

Recap: Equations of radiative equilibrium

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- **The 1st equation of radiative equilibrium:**

$$F(x) = F(0) = \text{const} = \sigma T_{eff}^4$$

i.e. the total flux must be constant at all depths of the photosphere (**conservation of energy**):
 $dF/dt=0$ or $dF/dx=0$ or $dF/d\tau=0$

- **The 2nd equation of radiative equilibrium:**

the **total energy absorbed** (RHS) must equal the **total energy re-emitted** (LHS) if no heating or cooling is taking place:

$$\int_0^{\infty} \kappa_{\lambda} S_{\lambda} d\lambda = \int_0^{\infty} \kappa_{\lambda} J_{\lambda} d\lambda$$

- **The 3rd radiative equilibrium condition:**

$$\int_0^{\infty} \frac{dK_{\lambda}}{d\tau_{\lambda}} d\lambda = \frac{F(\tau)}{4\pi}$$

- All the three radiative equilibrium conditions are not independent.
 S_{λ} that is a solution of one will be the solution of all three.

The depth dependence of the source function

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- In a grey atmosphere, with $K(\tau) = \int_0^\infty K_\lambda d\lambda$, the 3rd equation implies:

a new unknown function $K(\tau)$

$$\frac{dK(\tau)}{d\tau} = \frac{F(\tau)}{4\pi}$$

- We can differentiate this, and insert our earlier result:

$$\frac{d^2K(\tau)}{d\tau^2} = \frac{1}{4\pi} \frac{dF(\tau)}{d\tau} = J(\tau) - S(\tau) = 0 \quad [1]$$

- Integration of the equation with respect to τ gives $K(\tau) = c_1\tau + c_2$ where $dK/d\tau = c_1 = F/4\pi$ [2]

- For a given F , we now have two equations, [1] and [2], to determine the three unknowns: J , S and K (or c_2). We need an additional relation between two of these variables in order to determine all three.

Eddington approximation (1)

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- Previously we have seen that for the determination of the flux **the anisotropy in the radiation field is very important** because in the flux integral the inward-going intensities are subtracted from the outward-going ones, due to the factor $\cos\theta$.
- But for K , a small anisotropy is unimportant because the intensities are multiplied by the factor $\cos^2\theta$, which does **not** change sign for inward and outward radiation.
- To evaluate K or c_2 , we can approximate the radiation field by an isotropic radiation field of the mean intensity J : $I = J$ (**by definition**). From the definition of K_λ we obtain

$$4\pi K_\lambda = \oint I_\lambda(\tau_\lambda, \theta) \cos^2 \theta d\omega = J_\lambda(\tau_\lambda) \oint \cos^2 \theta d\omega = \frac{4\pi}{3} J_\lambda(\tau_\lambda)$$

or after division by 4π ,

$$K_\lambda(\tau_\lambda) = \frac{1}{3} J_\lambda(\tau_\lambda)$$

$$d\omega = \sin\theta d\theta d\varphi$$

This approximation for the K -function is known as the **Eddington approximation**.

Eddington approximation (2)

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- Inserting the Eddington approximation into the above equation we find $\frac{dK(\tau)}{d\tau} = \frac{F(\tau)}{4\pi}$

$$\frac{dK(\tau)}{d\tau} = \frac{1}{3} \frac{dJ(\tau)}{d\tau} = \frac{F(\tau)}{4\pi} = c_1 \qquad \frac{dJ(\tau)}{d\tau} = \frac{3}{4\pi} F(\tau)$$

- Since the mean intensity J equals the source function S in a grey atmosphere, integrating the latter result we obtain

$$S(\tau) = \frac{3}{4\pi} \tau F(0) + C = J(\tau) \qquad F(x) = F(0) = \text{const}$$

- From the conditions of radiative equilibrium, we finally obtained **the law for the depth dependence of the source function** (for a grey atmosphere assuming the Eddington approximation). We can evaluate C using boundary condition for the known emerging flux (there is no flux going into the star), plus we assume the outward intensity does not depend upon θ :

Eddington approximation (3)

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- Boundary condition: there is no flux going into the star, i.e. $I(0, \theta) = I^- = 0$ for $\pi/2 < \theta < \pi$
- We also assume that the outward intensity does not depend upon θ , i.e. $I(0, \theta) = I^+ = \text{const}$ for $0 < \theta < \pi/2$

- It gives
$$J(0) = \frac{1}{2\pi} I^+ = \frac{1}{2\pi} F(0)$$

- Hence $C = J(0) = F(0)/2\pi$ so:

$$S(\tau) = \frac{1}{\pi} \left(\frac{3}{4}\tau + \frac{1}{2} \right) F(0)$$

$$S(\tau) = \frac{3}{4\pi} \tau F(0) + C = J(\tau)$$

$$S(\tau) = \frac{3}{4\pi} \left(\tau + \frac{2}{3} \right) F(0)$$

- To find the depth dependence of T , we also need to assume **LTE**.

Temperature structure of the grey atmosphere

In LTE, the source function is the Planck function, $S(\tau) = B(\tau) = \sigma T^4 / \pi$

$$B(\tau) = \frac{\sigma}{\pi} T^4(\tau) = \frac{3}{4\pi} \left(\tau + \frac{2}{3}\right) F(0)$$

$$S(\tau) = \frac{3}{4\pi} \left(\tau + \frac{2}{3}\right) F(0)$$

Recall that $F(0) = \sigma T_{\text{eff}}^4$, by definition, so

$$\frac{1}{\pi} \sigma T^4(\tau) = \frac{3}{4\pi} \left(\tau + \frac{2}{3}\right) \sigma T_{\text{eff}}^4 \quad \text{or}$$

$$T^4(\tau) = \frac{3}{4} \left(\tau + \frac{2}{3}\right) T_{\text{eff}}^4$$

We derived the **temperature dependence on optical depth**.

Note $T(\tau = 2/3) = T_{\text{eff}}$ as we obtained earlier, and $T^4(\tau = 0) = T_{\text{eff}}^4 / 2$

A complete solution of the **grey** case, using **accurate** boundary conditions, without **Eddington** approximation, leads to a solution only slightly different from this, usually expressed as

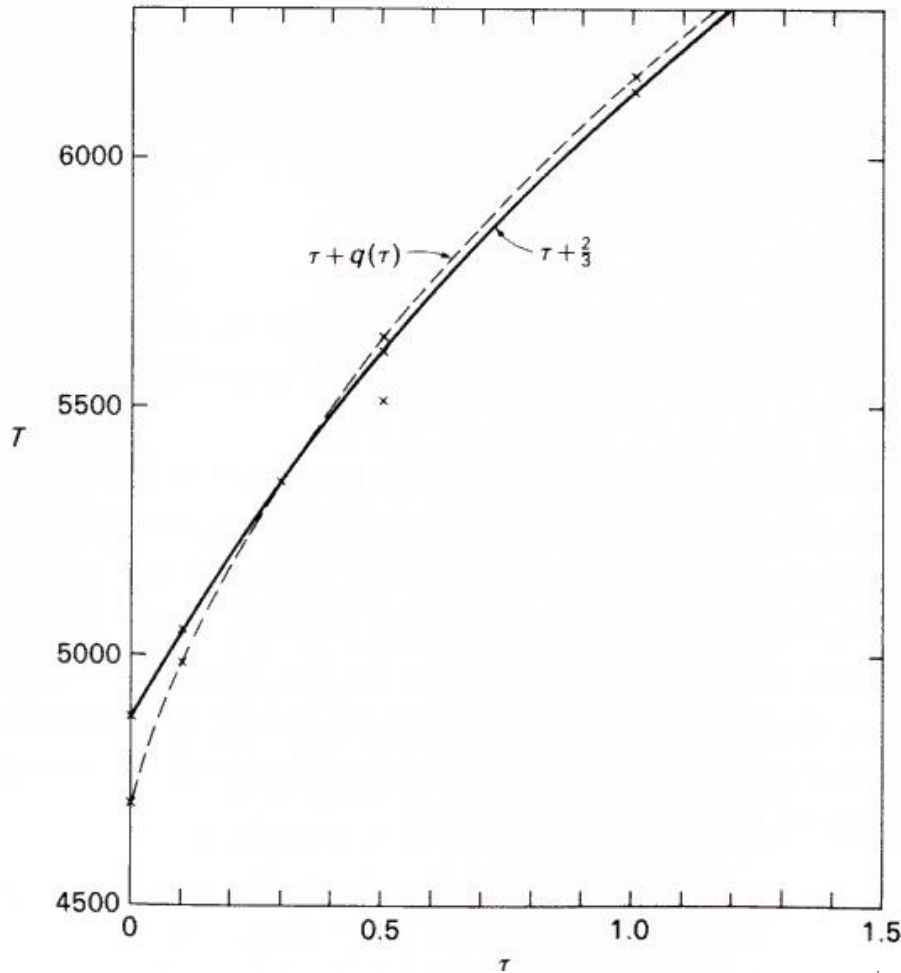
$$T^4(\tau) = \frac{3}{4} [\tau + q(\tau)] T_{\text{eff}}^4$$

Here $q(\tau)$ is a slowly varying function (**Hopf function**), with

$$q = 1/\sqrt{3} = 0.577 \text{ at } \tau = 0 \text{ to } q = 0.710 \text{ at } \tau = \infty.$$

Grey Temperature Structure

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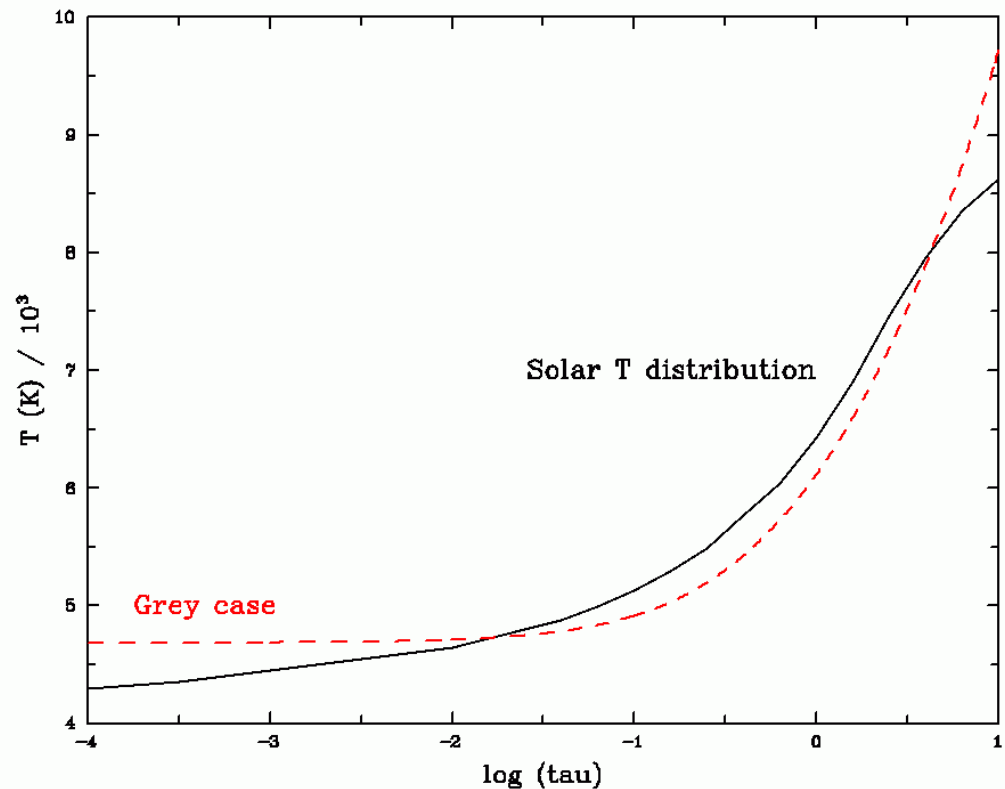
Comparison between $T(\tau)$ in the Solar atmosphere using the simplifying **Eddington assumption** (solid) versus the **exact grey case** (dashed) using the Hopf function, $q(\tau)$:

$$q(\tau) \approx 0.710 - 0.133e^{-2\tau}$$

How realistic is this?

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- How good an approximate is the **grey** atmosphere? Next we must look at the frequency dependence of the sources of opacity.
- The **grey** temperature distribution is shown here versus the observed Solar temperature distribution as a function of optical depth τ at 5000\AA (D. Gray, Table 9.2)
- The poor match is because the opacity is **wavelength dependent**, as we shall see next lecture.



Summary

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- **Three equations of radiative equilibrium** can be derived:
 - (a) constant flux with depth;
 - (b) energy absorbed equals energy emitted;
 - (c) the K -integral is linear in τ .
- From these, the **grey** temperature distribution $T(\tau)$ may be derived, assuming:
 - (a) the **Eddington approximation** and
 - (b) **LTE**, in reasonable agreement with the exact case.
- On the next lecture, we will discuss LTE in more detail.

Local Thermodynamic Equilibrium (LTE)



MAXWELLIAN VELOCITY DISTRIBUTION
BOLTZMANN EQUATION
SAHA EQUATION

Thermodynamic Equilibrium (TE)

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- Interaction of radiation and matter is the most important physical process in stellar atmospheres.
- To find I_λ we need to know α_λ and ε_λ (or k_λ and j_λ) – absorption and emission coefficients.
- To find α_λ and ε_λ , density ρ , temperature T , and chemical composition X are **not** enough. We need to know **distributions of atoms over levels and ionization states**, which depend on radiation I_λ .
- In TE, ρ , T , and X fully determine α_λ and ε_λ .

Local Thermodynamic Equilibrium

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In **Thermodynamic Equilibrium**:

1. All particles have **Maxwellian** distribution in velocities (with the same temperature T).
2. Atom populations follow **Boltzmann** law (same T).
3. Ionization is described by **Saha** formula (same T).
4. Radiation intensity is given by the **Planck** function (same T).
5. The principle of detailed equilibrium is valid (the number of direct processes = number of inverse processes).

In **Local thermodynamic equilibrium (LTE)**,
1-3 are applied **locally**.

The radiation spectrum can in principle be very far from
Planck function.

LTE

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In the study of stellar atmospheres, the assumption of Local Thermodynamic Equilibrium (**LTE**) is described by:

1. Electron and ion velocity distributions are **Maxwellian**.
2. Excitation equilibrium is given by **Boltzmann** equation (introduced today).
3. Ionization equilibrium is given by **Saha** equation (introduced today).
4. The source function is **given** by the **Planck** function

$$S_\lambda = I_\lambda = B_\lambda(T) \quad \text{i.e. Kirchoff's law} \quad j_\lambda = \kappa_\lambda B_\lambda(T)$$

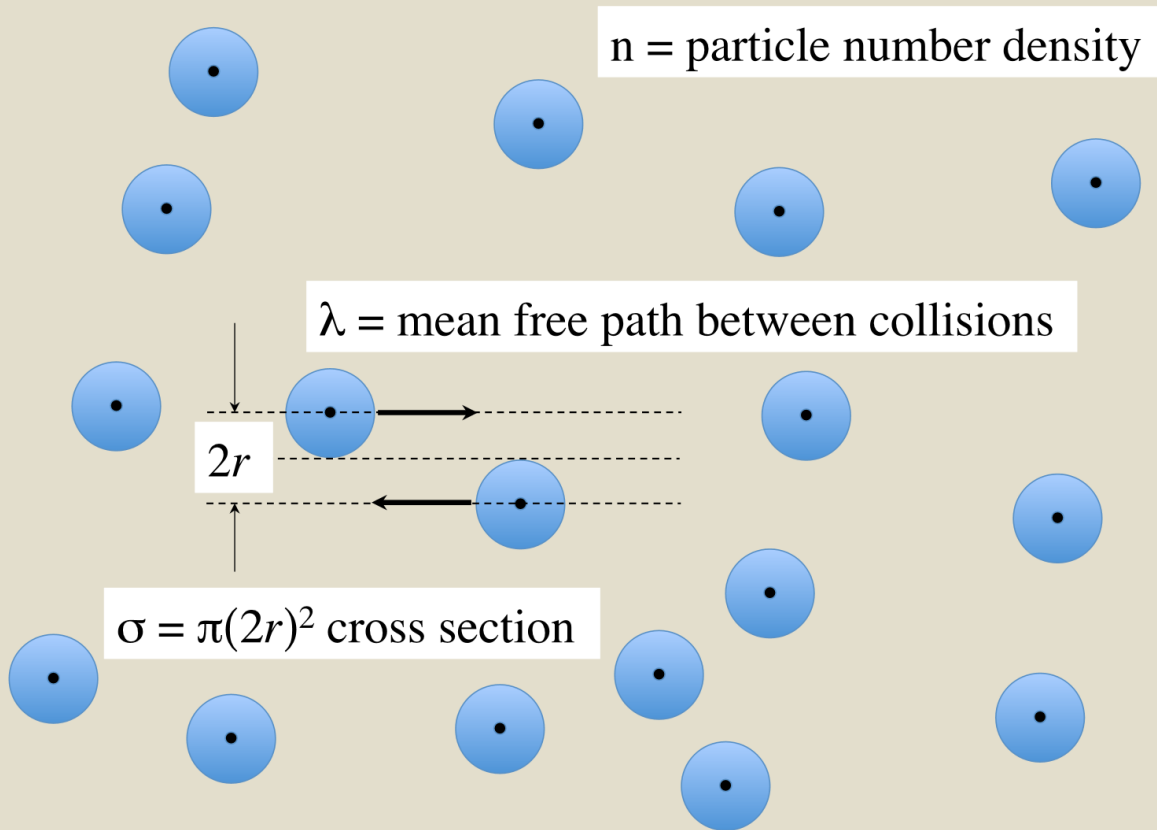
Is LTE a valid assumption?

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- For LTE to be valid, the photon and particle **mean free paths** need to be much smaller than the length scale over which these temperature changes significantly.
- Radiation cannot play a role in defining atom populations and ionization state. Collisions should dominate.
- Generally, **when collisional processes dominate over radiative processes in the excitation and ionization of atoms, the state of the gas is close to LTE.**
- Consequently, **LTE is a good assumption in stellar interiors, but may break down in the atmosphere.** If LTE is no longer valid, all processes need to be calculated in detail via **non-LTE**. This is much more complicated, but needs to be considered in some cases (see later in course).

Mean Free Path

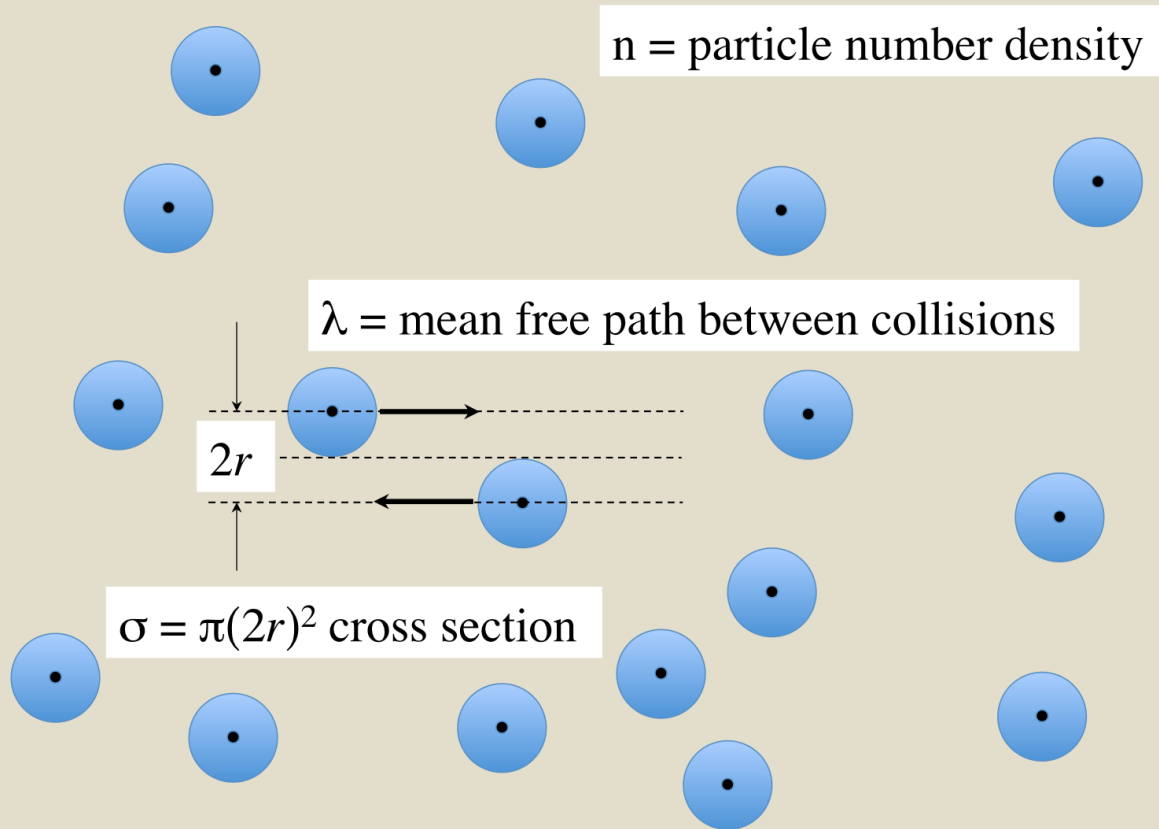
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- In the Sun, the characteristic distance over which the temperature varies (the temperature scale height) is $\sim 500\text{km}$.
- **How does this scale compare with the average distance travelled by an atom before hitting another atom?**
- Two hydrogen atoms will collide if their centres pass within a radius of 2 Bohr radii ($2a_0$) of each other. The collision cross-section of the H atom is $\sigma = \pi(2a_0)^2 = 3.5 \times 10^{-16} \text{ cm}^2$.
- The mean free path between collisions is $\lambda = 1/(\sigma n(\text{H}))$.

Mean free path in the solar photosphere

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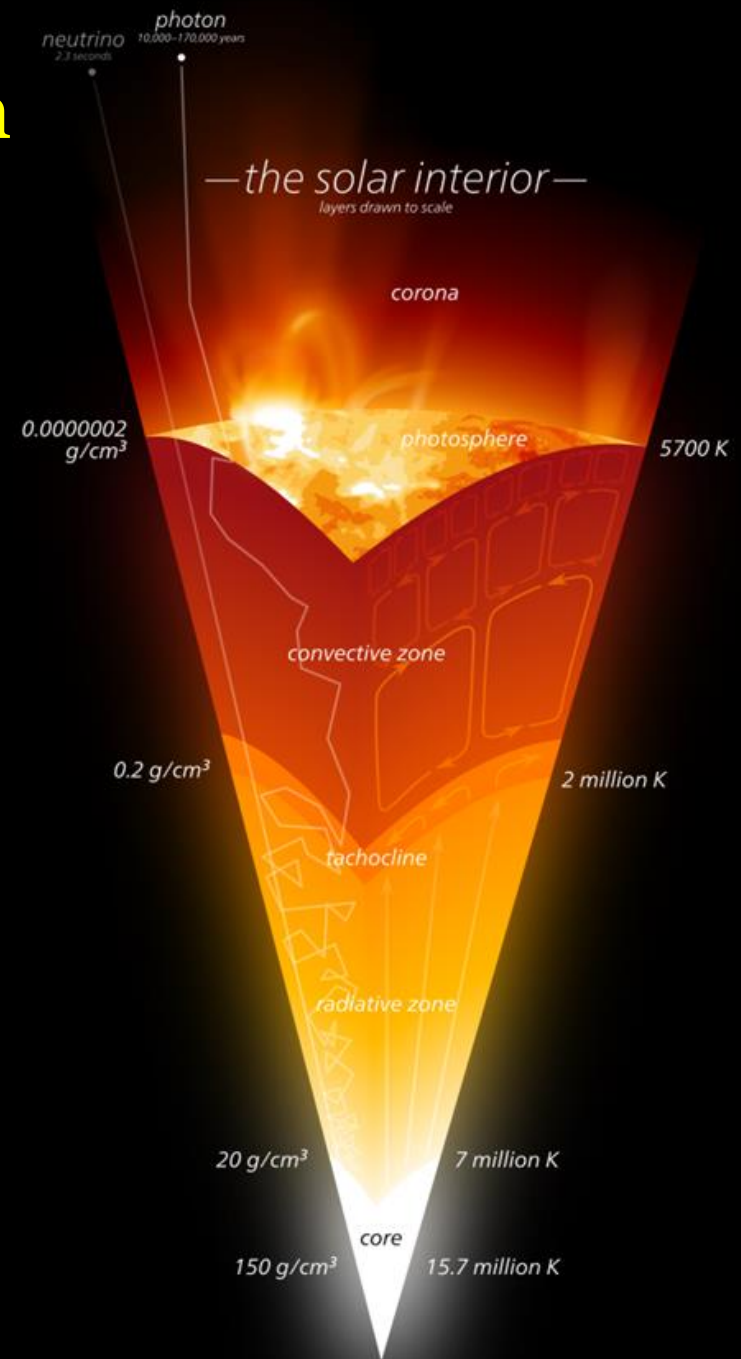
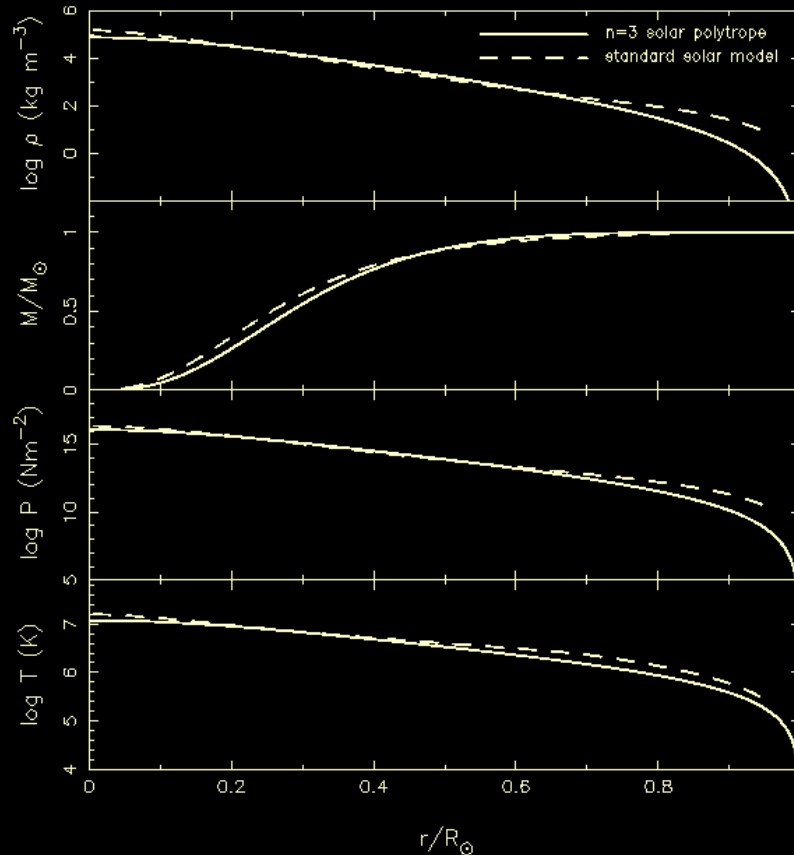


- The density of the Solar photosphere is $\rho = 2.5 \times 10^{-7}$ g/cm³ so the number of H atoms/cm³ is $n(\text{H}) = \rho/m_{\text{H}} = 1.5 \times 10^{17}$ cm⁻³ where m_{H} is the mass of the H atom.
- Then the mean free path between collisions is $\lambda = 1/(\sigma n(\text{H})) = 0.02$ cm. i.e. **atoms are confined within a limited volume of space in the photosphere at effectively fixed temperature** (relative to the temperature scale height).

In the upper layers, $\rho \rightarrow 0$, $\lambda \uparrow$, radiation dominates over collisions \rightarrow **out of LTE**

Mean Free Path in the Sun

Since the photosphere is the layer visible from Earth, photons must be able to escape freely into space. After $\sim 10^{21}$ scatterings and re-emissions (thousands years!) from the centre. Calculate the time needed for a photon to escape!



The Random Walk

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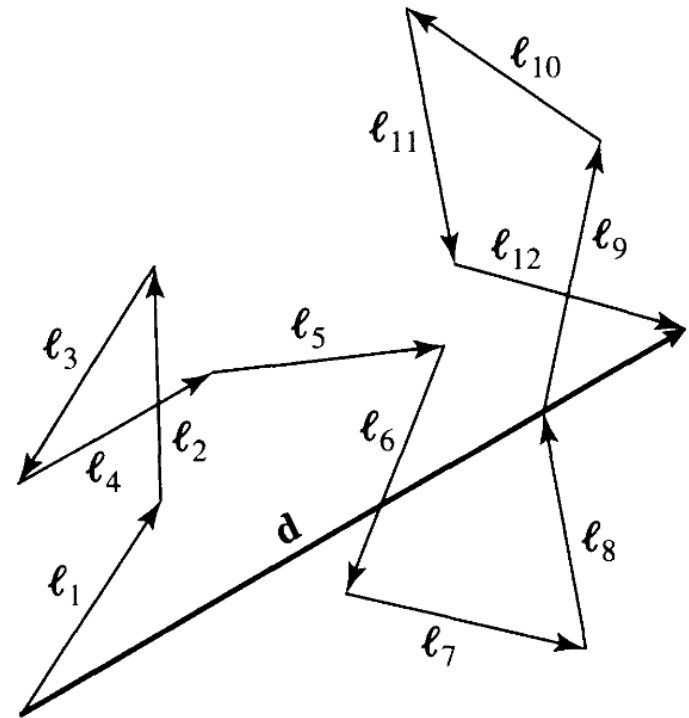
- As the photons diffuse upward through the stellar material, they follow a haphazard path called a **random walk**. Figure shows a photon that undergoes a net vector displacement d as the result of making a large number N of **randomly** directed steps, each of length l ($=\lambda$, the mean free path).
- It can be shown that for a random walk, the displacement d is related to the size of each step, l , by

$$d = l\sqrt{N}.$$

- This implies that the distance from the centre of a star to the surface is

$$D = l \times N$$

- This is why the transport of energy through a star by radiation may be extremely **inefficient**.



LTE

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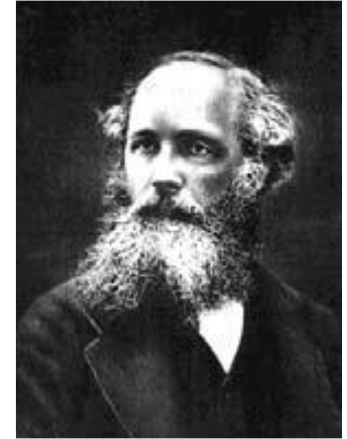
As noticed above, **LTE** is described by:

1. **Maxwellian** electron and ion velocity distributions.
2. Excitation equilibrium given by **Boltzmann** equation.
3. Ionization equilibrium given by **Saha** equation.

Let's discuss them.

Maxwellian velocity distribution

Gas pressure is produced by the motions of the gas particles. The velocities of particles are distributed in a Maxwellian distribution (also called the Maxwell–Boltzmann distribution).



$$\frac{dN(\mathbf{v})}{N_{\text{total}}} = \left(\frac{2}{\pi}\right)^{1/2} \left(\frac{m}{kT}\right)^{3/2} \mathbf{v}^2 e^{-m\mathbf{v}^2/2kT} d\mathbf{v}$$

Because the particles produce Doppler shifts, the line-of-sight velocities have a distribution that is an important special case for spectroscopy:

$$\frac{dN(v_R)}{N_{\text{total}}} = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-mv_R^2/2kT} dv_R$$

where v_R is the radial (line of sight) velocity component.

Maxwellian velocity distribution

The maximum of the speed distribution occurs at v_1 (the most probable velocity):

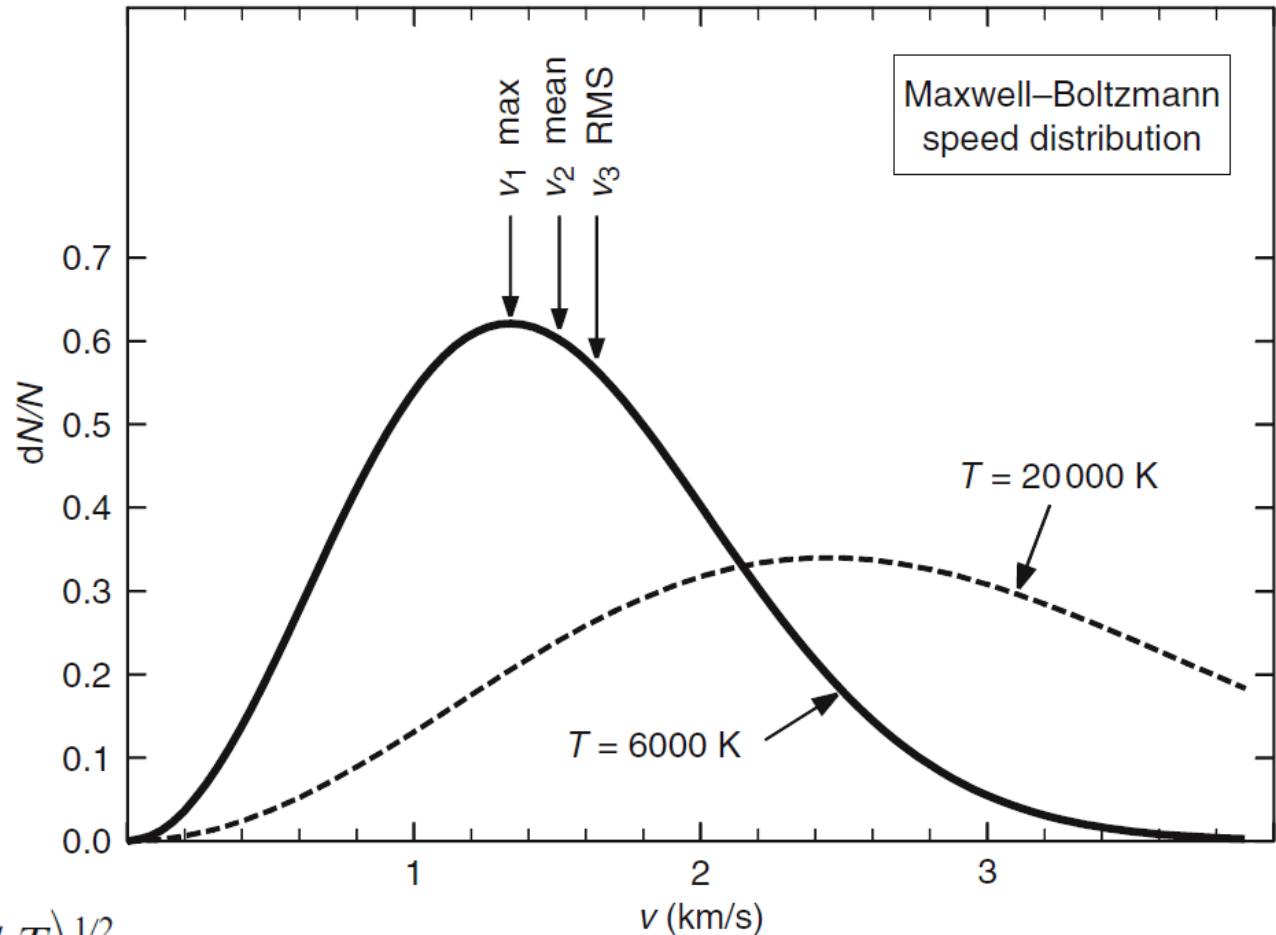
$$v_1 = \left(\frac{2kT}{m} \right)^{1/2}$$

The average velocity, v_2 , is

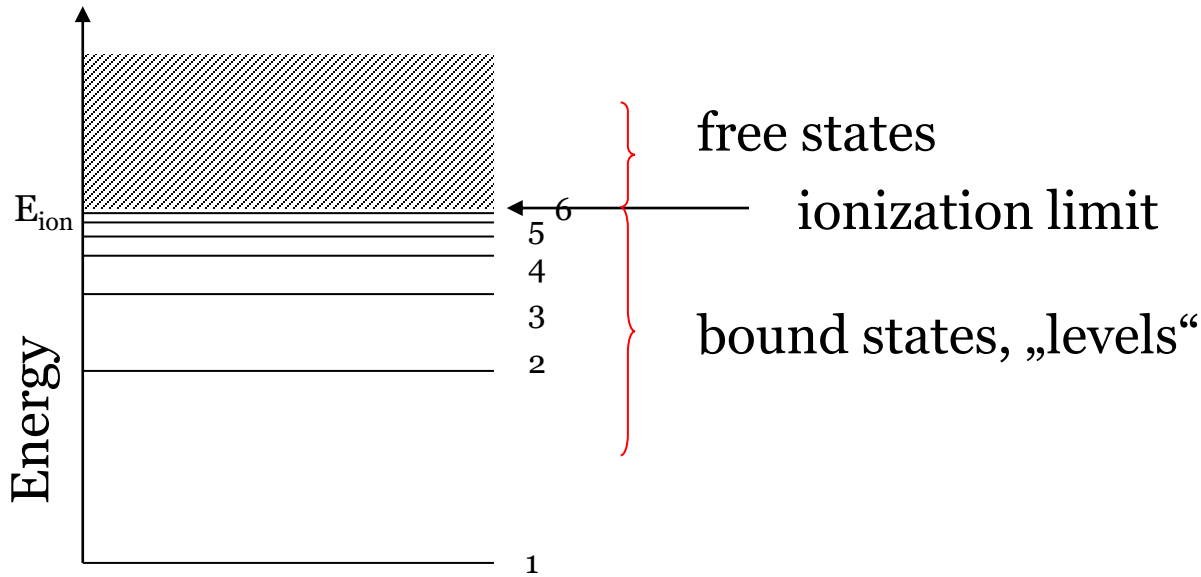
$$v_2 = \left(\frac{8kT}{\pi m} \right)^{1/2} = 1.128v_1$$

The root mean square velocity, v_3 , is

$$v_3 = \left(\frac{3kT}{m} \right)^{1/2} = 1.225v_1$$



Boltzmann equation



For excited levels u and l of e.g. atomic hydrogen, the **Boltzmann equation** relates their population (occupation) numbers as follows:

$$\frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-(E_u - E_l)/kT}$$

where $\chi_{ul} = E_u - E_l$ is the energy difference between the levels, g_u & g_l are their **statistical weights** (see next slide), $k = 8.6174 \times 10^{-5}$ eV/K is the Boltzmann constant.

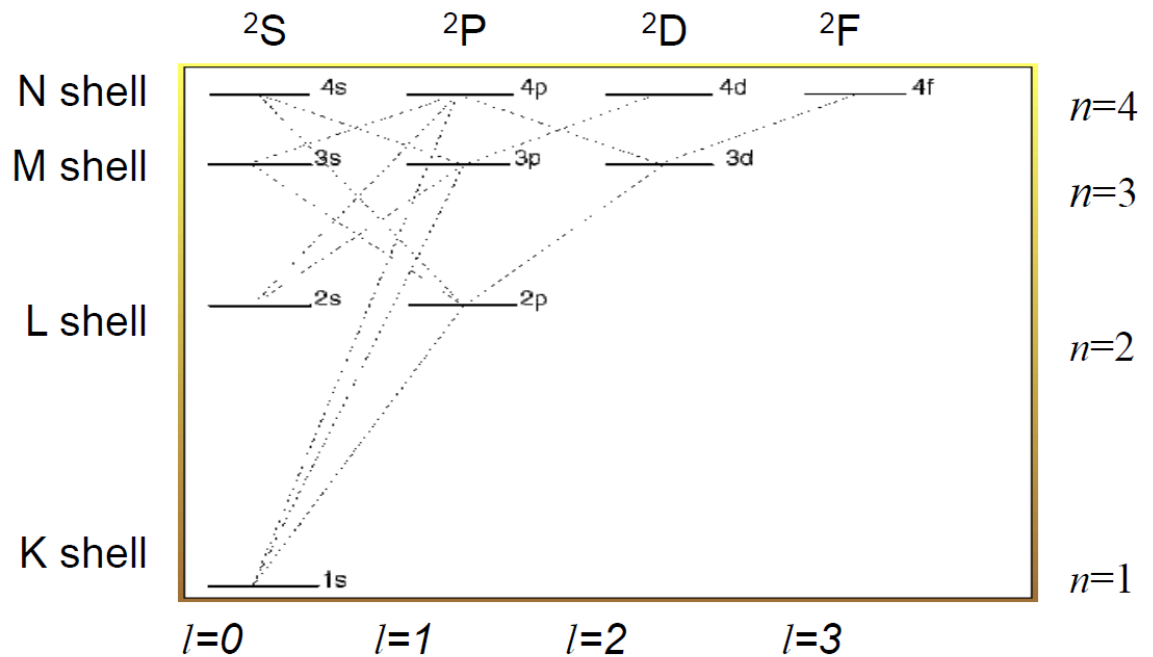
Boltzmann equation may also be written as:

$$\log \frac{N_u}{N_l} = \log \frac{g_u}{g_l} - \frac{5040}{T} \chi_{ul} (eV)$$

$$\Theta = 5040/T$$

In the “ground state” ($n=1$), “first excited state” ($n=2$), and all other excited states of H **more than one quantum state may have the same energy.**

The number of these for orbital n is the **statistical weight, g_n** , (also known as the degeneracy).



Hydrogen

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For H, orbital n has a statistical weight of $g_n = 2n^2$ – the various permutations for $n=1$ and $n=2$ are listed here, with statistical weights $g_1=2$ and $g_2=8$, respectively.

$l=0\dots n-1$ azimuthal quantum number
 m_l =magnetic quantum number with $-l \leq m_l \leq l$
 m_s =electron “spin” angular momentum $\pm 1/2$

Transition energy between levels u and l :

$$\chi_{ul} = C \left(\frac{1}{u^2} - \frac{1}{l^2} \right)$$

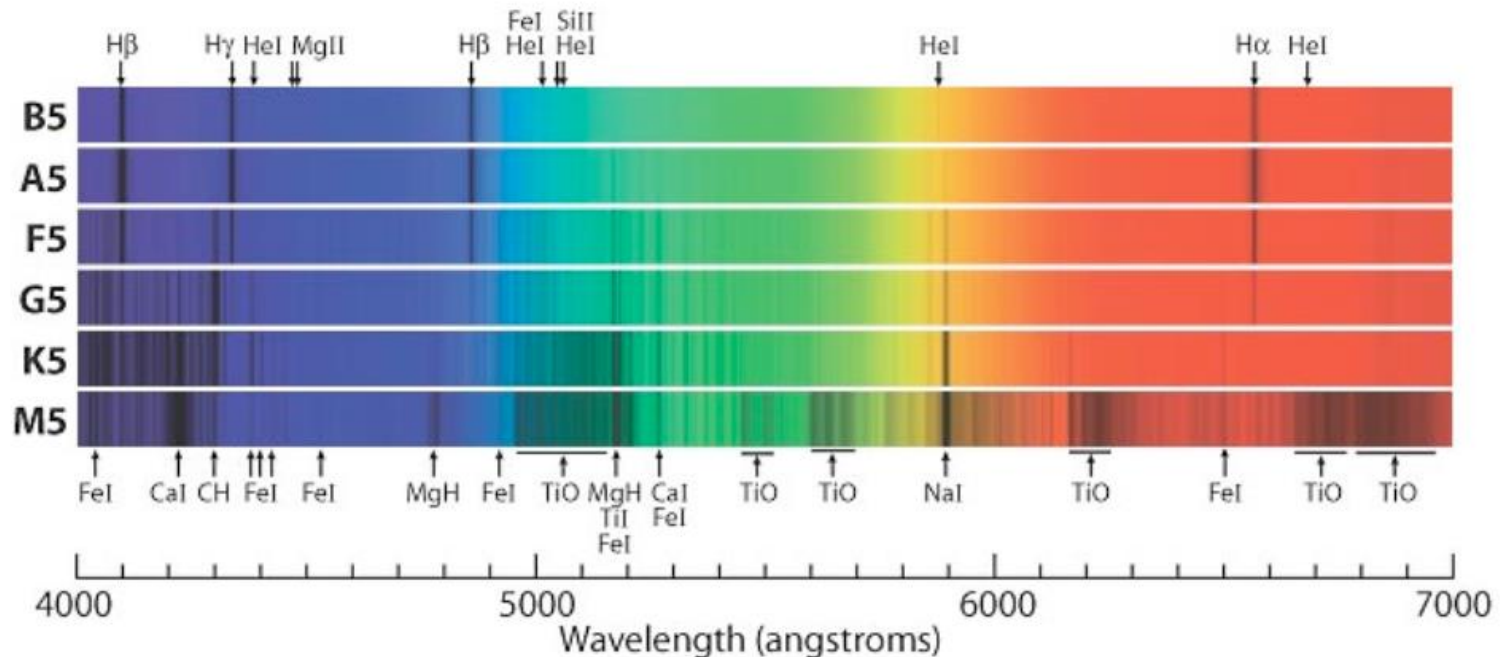
where $C = \chi_{\text{ion}} = -13.6$ eV

Ground States s_1				Energy E_1
n	l	m_l	m_s	(eV)
1	0	0	+1/2	-13.6
1	0	0	-1/2	-13.6
First Excited States s_2				Energy E_2
n	l	m_l	m_s	(eV)
2	0	0	+1/2	-3.40
2	0	0	-1/2	-3.40
2	1	1	+1/2	-3.40
2	1	1	-1/2	-3.40
2	1	0	+1/2	-3.40
2	1	0	-1/2	-3.40
2	1	-1	+1/2	-3.40
2	1	-1	-1/2	-3.40

Balmer lines

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An exceptionally high T is required for a significant number of H atoms to have electrons in their 1st excited states. The Balmer lines (involving an upward transition from $n=2$ orbital) reach a peak strength at spectral class A ($\approx 10000\text{K}$)

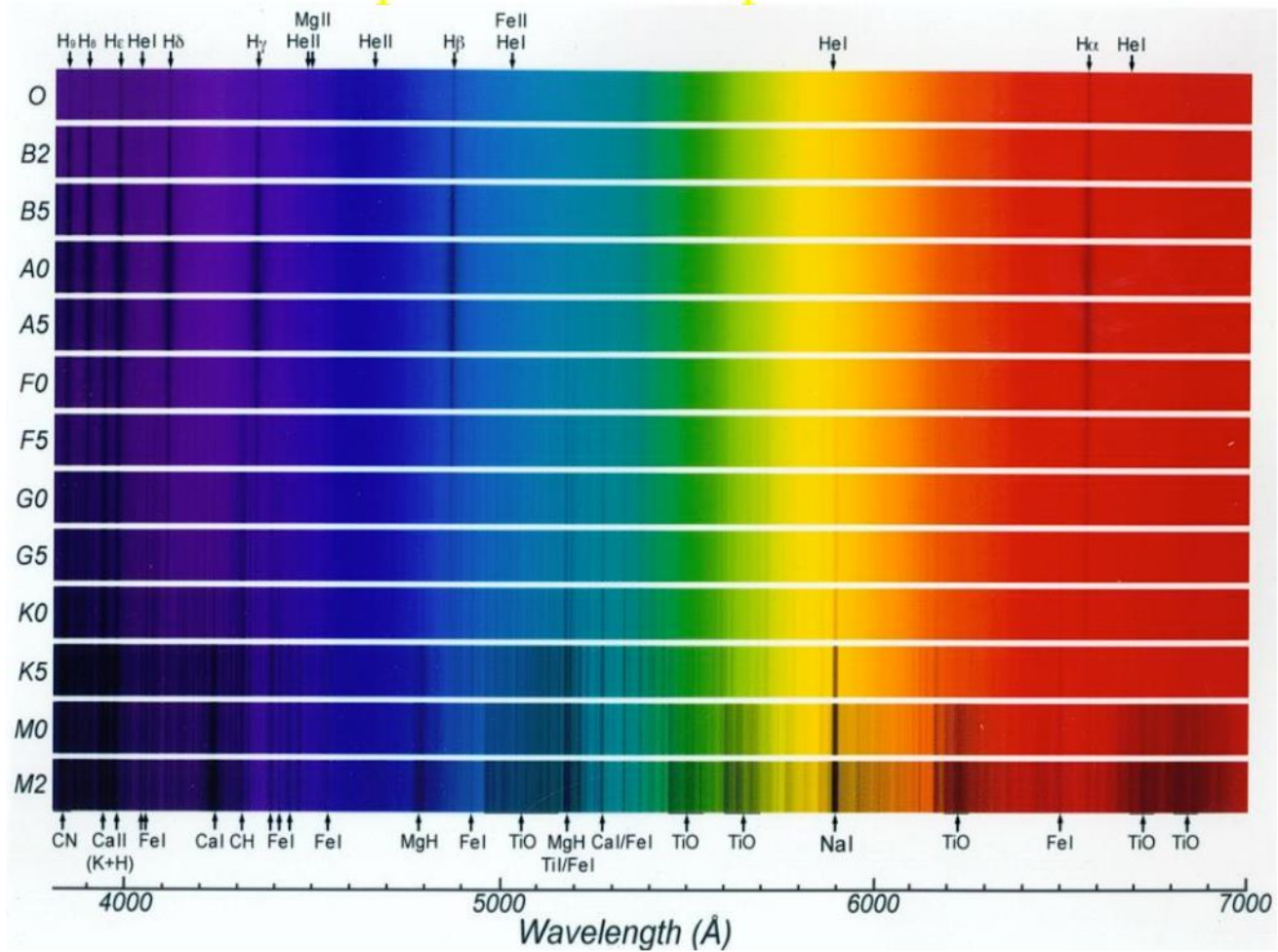


so why do the Balmer lines **diminish** in strength at higher temperatures?

We need Saha equation to answer this question.

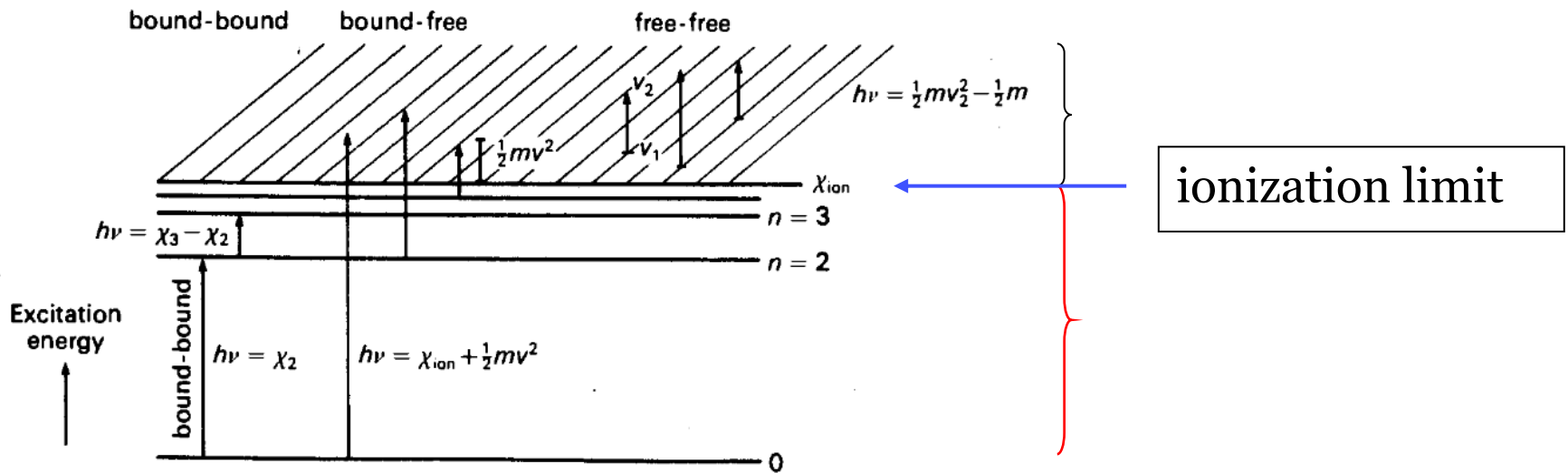
Balmer lines

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The degree of ionization of an atom

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The degree of ionization of any atom or ion can be obtained from [the Saha equation](#), which can be derived from [the Boltzmann formula](#) if we extend it to states with positive energies, i.e., to free electrons with the appropriate statistical weights (the [upper](#) state is now an ion plus free electron, with energy $\chi_{\text{ion}} + 1/2m_e v^2$).

The statistical weight of the ion in the ground state plus electron is the product of the statistical weight of the ion g_1^+ and the statistical weight of the electron g_e : $g_{\text{ion}+e} = g_1^+ g_e$

The Saha Equation

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The statistical weight of the ion in the ground state plus electron is the product of the statistical weight of the ion g_1^+ and the statistical weight of the electron g_e :

$$g_{\text{ion}+e} = g_1^+ g_e$$

The (differential) statistical weight of the electron, g_e , i.e. the number of available states in interval $(v, v+dv)$ is (from quantum mechanics)

$$g_e = \frac{1}{N_e} \frac{8\pi m_e^3 v^2 dv}{h^3}$$

The $1/N_e$ factor comes from the space volume element. It is the volume per electron.

Inserting this into Boltzmann's equation, we arrive at **the Saha equation**:

$$\frac{N_1^+}{N_1} = \frac{2g_1^+}{N_e g_1} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{\text{ion}}/kT}$$

This relates **the ground** state populations of the atom and ion.

The Saha Equation

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The Saha equation (Meghnad Saha 1920):

$$\frac{N_1^+}{N_1} = \frac{2g_1^+}{N_e g_1} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{ion}/kT}$$

This relates **the ground** state populations of the atom and ion.



To derive the ratio of the **total number of ions** (N^+) to the **total number of atoms** (N^0) we can use the conventional Boltzmann formula for each level n of the atom and ion, N_n/N_1 and N_n^+/N_1^+ i.e.:

$$\frac{N_n}{N_1} = \frac{g_n}{g_1} e^{-\chi_n/kT}$$

$$\frac{N_n^+}{N_1^+} = \frac{g_n^+}{g_1^+} e^{-\chi_n^+/kT}$$

Partition function (1)

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If N^0 is the sum of *all neutral* particles in their different quantum states:

$$N^0 = N_1^0 + \sum_{n=2}^{\infty} N_n^0 = N_1^0 + \frac{N_1^0}{g_1} \sum_{n=2}^{\infty} g_n e^{-\chi_n/kT}$$

We find:

$$N^0 = \frac{N_1^0}{g_1} \left(g_1 + \sum_{n=2}^{\infty} g_n e^{-\chi_n/kT} \right) = \frac{N_1^0}{g_1} u^0(T)$$

where we have introduced u^0 , the **partition function** of the atom. This is the weighted sum of the number of ways it can arrange its electrons with the same energy - e.g. all H is in the ground state for the Solar case, so $u^0 \approx 2$ (the ground state statistical weight). Similarly for the ion,

$$N^+ = N_1^+ + \frac{N_1^+}{g_1^+} u^+(T) \qquad u^+(T) = g_1^+ + \sum_{n=2}^{\infty} g_n^+ e^{-\chi_n^+/kT}$$

For H^+ , $u^+=1$, since no electrons left.

Partition function (2)

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If we multiply N_1^+/N_1^0 from earlier by N^+/N_1^+ and N_1^0/N^0 we again obtain the **Saha equation**:

$$\frac{N^+ N_e}{N^0} = \frac{2u^+}{u^0} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{ion}/kT} = 4.83 \times 10^{15} \frac{u^+}{u^0} T^{3/2} e^{-\chi_{ion}/kT}$$

In logarithmic form Saha equation can be written as:

$$\log \frac{N^+}{N^0} = \log \frac{u^+}{u^0} + \log 2 + \frac{5}{2} \log T - \chi_{ion} \Theta - \log P_e - 0.48$$

where χ_{ion} is measured in eV, $\Theta=5040/T$ and the electron pressure P_e is related to the electron density via the ideal gas law ($P_e=N_e kT$). In stellar atmospheres, P_e lies in the range 1 dyn/cm² (cool stars) to 1000 dyn/cm² (hot stars).

High temperature favours ionization, high pressure favours recombination.

Note that 1dyn/cm²=0.1N/m² (SI units), so for SI calculations the final constant is -1.48 instead of -0.48

Partition functions (Gray App D2)

Table D.2. *Partition functions, $\log u(T)$.*

	θ										$\log g_0$
	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6	1.8	2.0	
H	0.368	0.303	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301
He	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
He ⁺	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301	0.301
Li	–	0.987	0.488	0.359	0.320	0.308	0.304	0.302	0.302	0.302	0.301
Be	–	0.328	0.087	0.025	0.007	0.002	0.001	0.000	0.000	0.000	0.000
Be ⁺	0.541	0.334	0.307	0.302	0.301	0.301	0.301	0.301	0.301	0.301	0.301
B	1.191	0.831	0.786	0.778	0.777	0.777	0.777	0.777	0.777	0.776	0.778
B ⁺	0.435	0.051	0.006	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
C	1.163	1.037	0.994	0.975	0.964	0.958	0.954	0.951	0.950	0.948	0.954
C ⁺	0.853	0.782	0.775	0.774	0.773	0.772	0.771	0.770	0.769	0.767	0.778
C ⁺⁺	0.143	0.010	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
N	1.060	0.729	0.645	0.616	0.606	0.603	0.602	0.602	0.602	0.602	0.602
N ⁺	1.073	0.993	0.965	0.953	0.946	0.942	0.939	0.937	0.934	0.932	0.954
O	1.095	0.991	0.964	0.953	0.947	0.944	0.941	0.939	0.937	0.935	0.954
O ⁺	0.895	0.655	0.614	0.604	0.602	0.602	0.602	0.602	0.602	0.602	0.602
F	0.788	0.772	0.768	0.765	0.762	0.759	0.756	0.753	0.750	0.747	0.778
F ⁺	1.034	0.968	0.949	0.940	0.935	0.930	0.926	0.923	0.919	0.915	0.954
Ne	0.002	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Ne ⁺	0.771	0.766	0.760	0.754	0.748	0.743	0.737	0.732	0.727	0.723	0.778
Na	4.316	1.043	0.493	0.357	0.320	0.309	0.307	0.306	0.306	0.306	0.301
Na ⁺	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
Mg	2.839	0.478	0.110	0.027	0.007	0.002	0.001	0.001	0.001	0.000	0.000

$$\Theta = 5040/T$$

Partition functions (Gray, old edition)

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$$\log u(T) = c_0 + c_1 \log \Theta + c_2 \log^2 \Theta + c_3 \log^3 \Theta + c_4 \log^4 \Theta$$

$$\Theta = 5040/T$$

Element		c_0	c_1	c_2	c_3	c_4					
H	1	0.30103	-0.00001								
He	2	0.00000	0.00000								
He ⁺		0.30103	0.00000								
Li	3	0.31804	-0.20616	0.91456	-1.66121	1.04195					
Be	4	0.00801	-0.17135	0.62921	-0.58945						
Be ⁺		0.30389	-0.00819								
B	5	0.78028	-0.01622								
B ⁺		0.00349	-0.01035								
C	6	0.96752	-0.09452	0.08055							
C ⁺		0.77239	-0.02540								
N	7	0.60683	-0.08674	0.30565	-0.28114						
N ⁺		0.94968	-0.06463	-0.01291							
O	8	0.05033	-0.05703								
O ⁺		0.60405	-0.03025	0.04525							
F	9	0.76284	-0.03582	-0.05619							
Ne	10	0.00000	0.00000								
Ne ⁺		0.74847	-0.06562	-0.07088							
Na	11	0.30955	-0.17778	1.10594	-2.42847	1.70721					
Mg	12	0.00556	-0.12840	0.81506	-1.79635	1.26292					
Mg ⁺		0.30257	-0.00451								
Al	13	0.76786	-0.05207	0.14713	-0.21376						
Al ⁺		0.00334	-0.00995								
Si	14	0.97896	-0.19208	0.04753							
Si ⁺		0.75647	-0.05490	-0.10126							
P	15	0.64618	-0.31132	0.68633	-0.47505						
P ⁺		0.93588	-0.18848	0.08921	-0.22447						
S	16	0.95254	-0.15166	0.02340							
S ⁺		0.61971	-0.17465	0.48283	-0.39157						
Cl	17	0.74465	-0.07389	-0.06965							
Cl ⁺		0.92728	-0.15913	-0.01983							
K	19	0.34419	-0.48157	1.92563	-3.17826	1.83211					
Ca	20	0.07460	-0.75759	2.58494	-3.53170	1.65240					
Ca ⁺		0.34383	-0.41472	1.01550	0.31930						
Sc	21	1.08209	-0.77814	1.78504	-1.39179						
Sc ⁺		1.35894	-0.51812	0.15634							
Ti	22	1.47343	-0.97220	1.47986	-0.93275						
Ti ⁺		1.74561	-0.51230	0.27621							
V	23	1.68359	-0.82055	0.92361	-0.78342						
V ⁺		1.64112	-0.74045	0.49148							
Cr	24	1.02332	-1.02540	2.02181	-1.32723						
Cr ⁺		0.85381	-0.71166	2.18621	-0.97590	-2.72893					
Mn	25	0.80810	-0.39108	1.74756	-3.13517	1.93514					
Mn ⁺		0.88861	-0.36398	1.39674	-1.86424	-2.32389					
Fe	26	1.44701	-0.67040	1.01267	-0.81428						
Fe ⁺		1.63506	-0.47118	0.57918	-0.12293						
Co	27	1.52929	-0.71430	0.37210	-0.23278						
Ni	28	1.49063	-0.33662	0.08553	-0.19277						
Ni ⁺		1.03800	-0.69572	0.53893	0.28861						

Ionization Potentials

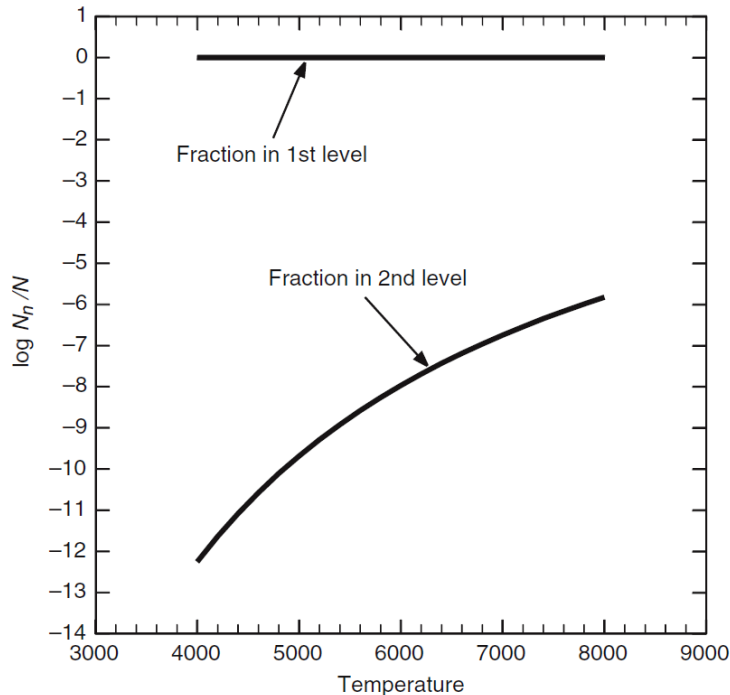
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Atom	Stage of ionization													
	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
1 H	13.598 44													
2 He	24.587 41	54.417 78												
3 Li	5.391 72	75.640 18	122.454											
4 Be	9.322 63	18.211 16	153.897	217.713										
5 B	8.298 03	25.154 84	37.931	259.366	340.22									
6 C	11.260 30	24.383 32	47.888	64.492	392.08	489.98								
7 N	14.534 14	29.601 3	47.449	77.472	97.89	552.06	667.03							
8 O	13.618 06	35.117 30	54.936	77.413	113.90	138.12	739.29	871.41						
9 F	17.422 82	34.970 82	62.708	87.140	114.24	157.17	185.19	953.91	1 103.1					
10 Ne	21.564 54	40.963 28	63.45	97.12	126.21	157.93	207.28	239.10	1 195.8	1 362.2				
11 Na	5.139 08	47.286 4	71.620	98.91	138.40	172.18	208.50	264.25	299.9	1 465.1	1 648.7			
12 Mg	7.646 24	15.035 28	80.144	109.265	141.27	186.76	225.02	265.96	328.1	367.5	1 761.8	1 963		
13 Al	5.985 77	18.828 56	28.448	119.99	153.83	190.49	241.76	284.66	330.1	398.8	442.0	2 086	2 304	
14 Si	8.151 69	16.345 85	33.493	45.142	166.77	205.27	246.49	303.54	351.1	401.4	476.4	523	2 438	2 673
15 P	10.486 69	19.769 4	30.203	51.444	65.03	220.42	263.57	309.60	372.1	424.4	479.5	561	612	2 817
16 S	10.360 01	23.337 9	34.79	47.222	72.59	88.05	280.95	328.75	379.6	447.5	504.8	564	652	707
17 Cl	12.967 64	23.814	39.61	53.465	67.8	97.03	114.20	348.28	400.1	455.6	529.3	592	657	750
18 Ar	15.759 62	27.629 67	40.74	59.81	75.02	91.01	124.32	143.46	422.5	478.7	539.0	618	686	756
19 K	4.340 66	31.63	45.806	60.91	82.66	99.4	117.56	154.88	175.8	503.8	564.7	629	715	787
20 Ca	6.113 16	11.871 72	50.913	67.27	84.50	108.78	127.2	147.24	188.5	211.3	591.9	657	727	818
21 Sc	6.561 44	12.799 67	24.757	73.489	91.65	111.68	138.0	158.1	180.0	225.2	249.8	688	757	831
22 Ti	6.828 2	13.575 5	27.492	43.267	99.30	119.53	140.8	170.4	192.1	215.9	265.1	292	788	863
23 V	6.746 3	14.66	29.311	46.71	65.28	128.1	150.6	173.4	205.8	230.5	255.1	308	336	896
24 Cr	6.766 64	16.485 7	30.96	49.16	69.46	90.64	161.18	184.7	209.3	244.4	270.7	298	355	384
25 Mn	7.434 02	15.639 99	33.668	51.2	72.4	95.6	119.20	194.5	221.8	248.3	286.0	314	344	404
26 Fe	7.902 4	16.187 8	30.652	54.8	75.0	99.1	124.98	151.06	233.6	262.1	290.2	331	361	392
27 Co	7.881 0	17.083	33.50	51.3	79.5	103	131	160	186.2	276.2	305	336	379	411
28 Ni	7.639 8	18.168 84	35.19	54.9	75.5	108	134	164	193	224.6	321	352	384	430
29 Cu	7.726 38	20.292 40	36.841	55.2	79.9	103	139	167	199	232	266	369	401	435
30 Zn	9.394 05	17.964 40	39.723	59.4	82.6	108	136	175	203	238	274	311	412	454

Degree of ionization of H in stars

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We can use the [Saha](#) equation to study the degree of ionization of H in general in stellar photospheres. The fraction of ionized hydrogen to the total is defined below. We find that H switches from **mostly neutral below 7000K** to **mostly ionized above 11000K** for typical N_e . This allows us to understand why hydrogen lines are strongest in A-type stars, with temperatures of 7500-10000K.



$$\frac{H^+}{H} = \frac{H^+}{H^0 + H^+} = \frac{H^+/H^0}{1 + H^+/H^0}$$

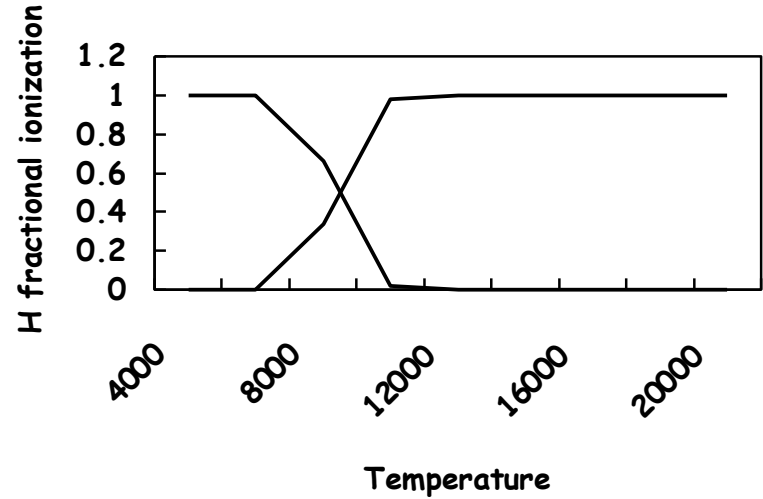
$$\frac{N^+ N_e}{N^0} = 2.4 \times 10^{15} T^{3/2} e^{-158000/T}$$

Using 1eV per particle, the hydrogen is heated from 0 to 10^4 K. Supplying 13.6 eV more, the temperature increases only up to 2×10^4 K. Ionization is an extremely energy consuming process. Ionization happens within a very small temperature interval.

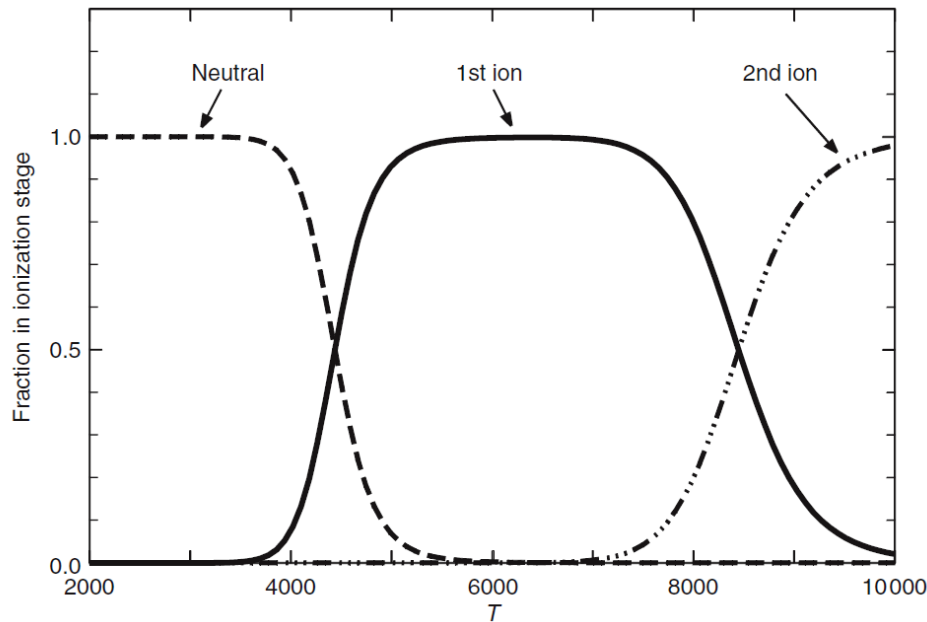
Degree of ionization in stars

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Hydrogen:



Iron:

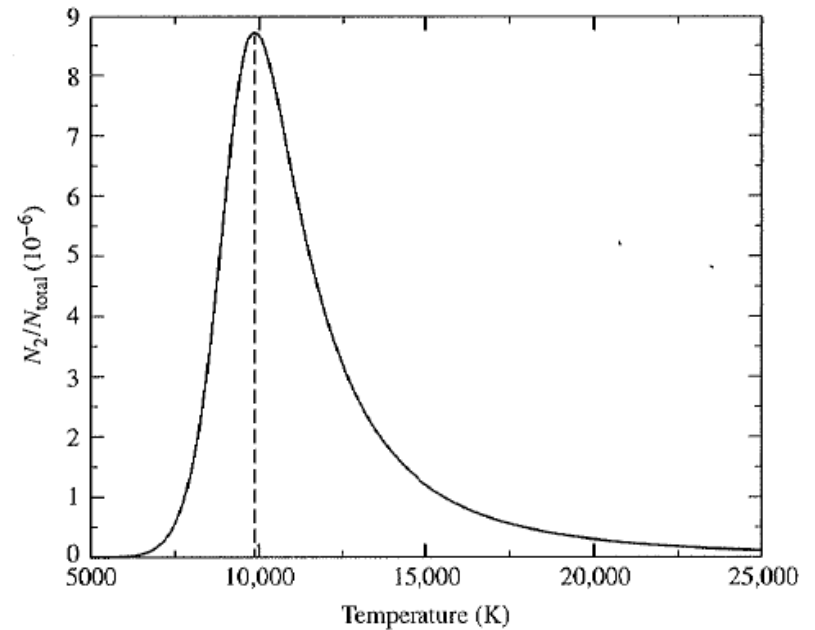


As temperature increases, ionization occurs rather abruptly. In a stellar photosphere, elements exist mainly in just two ionization stages.

Strong Balmer lines in A stars – why?

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From today's first example, a very high T was required to populate level $n=2$ of **H** relative to the ground state. We can now use the Boltzmann & Saha equations to measure $H(n=2)/H(\text{total})$ as a function of T . For increasing T , the $n=2$ population increases due to the Boltzmann equation, reaching a maximum value around 10,000K (equivalent to A spectral type) and **then reduces** as H becomes mostly ionized. This is why A stars have strong Balmer lines.



Note: **He** in stellar atmospheres complicates this calculation since ionized **He** provides **excess** electrons with which **H** ions can recombine, so it takes higher temperatures to achieve the same degree of ionization.

Strong lines in Solar photosphere

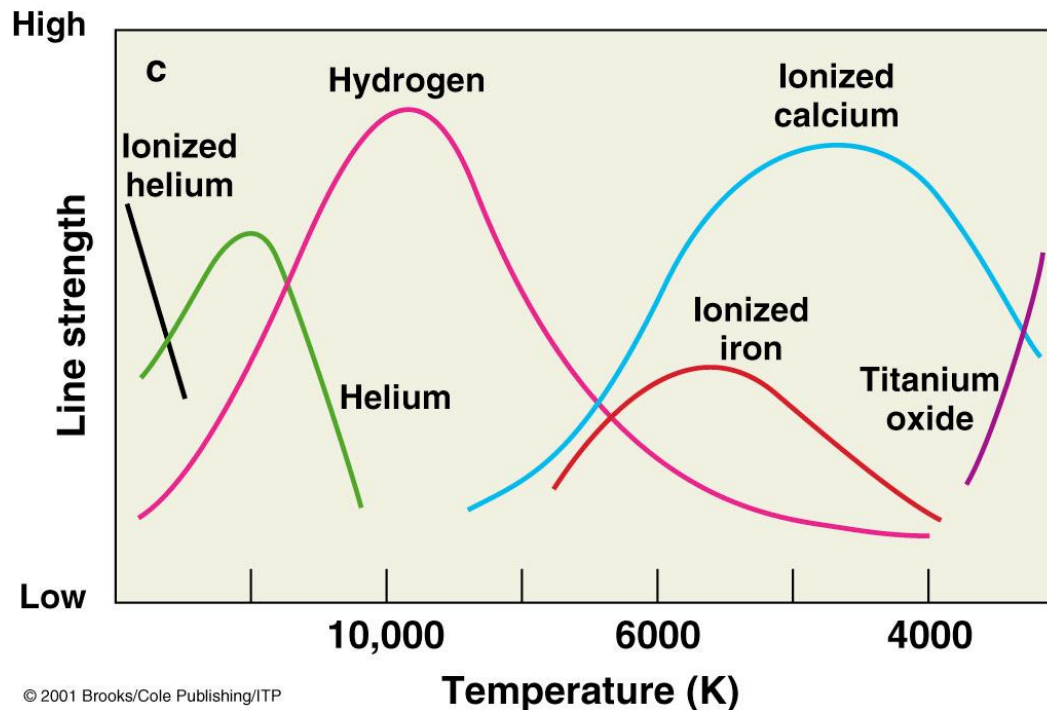
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λ	Element	$W(\text{\AA})$	Name	λ	Element	$W(\text{\AA})$	Name
3581.21	Fe I	2.14	N	4920.51	Fe I	0.43	
3719.95	Fe I	1.66		4957.61	Fe I	0.45	
3734.87	Fe I	3.03	M	5167.33	Mg I	0.65	b_4
3749.50	Fe I	1.91		5172.70	Mg I	1.26	b_2
3758.24	Fe I	1.65		5183.62	Mg I	1.58	b_1
3770.63	H ₁₁	1.86		5232.95	Fe I	0.35	
3797.90	H ₁₀	3.46		5269.55	Fe I	0.41	
3820.44	Fe I	1.71	L	5324.19	Fe I	0.32	
3825.89	Fe I	1.52		5238.05	Fe I	0.38	
3832.31	Mg I	1.68		5528.42	Mg I	0.29	
3835.39	H ₉	2.36		5889.97	Na I	0.63	D ₂
3838.30	Mg I	1.92		5895.94	Na I	0.56	D ₁
3859.92	Fe I	1.55		6122.23	Ca I	0.22	
3889.05	H ₈	2.35		6162.18	Ca I	0.22	
3933.68	Ca II	20.25	K	6562.81	H $_{\alpha}$	4.02	C
3968.49	Ca II	15.47	H	6867.19	O ₂	tell	B
4045.82	Fe I	1.17		7593.70	O ₂	tell	A
4101.75	H ₈	3.13	h	8194.84	Na I	0.30	
4226.74	Ca I	1.48	g	8498.06	Ca II	1.46	
4310 ± 10	—	—	G	8542.14	Ca II	3.67	
4340.48	H $_{\gamma}$	2.86		8662.17	Ca II	2.60	
4383.56	Fe I	1.01		8688.64	Fe I	0.27	
4861.34	H $_{\beta}$	3.68		8736.04	Mg I	0.29	
4891.50	Fe I	0.31					

Ca II in the Sun

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The photosphere of the Sun has only two calcium atoms for every million H atoms, yet the Ca II **H and K lines** (produced by the ground state of singly ionized calcium, Ca^+) are **stronger** than the Balmer lines of H (produced by the 1st excited state of neutral H). Why?



Saha-Boltzmann applied to Ca

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From the Saha equation we can find that H is essentially neutral in the Solar photosphere:

$P_e=200 \text{ dyn/cm}^2$, $\chi_{\text{ion}}=13.6 \text{ eV}$, $\Theta=5040/(T_{=5777})=0.872$, the partition function $u^0=2$, $u^+=1$ (i.e. $\log u^+=0$)

$$\log \frac{N^+}{N^0} = \log u^+ - \log u^0 + \log 2 + \frac{5}{2} \log T - \chi_{\text{ion}} \Theta - \log P_e - 0.48 = -5.235 \rightarrow N^+/N^0 \approx 0.0006\%$$

yet from the Boltzmann formula $\log \frac{N_u}{N_l} = \log \frac{g_u}{g_l} - \frac{5040}{T} \chi_{ul}(\text{eV})$: $H(n=2)/H(n=1)=5 \times 10^{-9}$

i.e. **very little** H is available to produce Balmer absorption lines.

For Ca, $\chi_{\text{ion}}=6.1 \text{ eV}$, and partition functions may be determined from tables (Slide 104) via

$$\log u(T) = c_0 + c_1 \log \Theta + c_2 \log^2 \Theta + c_3 \log^3 \Theta + c_4 \log^4 \Theta$$

For $\Theta=5040/T=0.872$, the partition function of neutral Ca

$$\log u^0(T) = 0.075 - 0.757 \log \Theta + 2.58 \log^2 \Theta + 3.53 \log^3 \Theta - 1.65 \log^4 \Theta$$

i.e. $u^0=1.3$. Similarly, $u^+=2.3$.

$$\log \frac{\text{Ca}^+}{\text{Ca}^0} = \log \frac{2.3}{1.3} + \log 2 + 9.40 - 5.34 - 1.18 - 0.48 = +2.95 \rightarrow \text{Ca}^+/\text{Ca}^0 \approx 900$$

Essentially **all** Calcium is singly ionized.

Saha-Boltzmann applied to Ca

Essentially **all** Calcium is singly ionized.

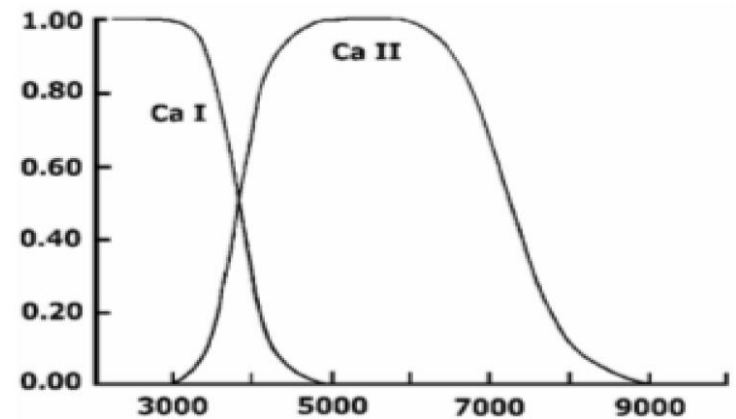
$N(\text{Ca}^+)$ in the first excited state relative to the ground state ($g_1=2$, $g_2=4$, $\chi=3.12\text{eV}$) is $1/265$ from Boltzmann eqn, so nearly all Calcium in the Sun's photosphere is in the ground state of Ca^+ .

Combining these results:

$$N(\text{Ca}^+_{\text{g.s.}})/N(\text{H}_{n=2}) = N(\text{Ca}^+_{\text{g.s.}})/N(\text{Ca}) \times N(\text{Ca})/N(\text{H}) \times N(\text{H})/N(\text{H}_{n=2}) = 400$$

There are **400 times** more Ca^+ ions with electrons in the ground-state (which produce the **Ca II H&K** lines) than there are neutral **H** atoms in the first excited state (which produce Balmer lines).

The Ca II lines in the Sun are so strong due to T dependence of excitation and ionization (**not** high **Ca/H** abundance).



More from Saha

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- Another observational effect that can be understood using the Saha equation is that **supergiants and giants have lower temperatures than dwarfs of the same spectral type.**
- Spectral classes are defined by line ratios of different ions, e.g. He II 4542A / He I 4471 for O stars. At higher temperatures the fraction of He II will increase relative to He I, so the above ratio will increase.
- However, supergiants have lower surface gravities (or pressure) than main-sequence stars, so from Saha equation a **lower P_e** at the same temperature will give a **higher ion fraction**, N^+/N^0
- Assuming a given spectral class corresponds to a fixed ratio N^+/N^0 , a star with a lower pressure can have a lower T_{eff} for the same ratio and spectral class

Summary

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- **LTE = Maxwell + Boltzmann + Saha.**
- **Boltzmann** equation describes degree of excitation of an atom or ion, e.g. $N(H_{n=2})/N(H_{n=1})$.
- **Saha** equation describes degree of ionization of successive ions, e.g. $N(He^+)/N(He^0)$ or $N(He^{2+})/N(He^+)$.
- The **Partition function** is the weighted sum of the number of ways an atom or ion can arrange its electrons with the same energy.
- Ionization is an extremely energy consuming process. Ionization happens within a very small temperature interval.
- **Saha-Boltzmann** explains the spectral type (or temperature) dependence of lines in stellar atmospheres, e.g. Strongest Balmer series at spectral type A and strong CaII lines in Solar-type stars.