deepen, which occurs with little change in the line equivalent width and so produces a PLATEAU in the curve of growth, $W_\lambda \sim (\ln N)^{1/2}$.
- Ultimately, the damping wings become optically thick, increasing the equivalent width, $W_\lambda \sim (N)^{1/2}$. This is the DAMPING or SQUARE ROOT part of the curve of growth.

Curve of Growth

Curve of growth for the K line of Ca II. As N increases, the functional dependence of the equivalent width changes.



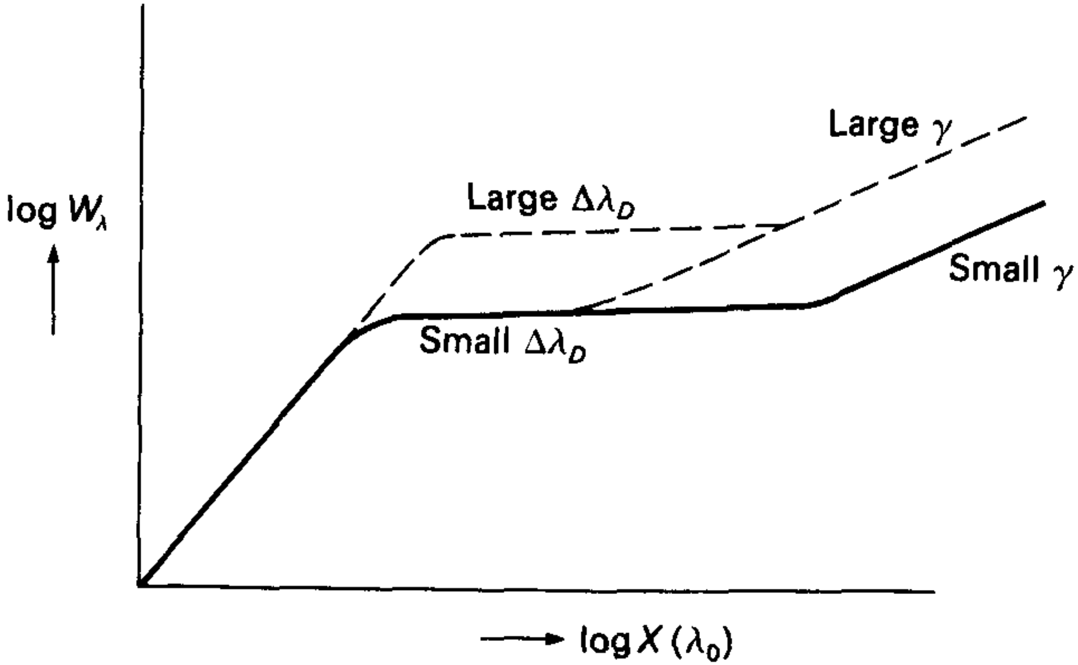
Methodology

- Using the curve of growth and a measured equivalent width we can derive the number of absorbing atoms.
- The Boltzmann and Saha equations convert this value into the total number of atoms of that element in the photosphere → abundance.
- To reduce errors, it is advisable to locate several lines on a curve of growth

Thermal and Pressure effects

The exact form of the curve of growth depends on the ratio of pressure to thermal broadening, $\alpha = \gamma / 2\Delta\lambda_D$.

For increasing Doppler line width, saturation occurs for larger W_λ, whilst the damping part will start earlier if α (i.e. γ) is larger.



Transfer Equation including lines

SCATTERING IN LINES
THE MILNE-EDDINGTON MODEL
RESIDUAL FLUX OF THE LINE
ABSORPTION AND SCATTERING LINES
SCHUSTER MECHANISM FOR LINE EMISSION

Summary of simple line transfer

Simple line transfer:

The total absorption coefficient within an arbitrary line is the sum of the line (α_L) and continuum (α_C) contributions i.e. $\alpha_\lambda = \alpha_L + \alpha_C$ as is the total emission coefficient ($\varepsilon_\lambda = \varepsilon_L + \varepsilon_C$). Hence,

$$S_\lambda = (\varepsilon_L + \varepsilon_C)/(\alpha_L + \alpha_C)$$

and

$$d\tau_\lambda = -(\alpha_L + \alpha_C)\, dz \qquad \tau_\lambda = \tau_L + \tau_C$$

So, we can write the transfer equation as usual: $\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$

The surface specific intensity

and surface flux

are obtained as previously.

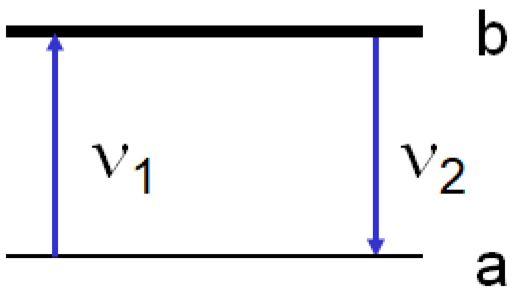
$$I_\lambda(0,\theta) = \int_0^\infty S_\lambda(\tau_\lambda) e^{-\tau_\lambda \sec\theta} \sec\theta \; d\tau_\lambda$$

$$F_\lambda(0) = 2\pi \int_0^1 I_\lambda(0,\theta)\mu d\mu \qquad \mu = \cos\theta$$

Again, we need to know $S(\tau)$ to evaluate these integrals.

Scattering in lines

- Special case:
  Coherent scattering: $\nu_1 = \nu_2$
- Common case:
  2-level atom absorbs photon with frequency ν_1, re-emits photon with frequency ν_2; frequencies not exactly equal, because
  - levels a and b have non-vanishing energy width
  - Doppler effect because atom moves
- Non-coherent scattering requires a redistribution function



Transfer Equation including lines

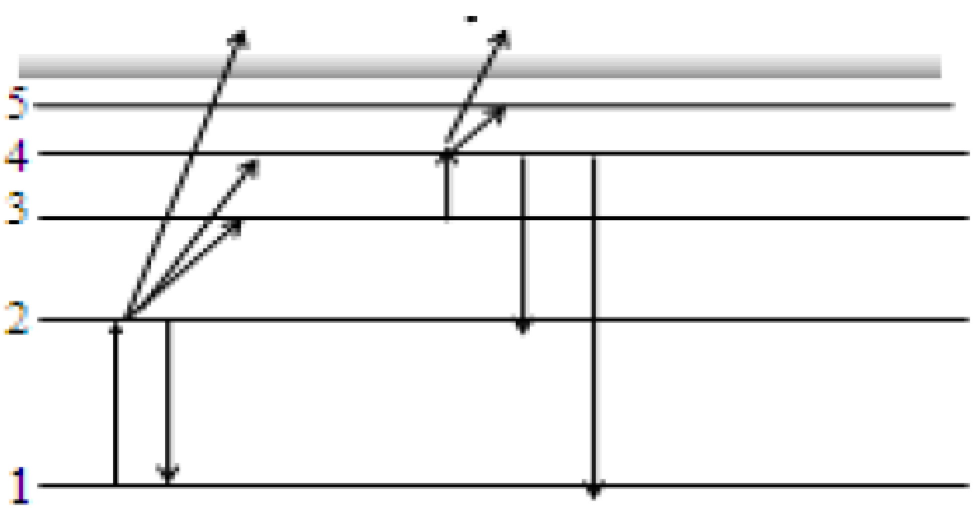
Classical approach:

absorption of photons by line has two parts

1. (**1-ζ**) of absorbed photons are scattered
   (e$^-$ returns to original state)
2. **ζ** of absorbed photons are destroyed
   (into thermal energy of gas)
   (for LTE: ζ =1)

Resonance lines (to/from the ground level)

A photon 1→2 returns back to the radiation field, thus dominates **Scattering**



Subordinate lines (to/from higher levels)

A photon 3→4 disappears,

thus dominates **True absorption**

Scattering

- Pure Absorption and Thermal Emission:

$$\boldsymbol{S(\tau) = \frac{\varepsilon}{\alpha}} \qquad \textbf{LTE:}\ \boldsymbol{\varepsilon_{th} = \alpha_{th}\, B(\tau)}$$

- Pure Scattering:

  For the case of pure scattering, the associated emission becomes completely insensitive to the thermal properties of the gas, and instead depends only on the local radiation field. If the scattering is roughly isotropic, the scattering emissivity $\boldsymbol{\varepsilon}_{\rm sc}$ in any direction depends on both the opacity and the angle-averaged mean-intensity $\boldsymbol{\varepsilon}_{\rm sc} = \kappa_{\rm sc}\rho J = \alpha_{\rm sc} J$

  This implies then that, for pure-scattering,

$$\boldsymbol{S(\tau) = J(\tau)}$$

- Source Function for Scattering and Absorption:

  The total opacity consists of both scattering and absorption, $\alpha \equiv \alpha_{\rm abs} + \alpha_{\rm sc}$

  The total emissivity likewise contains both thermal and scattering components $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}_{\rm th} + \boldsymbol{\varepsilon}_{\rm sc} = \alpha_{\rm th} B + \alpha_{\rm sc} J$. The general source function

  absorption fraction $\zeta \equiv \dfrac{\alpha_{\rm abs}}{\alpha_{\rm abs} + \alpha_{\rm sc}}$

$$\boldsymbol{S(\tau) = \zeta B + (1 - \zeta) J(\tau)}$$

The Milne-Eddington model (1)

Consider a case where at the given frequency the total opacity is a combination of both continuum and line processes:

Total absorption coefficient is $\alpha_\nu = \alpha_\nu^C + \alpha_\nu^L + \sigma$ ← scattering in the continuum

$\alpha_\nu^L = \alpha_\nu \times \phi_\nu =$ line opacity × line profile

The total optical depth is $d\tau_\nu = -(\alpha_\nu^C + \alpha_\nu^L + \sigma)\, ds$

(**larger** than in the continuum!)

The correponding emissivities $\varepsilon_\nu = \varepsilon_\nu^C + \varepsilon_\nu^L + \sigma J_\nu$

$S(\tau) = \frac{\varepsilon}{\alpha}$

$$\mu \frac{dI_\nu(\mu)}{d\tau_\nu} = I_\nu(\mu, \tau_\nu) - S_\nu(\tau_\nu)$$

Recall radiative transfer equation

$$\mu \frac{dI_\nu}{ds} = -\alpha_\nu I_\nu + \varepsilon_\nu$$

The Milne-Eddington model

Transfer equation:

$\varepsilon = \varepsilon_{th} + \varepsilon_{sc} = \alpha_{th} B + \alpha_{sc} J$

-absorbed **+thermal** +scattered

$$\mu \frac{dI_\nu}{ds} = -(\alpha_\nu^C + \alpha_\nu^L + \sigma)I_\nu + \varepsilon_\nu^C + \sigma J_\nu + \zeta \alpha_\nu^L B_\nu + (1-\zeta)\alpha_\nu^L J_\nu$$

+therm. line em. +scat. line emission (coherent)

Without dealing with the general case for the computation of all coefficients we assume:

- LTE in the continuum $\varepsilon_\nu^C = \alpha_\nu^C B_\nu(T)$
- scattering negligible in the continuum $\sigma \ll \alpha_\nu^C$

The following slides with light-grey backgrounds (like in this box) are for self-study. The derivation of equations will not be asked at the exam but will help understand the important results and conclusions.

The Milne-Eddington model (2)

$$\mu \frac{dI_\nu}{ds} = -(\alpha_\nu^C + \alpha_\nu^L) I_\nu + \alpha_\nu^C B_\nu + \zeta \alpha_\nu^L B_\nu + (1-\zeta)\alpha_\nu^L J_\nu$$

Using $\beta_\nu \equiv \frac{\alpha_\nu^L}{\alpha_\nu^C}$ $\quad d\tau_\nu = -(\alpha_\nu^C + \alpha_\nu^L)\, ds = -\alpha_\nu^C (1+\beta_\nu)\, ds$

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - B_\nu \frac{1+\zeta\beta_\nu}{1+\beta_\nu} - \frac{(1-\zeta)\beta_\nu}{1+\beta_\nu} J_\nu = I_\nu - \lambda_\nu B_\nu - (1-\lambda_\nu)\, J_\nu$$

destruction probability → $\lambda_\nu \equiv \frac{1+\zeta\beta_\nu}{1+\beta_\nu}$

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \lambda_\nu B_\nu - (1-\lambda_\nu)\, J_\nu$$

Milne-Eddington Equation. Solve at each frequency point across profile.

The Milne-Eddington model (3)

$$\mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \lambda_\nu B_\nu - (1 - \lambda_\nu) J_\nu$$

Milne-Eddington assumptions (for analytical solution):

1. β_ν, λ_ν and ζ are constant with depth
2. B_ν is linear in continuum optical depth: $B_\nu = a + b\tau_c$

$$d\tau_c = \frac{d\tau_\nu}{1 + \beta_\nu} \qquad \tau_c = \frac{\tau_\nu}{1 + \beta_\nu}$$

Also, the Eddington approximation $K_\lambda(\tau_\lambda) = \frac{1}{3} J_\lambda(\tau_\lambda)$

Recap: Eddington approximation

Lecture 6

Lecture 18

K-integral and radiation pressure

- K-integral is related to the radiation pressure: $K_\lambda = \frac{1}{4\pi}\oint I_\lambda \cos^2\theta \, d\omega$
- A photon has momentum $p_\lambda = E_\lambda / c$
- Consider photons transferring momentum to a solid wall. Force:

$$F = \frac{dp_{\lambda\perp}}{dt} = \frac{1}{c}\frac{dE_\lambda}{dt}\cos\vartheta$$

- Pressure: $dP_\lambda = \frac{F}{d\sigma} = \frac{1}{c}\frac{dE_\lambda \cos\vartheta}{dt \quad d\sigma} = \frac{1}{c} I_\lambda \cos^2\vartheta \; d\omega \, d\lambda$

$$I_\lambda = \frac{dE_\lambda}{\cos\theta \, d\lambda \, d\sigma \, d\omega \, dt}$$

$$P(\lambda) = \frac{1}{c}\oint_{4\pi} I_\lambda \cos^2\vartheta \; d\omega = \frac{4\pi}{c} K_\lambda$$

Eddington approximation (1)

- Previously we have seen that for the determination of the flux the anisotropy in the radiation field is very important because in the flux integral the inward-going intensities are subtracted from the outward-going ones, due to the factor $\cos\theta$.
- But for K, a small anisotropy is unimportant because the intensities are multiplied by the factor $\cos^2\theta$, which does **not** change sign for inward and outward radiation.
- In order to evaluate K or c_2 we can approximate the radiation field by an isotropic radiation field of the mean intensity J: $I = J$ (by definition). From the definition of K_λ we obtain

$$4\pi \, K_\lambda = \oint I_\lambda(\tau_\lambda, \theta)\cos^2\theta \, d\omega = J_\lambda(\tau_\lambda)\oint \cos^2\theta \, d\omega = \frac{4\pi}{3} J_\lambda(\tau_\lambda)$$

or after division by 4π,

$$K_\lambda(\tau_\lambda) = \frac{1}{3} J_\lambda(\tau_\lambda)$$

This approximation for the K-function is known as the **Eddington approximation**.

Recap: Moments of intensity

- The mean intensity J_λ is the directional average (over 4π steradians) of the specific intensity **[0-th moment of intensity]**:

$$J_\lambda \equiv \frac{1}{4\pi}\oint I_\lambda d\omega = \frac{2\pi}{4\pi}\int_{-1}^{1} I(\mu)\,d\mu = \frac{1}{2}\int_{-1}^{1} I(\mu)\,d\mu$$

- **Eddington** flux H_λ,, is the directional average (over 4π steradians) of the projection of the specific intensity **[1st moment of intensity]**:

$$H_\lambda = \frac{1}{4\pi}\oint I_\lambda \cos\theta\, d\omega = \frac{2\pi}{4\pi}\int_{-1}^{1} I(\mu)\,\mu\, d\mu = \frac{1}{2}\int_{-1}^{1} I(\mu)\,\mu\, d\mu$$

F_λ - astrophysical flux
H_λ - Eddington flux
$F_\lambda = \pi \mathrm{F}_\lambda = 4\pi H_\lambda$

- K-integral **[2nd moment of intensity]** :

$$K_\lambda = \frac{1}{4\pi}\oint I_\lambda \cos^2\theta\, d\omega = \frac{2\pi}{4\pi}\int_{-1}^{1} I(\mu)\,\mu^2\, d\mu = \frac{1}{2}\int_{-1}^{1} I(\mu)\mu^2 d\mu$$

The Milne-Eddington model (4)

$$\frac{1}{2}\int_{-1}^{+1} \ldots [\mu]\, d\mu \;\times \qquad \mu \frac{dI_\nu}{d\tau_\nu} = I_\nu - \lambda_\nu B_\nu - (1-\lambda_\nu)\, J_\nu \qquad \times \frac{1}{2}\int_{-1}^{+1} \ldots [\mu]\, d\mu$$

Multiply both sides by $\mathrm{d}\mu$ and $\mu\,\mathrm{d}\mu$ and integrate:

$$\frac{dH_\nu}{d\tau_\nu} = J_\nu - \lambda_\nu B_\nu - (1-\lambda_\nu)\, J_\nu = \lambda_\nu (J_\nu - B_\nu)$$

$$\int_0^\infty \frac{dK_\lambda}{d\tau_\lambda} d\lambda = \frac{F(\tau)}{4\pi} = H(\tau)$$

The third radiative equilibrium condition

$$\frac{dK_\nu}{d\tau_\nu} = H_\nu = \frac{1}{3}\frac{dJ_\nu}{d\tau_\nu}$$

F_λ - astrophysical flux
H_λ - Eddington flux
$\mathcal{F}_\lambda = \pi F_\lambda = 4\pi H_\lambda$

Differentiate again

$$\frac{d^2K_\nu}{d\tau_\nu^2} = \lambda_\nu (J_\nu - B_\nu) = \frac{1}{3}\frac{d^2J_\nu}{d\tau_\nu^2}$$

Eddington approximation

The Milne-Eddington model (5)

$$\frac{1}{3}\frac{d^2 J_\nu}{d\tau_\nu^2} = \lambda_\nu (J_\nu - B_\nu)$$

B_ν is linear in τ, so zero second derivative $\frac{d^2 B_\nu}{d\tau_\nu^2} = 0$

$$\frac{1}{3}\frac{d^2 J_\nu}{d\tau_\nu^2} = \frac{1}{3}\frac{d^2(J_\nu - B_\nu)}{d\tau_\nu^2} = \lambda_\nu (J_\nu - B_\nu)$$

This can be integrated to give

$$J_\nu - B_\nu = \mathcal{A} e^{-\sqrt{3\lambda_\nu}\tau_\nu} + \mathcal{B} e^{\sqrt{3\lambda_\nu}\tau_\nu}$$

Apply boundary condition at depth:

$$\tau_\nu \rightarrow \infty \quad \Rightarrow \quad J_\nu \rightarrow B_\nu \quad \Rightarrow \quad \boldsymbol{\mathcal{B} = 0}$$

The Milne-Eddington model (6)

$J_\nu - B_\nu =$

Now ap

From g

Lecture 19

Eddington approximation (3)

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- Boundary condition: there is no flux going into the star, i.e. $I(0,\theta) = I^- = 0$ for $\pi/2 < \theta < \pi$
- We also assume that the outward intensity does not depend upon θ, i.e. $I(0,\theta) = I^+ = \text{const}$ for $0 < \theta < \pi/2$
- It gives $J(0) = \frac{1}{2\pi} I^+ = \frac{1}{2\pi} F(0)$

$$S(\tau) = \frac{3}{4\pi}\tau F(0) + C = J(\tau)$$

- Hence $C = J(0) = F(0)/2\pi$ so:

$$S(\tau) = \frac{1}{\pi}\left(\frac{3}{4}\tau + \frac{1}{2}\right)F(0)$$

$$S(\tau) = \frac{3}{4\pi}\left(\tau + \frac{2}{3}\right)F(0)$$

- To find the depth dependence of T, we also need to assume **LTE**.

$q(\tau)$ is a slowly varying function (Hopf function), with $q = 1/\sqrt{3}$ at $\tau=0$

The Milne-Eddington model (7)

$$J_\nu - B_\nu = \mathcal{A}e^{-\sqrt{3\lambda_\nu}\tau_\nu} + \mathcal{B}e^{\sqrt{3\lambda_\nu}\tau_\nu}$$

Now apply boundary condition at at surface:

$$\tau_\nu = 0 \quad \Rightarrow \quad J_\nu = B_\nu + \mathcal{A}$$

From grey atmosphere solution, get $J(\tau=0)$:

Eddington approximation (2)

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- Inserting the Eddington approximation into the above equation we find

$$\frac{dK(\tau)}{d\tau} = \frac{F(\tau)}{4\pi}$$

$$\frac{dK(\tau)}{d\tau} = \frac{1}{3}\frac{dJ(\tau)}{d\tau} = \frac{F(\tau)}{4\pi} = c_1$$

$$\frac{dJ(\tau)}{d\tau} = \frac{3}{4\pi}F(\tau)$$

The Milne-Eddington model (8)

$$J_\nu - B_\nu = \mathcal{A}e^{-\sqrt{3\lambda_\nu}\tau_\nu} + \mathcal{B}e^{\sqrt{3\lambda_\nu}\tau_\nu}$$

Now apply boundary condition at at surface:

$$\tau_\nu = 0 \quad \Rightarrow \quad J_\nu = B_\nu + \mathcal{A}$$

From grey atmosphere solution, get $J(\tau=0)$:

$$J(\tau) = \frac{3}{4\pi}[\tau + q(\tau)]F(0) = 3H(0 + \frac{1}{\sqrt{3}}) = \sqrt{3}H$$

$$\left.\frac{1}{3}\frac{dJ_\nu}{d\tau_\nu}\right|_{\tau_\nu=0} = H_\nu(0) = \frac{1}{\sqrt{3}}J_\nu(0)$$

From $B_\nu = a + b\tau_c$ $\quad J_\nu(\tau_c = 0) = B_\nu + \mathcal{A} = a + \mathcal{A} = \left.\frac{1}{\sqrt{3}}\frac{dJ_\nu}{d\tau_\nu}\right|_{\tau_\nu=0}$

The Milne-Eddington model (9)

$$J_\nu = B_\nu + \mathcal{A}e^{-\sqrt{3\lambda_\nu}\tau_\nu} = a + b\tau_c + \mathcal{A}e^{-\sqrt{3\lambda_\nu}\tau_\nu}$$

$$\frac{1}{\sqrt{3}}\frac{dJ_\nu}{d\tau_\nu}\bigg|_{\tau_\nu=0} = \frac{1}{\sqrt{3}}\left[-\mathcal{A}\sqrt{3\lambda_\nu} + \frac{b}{1+\beta_\nu}\right] = a+\mathcal{A}$$

$$\tau_c = \frac{\tau_\nu}{1+\beta_\nu}$$

can now solve for $\mathcal{A}$!

$$\mathcal{A} = \frac{\frac{b}{1+\beta_\nu} - \sqrt{3}a}{\sqrt{3}+\sqrt{3\lambda_\nu}}$$

Define $p_\nu \equiv \frac{b}{1+\beta_\nu}$

$$J_\nu(\tau) = a + p_\nu\tau_\nu + \frac{p_\nu - \sqrt{3}a}{\sqrt{3}+\sqrt{3\lambda_\nu}}e^{-\sqrt{3\lambda_\nu}\tau_\nu}$$

The Milne-Eddington model (10)

Thus, we obtained the fully analytic solution for the mean intensity

$$J_\nu(\tau) = \underbrace{a + p_\nu \tau_\nu}_{B_\nu} + \frac{p_\nu - \sqrt{3}a}{\sqrt{3} + \sqrt{3\lambda_\nu}} e^{-\sqrt{3\lambda_\nu}\tau_\nu}$$

Thermalization depth

$$\tau_\nu \gtrsim \frac{1}{\sqrt{\lambda_\nu}}$$

$J_\nu \rightarrow B_\nu$

$J_\nu < B_\nu$ in outer parts of atmosphere

We can use this to obtain the emergent flux

$$H_\nu(0) = \frac{1}{\sqrt{3}} J_\nu(0) = \frac{a}{\sqrt{3}} + \frac{p_\nu - \sqrt{3}a}{3(1 + \sqrt{\lambda_\nu})} = \frac{p_\nu + a\sqrt{3\lambda_\nu}}{3(1 + \sqrt{\lambda_\nu})}$$

Residual flux of the line

$$H_\nu(0) = \frac{p_\nu + a\sqrt{3\lambda_\nu}}{3(1+\sqrt{\lambda_\nu})}$$

$$B_\nu = a + b\tau_c \qquad \beta_\nu \equiv \frac{\alpha_\nu^L}{\alpha_\nu^C} \qquad \tau_c = \frac{\tau_\nu}{1+\beta_\nu}$$

$$p_\nu \equiv \frac{b}{1+\beta_\nu} \qquad \lambda_\nu \equiv \frac{1+\zeta\beta_\nu}{1+\beta_\nu}$$

Residual flux (relative intensity)

$$r_\nu = \frac{F_\nu}{F_c} = \frac{H_\nu(0)}{H_c(0)}$$

for continuum H_c: $\beta_\nu = 0 \quad \Rightarrow \quad p_\nu = b \quad \lambda_\nu = 1$

$$H_c(0) = \frac{1}{3}\frac{(b + a\sqrt{3})}{2}$$

$$r_\nu = 2\frac{p_\nu + a\sqrt{3\lambda_\nu}}{(1+\sqrt{\lambda_\nu})(b + a\sqrt{3})}$$

Non-negligible scattering in continuum

$$H_\nu(0) = \frac{p_\nu + a\sqrt{3\lambda_\nu}}{3(1+\sqrt{\lambda_\nu})}$$

$B_\nu = a + b\tau_c$ $\quad \beta_\nu \equiv \frac{\alpha_\nu^L}{\alpha_\nu^C}$ $\quad \tau_c = \frac{\tau_\nu}{1+\beta_\nu}$

$p_\nu \equiv \frac{b}{1+\beta_\nu}$ $\quad \lambda_\nu \equiv \frac{\zeta^C + \zeta^L \beta_\nu}{1+\beta_\nu}$

for continuum H_c: $\quad \beta_\nu = 0 \quad \Rightarrow \quad p_\nu = b \quad \lambda_\nu = \zeta^C$

without proof

$$H_c(0) = \frac{(b + a\sqrt{3\zeta^C})}{3(1+\sqrt{\zeta^C})}$$

$$r_\nu = \left(\frac{p_\nu + a\sqrt{3\lambda_\nu}}{b + a\sqrt{3\zeta^C}}\right)\left(\frac{1+\sqrt{\zeta^C}}{1+\sqrt{\lambda_\nu}}\right)$$

Various special cases

$$r_\nu = 2\frac{p_\nu + a\sqrt{3\lambda_\nu}}{(1+\sqrt{\lambda_\nu})(b + a\sqrt{3})}$$

$$B_\nu = a + b\tau_c \qquad \beta_\nu \equiv \frac{\alpha_\nu^L}{\alpha_\nu^C} \qquad \tau_c = \frac{\tau_\nu}{1+\beta_\nu}$$

$$p_\nu \equiv \frac{b}{1+\beta_\nu} \qquad \lambda_\nu \equiv \frac{1+\zeta\beta_\nu}{1+\beta_\nu}$$

This general result contains interesting behaviours in various special cases:

a) case $\zeta = 1$ (LTE: pure absorption lines)

b) case $\zeta = 0$ (extreme non-LTE: pure scattering lines)

c) Schuster Mechanism: Line Emission from Continuum Scattering Layer

$$r_\nu = \left(\frac{p_\nu + a\sqrt{3\lambda_\nu}}{b + a\sqrt{3\zeta^C}}\right)\left(\frac{1+\sqrt{\zeta^C}}{1+\sqrt{\lambda_\nu}}\right)$$

Pure absorption lines (LTE)

$$r_\nu = 2\frac{p_\nu + a\sqrt{3\lambda_\nu}}{(1+\sqrt{\lambda_\nu})(b+a\sqrt{3})}$$

$$B_\nu = a + b\tau_c \quad \beta_\nu \equiv \frac{\alpha_\nu^L}{\alpha_\nu^C} \quad \tau_c = \frac{\tau_\nu}{1+\beta_\nu}$$

$$p_\nu \equiv \frac{b}{1+\beta_\nu} \quad \lambda_\nu \equiv \frac{1+\zeta\beta_\nu}{1+\beta_\nu}$$

a) pure absorption in line: $\zeta = 1$

$$\lambda_\nu \equiv \frac{1+\zeta\beta_\nu}{1+\beta_\nu} = 1 \quad \Rightarrow \quad r_\nu = \frac{p_\nu + a\sqrt{3}}{b + a\sqrt{3}} = \frac{\frac{b}{1+\beta_\nu} + a\sqrt{3}}{b + a\sqrt{3}}$$

For strong lines: $\beta_\nu \gg 1$

$$r_\nu = \frac{a\sqrt{3}}{b+a\sqrt{3}} = \frac{a}{b/\sqrt{3}+a} = \frac{B_\nu\,(\tau_\nu = 0)}{B_\nu\,(\tau_\nu = 1/\sqrt{3})} \neq 0$$

For grey atmosphere, strongest lines:

$$S_\lambda(\tau_\lambda) = \frac{3}{4\pi}(\tau_\lambda + \frac{2}{3})F_\lambda(0)$$

a/b=2/3 → $r_\nu \approx 0.54$

Non-zero because we see B_ν at upper level with non-zero temperature

Thus, in LTE, the residual flux is non-zero even for strong absorption lines. However, resonance lines such as Na D have R~10^{-3} – 10^{-4}

Pure scattering lines (extreme NLTE)

$$r_\nu = 2\frac{p_\nu + a\sqrt{3\lambda_\nu}}{(1+\sqrt{\lambda_\nu})(b + a\sqrt{3})}$$

$$B_\nu = a + b\tau_c \qquad \beta_\nu \equiv \frac{\alpha_\nu^L}{\alpha_\nu^C} \qquad \tau_c = \frac{\tau_\nu}{1+\beta_\nu}$$

$$p_\nu \equiv \frac{b}{1+\beta_\nu} \qquad \lambda_\nu \equiv \frac{1+\zeta\beta_\nu}{1+\beta_\nu}$$

b) pure scattering in line: $\zeta = 0$

$$\lambda_\nu \equiv \frac{1+\zeta\beta_\nu}{1+\beta_\nu} = \frac{1}{1+\beta_\nu}$$

$$r_\nu = 2\frac{\frac{b}{1+\beta_\nu} + a\sqrt{\frac{3}{1+\beta_\nu}}}{\left(1+\sqrt{\frac{1}{1+\beta_\nu}}\right)(b + a\sqrt{3})}$$

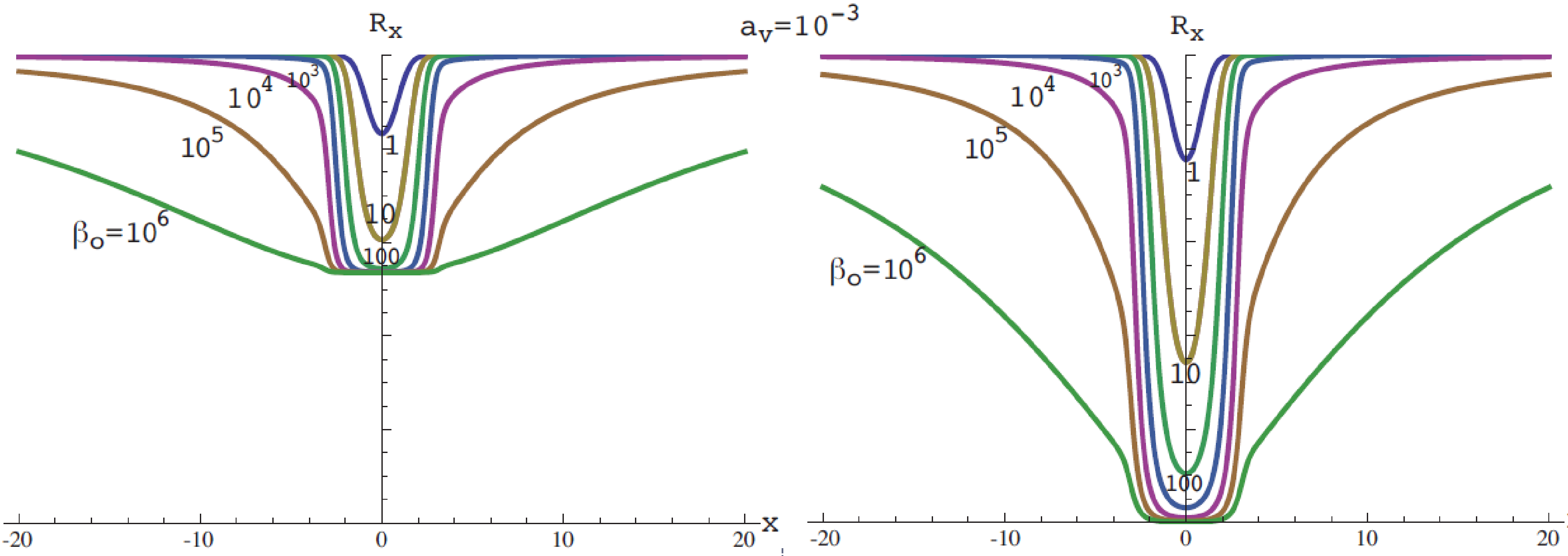
For strong lines: $\beta_\nu \gg 1$, $r_\nu \rightarrow 0$

Scattering removes all photons → no photon emerges from surface. Cores of strong scattering lines are **dark**!

The residual flux Rx vs frequency x

$\zeta = 1$ (LTE)

$\zeta = 0$ (non-LTE)



Line emission from continuum scattering layer

$$r_\nu = \left(\frac{p_\nu + a\sqrt{3\lambda_\nu}}{b + a\sqrt{3\zeta^C}}\right)\left(\frac{1+\sqrt{\zeta^C}}{1+\sqrt{\lambda_\nu}}\right)$$

$B_\nu = a + b\tau_c$ $\quad \beta_\nu \equiv \frac{\alpha_\nu^L}{\alpha_\nu^C}$ $\quad \tau_c = \frac{\tau_\nu}{1+\beta_\nu}$

$p_\nu \equiv \frac{b}{1+\beta_\nu}$ $\quad \lambda_\nu \equiv \frac{\zeta^C + \zeta^L\beta_\nu}{1+\beta_\nu}$

c) pure scattering in continuum: $\zeta^C = 0$

$$\lambda_\nu \equiv \frac{\zeta^C + \zeta^L\beta_\nu}{1+\beta_\nu} = \frac{\zeta^L\beta_\nu}{1+\beta_\nu} \quad \Rightarrow \quad r_\nu = \frac{\frac{1}{1+\beta_\nu} + \frac{a}{b}\sqrt{3\lambda_\nu}}{1+\sqrt{\lambda_\nu}}$$

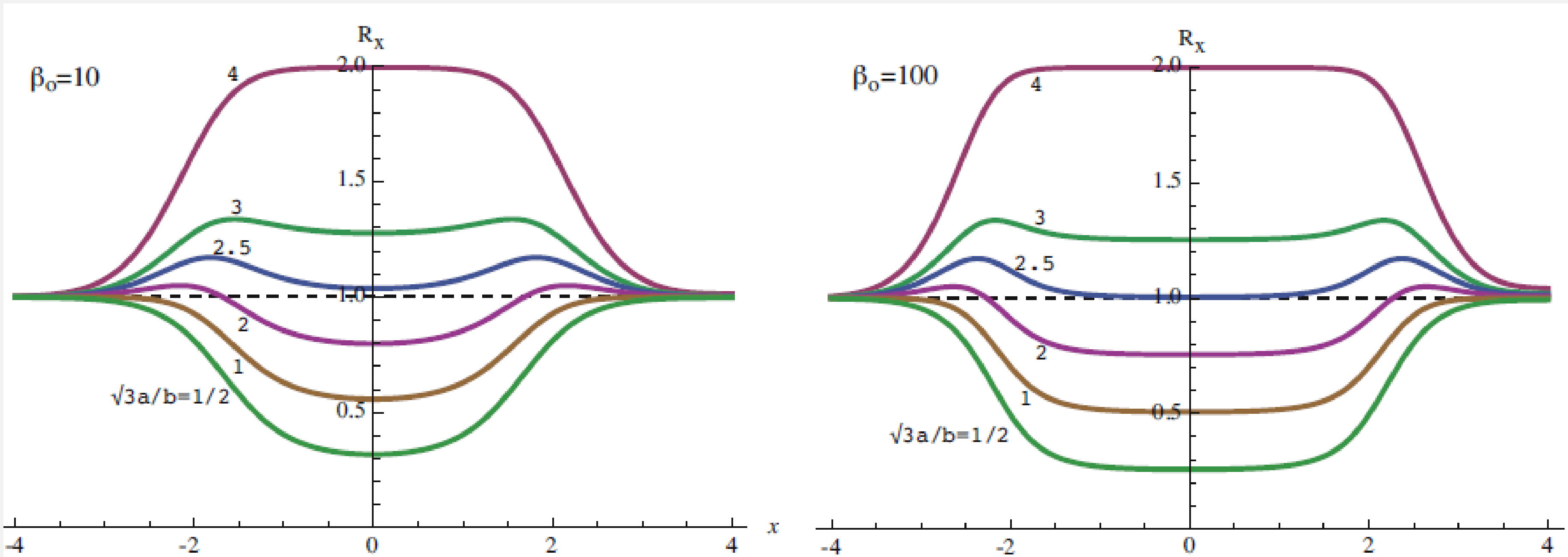
If the line opacity is also pure scattering, $\zeta^L = 0$, then $\lambda_\nu = 0$ $\quad r_\nu = \frac{1}{1+\beta_\nu} < 1$

always in absorption

But for $\zeta^L = 1$ and for strong lines $\beta_\nu \gg 1$ $\quad r_\nu \to \frac{\sqrt{3}a}{2b}$

For a weak temperature gradient with small b/a, can exceed unity, implying a net line emission instead of absorption.

Line profiles for Schuster model



Scattering makes the continuum source function low near the surface, $S_c(0) \sim J_c(0) \ll B(0)$, which implies a weak continuum flux. The line can potentially be brighter, but only if the decline from the negative temperature gradient term is not too steep.

Summary

- We obtained Transfer Equation including lines and taking into account Scattering in lines.
- We solved it using the Milne-Eddington model.
- We then obtained Residual flux of the line.
- Finally, we discussed interesting special cases such as pure absorption and pure scattering lines.
- We also tried to explain emission lines applying Schuster mechanism for line emission.