Linear vs Quadratic source function

Up to now we assumed a linear source function. More generally, if:

$$S_\lambda(\tau_\lambda) = \sum_{n=0} a_{n\lambda}\, \tau_\lambda^n$$

Then

$$I_\lambda(0,\theta) = \sum_{n=0} A_n \cos^n \theta \qquad\qquad A_n = a_{n\lambda} \int_0^\infty u^n\, e^{-u} du = a_{n\lambda} n$$

We still get $S_\lambda(0)$ at the limb, but a more complicated result at the centre. For example, a quadratic term requires the solution of

$$S(\tau_\lambda) = a_{0\lambda} + a_{1\lambda}\tau_\lambda + a_{2\lambda}\tau_\lambda^2$$

$$I_\lambda(0,\theta) = a_{0\lambda} + a_{1\lambda} \cos\theta + 2a_{2\lambda} \cos^2 \theta$$

At $\theta = 90°$, $\tau_\lambda=0$, whilst at $\theta=0°$, $\tau_\lambda \sim 1+2a_{1\lambda}/a_{2\lambda}$ providing $a_{2\lambda} << a_{1\lambda}$. The ratio of the limb-to-centre intensity is

$$\boldsymbol{I_\lambda(0,90°)/I_\lambda(0,0°) = a_{0\lambda}/(a_{0\lambda} + a_{1\lambda} + 2a_{2\lambda})}$$

Example for Solar Case:

The measured centre to limb variation of the solar intensity is

$$I_\lambda(0,\theta)/I_\lambda(0,0) = a_{0\lambda} + a_{1\lambda}\cos\theta + 2a_{2\lambda}\cos^2\theta$$

λ(μm)	a_0	a_1	$2a_2$
0.3	0.06	0.74	0.20
0.4	0.14	0.91	-0.05
0.6	0.35	0.88	-0.23
0.8	0.49	0.73	-0.22
1.5	0.56	0.64	-0.20
2.0	0.70	0.48	-0.18

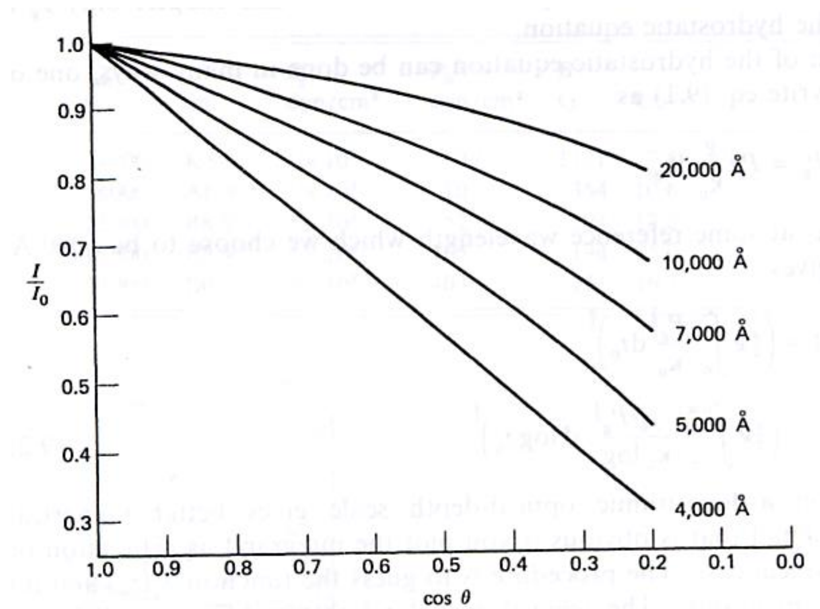
(Table 4.17, AQ 4th edition)

Wavelength dependence

Limb darkening is observed to be greatest at **shorter** wavelengths in the Sun. The temperature distribution of the upper atmosphere of the Sun can be obtained from limb darkening measurements, carried out via e.g. multi-filter images of the Solar continuum (between the lines).

Until recently, the Sun was the only star for which limb darkening was observed, since one needs to **spatially** resolve the disc (most other stars appear as point sources!) to measure limb darkening.

Other methods are now possible.



(Pierce & Waddell 1961).

Centre

Limb

Limb darkening for other stars

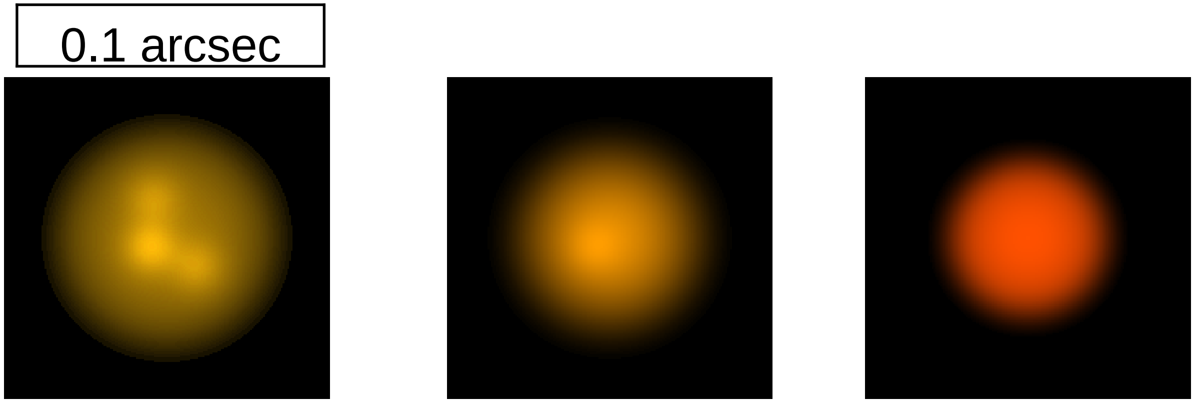
1. Direct interferometry, via high spatial resolution "imaging" – e.g. ESO/VLT interferometry or COAST array, providing a star is very large and nearby (a cool supergiant).

2. The light curve due to the gravitational micro-lensing of a background (generally Galactic bulge or Magellanic Cloud) star by a foreground source (e.g. PLANET team).

3. The light curve from an eclipsing binary system during secondary eclipse allows us to study limb darkening of the primary, although non-trivial! Similar approach followed by extra solar planets occulting parent star (e.g. HD209458).

Limb darkening from interferometry

COAST (Cambridge Optical Aperture Synthesis Telescope) spatial resolution of 20-30 milli-arcsec) has made limb darkening observations of M supergiant Betelgeuse at different wavelengths (using filters).



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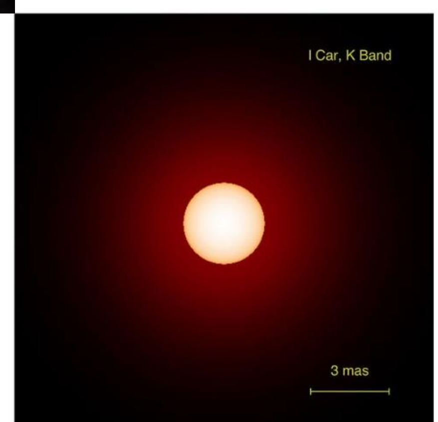
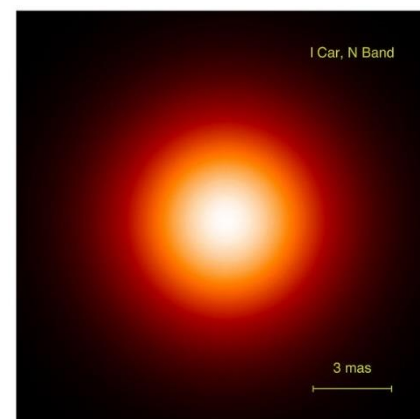
Limb darkening from interferometry

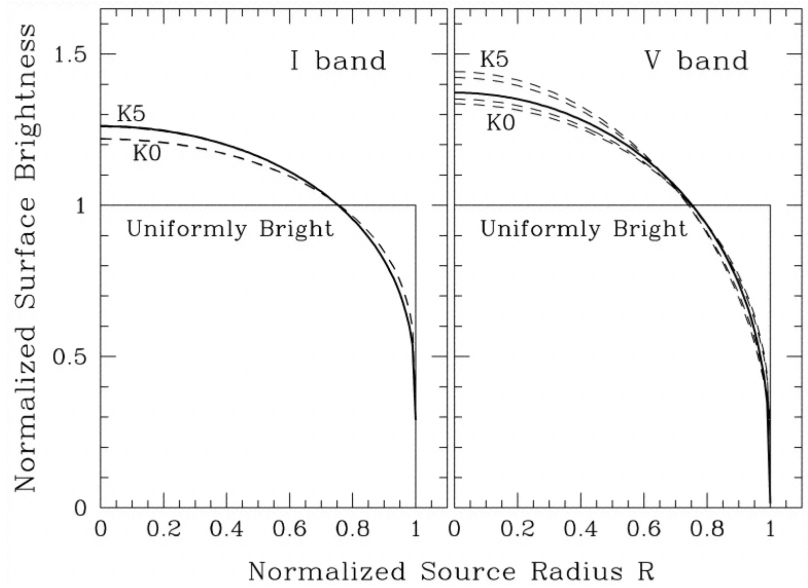
ESO's Very Large Telescope Interferometer (VLTI) is possible to achieve a resolution of 0.001 arcsec or even less. It has resolved the disc of the cepheid L Carinae.





Limb darkening from microlensing

- Galactic gravitational micro-lensing occurs when a foreground object (lens) passes in front of a background star (source). The gravitational deflection of light by the lens causes the flux from the source to be amplified.
- Microlensing surveys (e.g. PLANET, MACHO) have identified hundreds of such events towards the Galactic bulge and Magellanic Clouds.
- One such event, MACHO 97-BLG-28 was studied to reveal limb darkening information for the background K giant (Albrow et al. 1999).

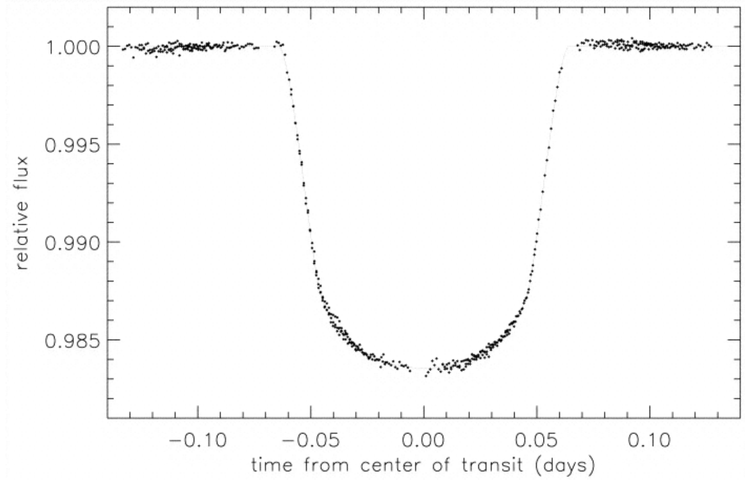


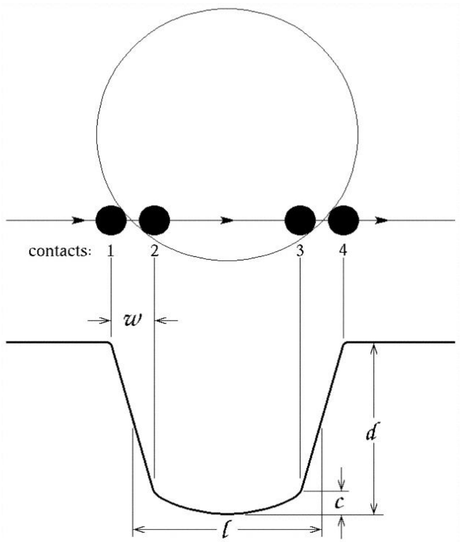
Thick lines show how much fainter the K giant becomes at its edges in the red I (left) and blue-green V filter (right). If the star emitted a uniform amount of light across its whole stellar disk, the profile would look like the straight solid black line instead

Limb darkening from eclipsing systems

- HD209458 is the first system in which extra-solar planet (P=3.5d, $0.6M_J$) has been observed to transit its (F8V) primary, allowing determination of limb darkening (Brown et al. 2001).

- More generally eclipsing binaries are problematic due to degeneracy with other parameters (Grygar et al. 1972). Accurate light curves needed for linear limb darkening parameters.





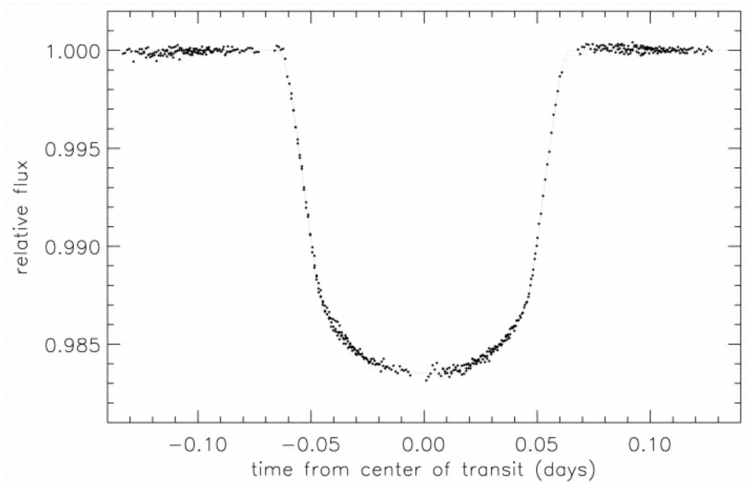
Limb darkening: current state

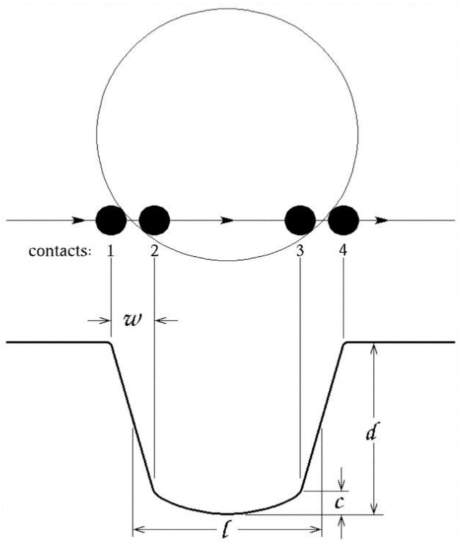
- Stars appear darker at their limbs than at their disk centers because at the limb we are viewing the higher and cooler layers of stellar photospheres.

- Limb darkening derived from state-of-the-art stellar atmosphere models systematically **fails** to reproduce recent transiting exoplanet light curves from the Kepler, TESS, and JWST telescopes – stellar brightness obtained from measurements drops less steeply towards the limb than predicted by models.

- Possible explanation: magnetic fields on the stellar surface are not taken into account:

  Kostogryz et al. (2024, NatAst):
  stellar atmosphere models computed with the use of a 3D radiative magneto-hydrodynamic code show that small-scale concentration of magnetic fields on the stellar surface affect limb darkening at a level allowing the authors to explain the observations.





Eddington-Barbier relation

FORMAL SOLUTION TO THE PLANE-PARALLEL TRANSFER EQUATION.

EDDINGTON-BARBIER RELATION.

GREY ATMOSPHERE.

Formal Solution to RTE (1)

The **plane-parallel** transfer equation
(for stars with thin photospheres)

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

The integrated form of the RTE is
[See D. Gray (page 127-129, 131) for more detail]:

$$I_\lambda(\tau_\lambda) = -\int_c^{\tau_\lambda} S_\lambda(t_\lambda) e^{-(t_\lambda - \tau_\lambda)\sec\theta} \sec\theta \, dt_\lambda$$

Here, the integration limit c (*which complicates the integral*), replaces $I_\nu(0)$ in the parallel-ray transfer equation (Lecture 5, slide 148):

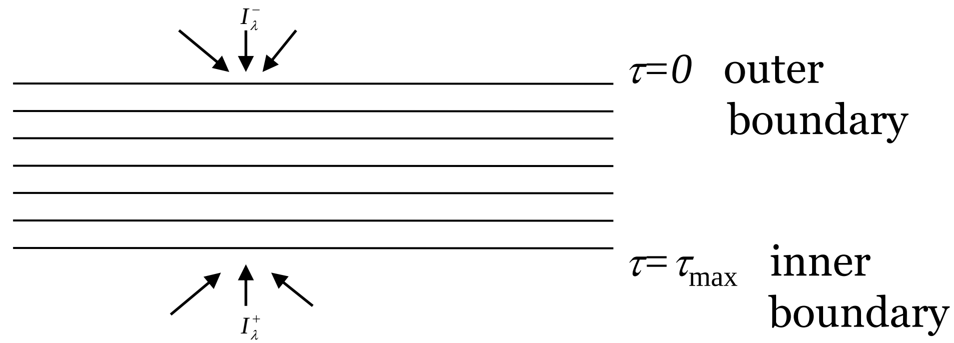
$$I_\lambda(\tau_\lambda) = \int_0^{\tau_\lambda} S_\lambda(t_\lambda) e^{-(\tau_\lambda - t_\lambda)} \, dt_\lambda + I_{\lambda 0} e^{-\tau_\lambda}$$

This is because the boundary conditions are different for radiation going in (θ>90°) and coming out (θ<90°) →

Formal Solution to RTE (2)

- The full intensity at the position τ_λ on the line of sight through the photosphere is

$$
\begin{aligned}
I_\nu(\tau_\nu) &= I_\nu^{\text{out}}(\tau_\nu) + I_\nu^{\text{in}}(\tau_\nu) \\
&= \int_{\tau_\nu}^{\infty} S_\nu \mathrm{e}^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, \mathrm{d}t_\nu \\
&\quad - \int_0^{\tau_\nu} S_\nu \mathrm{e}^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, \mathrm{d}t_\nu
\end{aligned}
$$



- An important special case occurs at the stellar surface. In this case

$$
\boxed{
\begin{aligned}
&I_\nu^{\text{in}}(0) = 0 \\
&I_\nu^{\text{out}}(0) = \int_0^{\infty} S_\nu \mathrm{e}^{-t_\nu \sec\theta} \sec\theta \, \mathrm{d}t_\nu
\end{aligned}
}
$$

where we assumed that the external radiation is **completely negligible** compared to the star's own radiation. **This Equation is the expression we need to compute the spectrum**.

- However, since the discs of most stars are spatially unresolved, we must deal with flux rather than intensity, so we will not deal with this equation any further.

Emergent Flux

From our lecture 6 (slide 160), the flux is [If there is no azimuthal (ϕ) dependence in I_λ]:

$$F = 2\pi \int_{-1}^{1} I(\mu)\,\mu\,d\mu \qquad\qquad \mu = \cos\theta$$

Netto = Outwards – Inwards.

Decomposition into two half-spaces:

$$F = 2\pi \int_0^1 I(\mu)\mu\,d\mu + 2\pi \int_{-1}^{0} I(\mu)\,\mu\,d\mu$$

$$= 2\pi \int_0^1 I(\mu)\,\mu\,d\mu - 2\pi \int_0^1 I(-\mu)\mu\,d\mu = \boldsymbol{F}^+ - \boldsymbol{F}^-$$

Eddington-Barbier relation

Special case: at the surface of a star $F^{-} = 0$, so that $F = F^{+}$

$$F_\lambda(0) = 2\pi \int_0^1 I_\lambda\,(0,\theta)\;\mu\, d\mu$$

From earlier, assuming a linear source function $S_\lambda(\tau_\lambda) = a_\lambda + b_\lambda \tau_\lambda$ yields

$$I_\lambda(0,\theta) = a_\lambda + b_\lambda \cos\theta = a_\lambda + b_\lambda \mu$$

In this case we obtain the "Eddington-Barbier" relation:

$$F_\lambda(0) = \pi(a_\lambda + 2/3\, b_\lambda) = \pi S_\lambda(\tau_\lambda = 2/3)$$

The emergent flux from the stellar surface is π times the Source function at an optical depth of 2/3

Grey atmosphere (1)

If we assume Local TE (LTE), then

$$F_\lambda(0) = \pi S_\lambda(\tau_\lambda = 2/3) = \pi B_\lambda[T(\tau_\lambda = 2/3)]$$

Let us assume the opacity is independent of λ, i.e. $\kappa_\lambda = \kappa$. We call such a (hypothetical) atmosphere a **grey atmosphere**. Then

$$F_\lambda(0) = \pi B_\lambda[T(\tau = 2/3)]$$

The energy distribution of F_λ is that of a blackbody corresponding to the temperature at the optical depth $\boldsymbol{\tau = 2/3}$.

The black body intensity is defined (following discovery by Max Planck in 1900) as either

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5} \frac{1}{e^{hc/\lambda kT} - 1} \qquad \text{or} \qquad B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

where c=2.99x10^{10} cm, h=6.57x20^{-27} erg s, k=1.38x10^{-16} erg/s.

Let's compute the Bolometric flux.

Bolometric flux of Black Body

$$B_\nu(T) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}$$

Note that: $B_\nu(T)\mathrm{d}\nu = B_\lambda(T)\mathrm{d}\lambda \Rightarrow B_\lambda = B_\nu\left|\frac{\mathrm{d}\nu}{\mathrm{d}\lambda}\right| = B_\nu \frac{c}{\lambda^2}$

Let us compute the bolometric flux:

$$F = \pi\int_0^\infty B_\nu(T)\mathrm{d}\nu = \pi\int_0^\infty \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/kT}-1}\mathrm{d}\nu = \pi\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\int_0^\infty \frac{x^3}{e^x-1}\mathrm{d}x = \pi\frac{2h}{c^2}\left(\frac{kT}{h}\right)^4\frac{\pi^4}{15} = \sigma_{SB}T^4$$

$$\sigma_{SB} = 2\frac{\pi^5k^4}{15c^2h^3} = 5.67\ 10^{-5}\ \text{erg cm}^{-2}\text{s}^{-1}\ \text{K}^{-4} - \text{ Stefan-Boltzmann constant}$$

Planck function is monotonic with temperature: $\boxed{\frac{\partial B_\nu(T)}{\partial T} = \frac{2h^2\nu^4}{c^2kT^2}\frac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2} > 0}$

$$F = \sigma T^4$$

Grey atmosphere (2)

If we assume Local TE (LTE), then

$$F_\lambda(0) = \pi S_\lambda(\tau_\lambda = 2/3) = \pi B_\lambda[T(\tau_\lambda = 2/3)]$$

Let us assume the opacity is independent of λ, i.e. $\kappa_\lambda = \kappa$. We call such a (hypothetical) atmosphere a grey atmosphere. Then

$$F_\lambda(0) = \pi B_\lambda[T(\tau = 2/3)]$$

The energy distribution of F_λ is that of a blackbody corresponding to the temperature at the optical depth $\boldsymbol{\tau = 2/3}$.

Thus, integrating over λ

$$F(0) = \int_0^\infty F_\lambda(0) d\lambda = \pi \int_0^\infty B_\lambda(T(\tau = 2/3)] d\lambda = \sigma T^4(\tau = 2/3)$$

From Stefan-Boltzmann, $F(0) = \sigma T_{eff}^4$, by definition, we find $T_{eff} = T(\tau = 2/3)$.
The "surface" of a star, which has temperature T_{eff} (by definition) is not at the very top of the atmosphere (where $\tau = 0$), but lies deeper down, at $\tau = 2/3$.

This can be considered as an *average* point of origin from the observed photons.

Summary

- Solution to plane-parallel transfer equation at surface explains limb darkening in Sun.
- Limb darkening in other stars can be estimated from interferometry, eclipsing binaries, microlensing.
- Eddington-Barbier relation.
- Grey atmosphere.
- Assuming a grey atmosphere , we found that the "surface" of a star, which has temperature T_{eff} (by definition) is not at the very top of the atmosphere (where $\tau = 0$), but lies deeper down, at $\tau = 2/3$.

Radiative Equilibrium

GREY ATMOSPHERE
THERMAL (RADIATIVE) EQUILIBRIUM
THE DEPTH DEPENDENCE OF THE SOURCE FUNCTION
EDDINGTON APPROXIMATION
TEMPERATURE STRUCTURE OF THE GREY ATMOSPHERE

Grey atmosphere

- Above we assumed that the opacity can be independent of λ, i.e. $\kappa_\lambda = \kappa$. We call such a (hypothetical) grey atmosphere.
- In the theory of stellar atmospheres, much of the technical effort goes into iteration schemes using equations of radiative equilibrium (which we will discuss today) to find the source function S_λ.
- Often, a starting point for such iterations is the **grey** case.

Thermal (radiative) equilibrium

- In stellar atmospheres, radiation dominates transfer of energy, so we can discuss (three) conditions of radiative equilibrium, which can be used to derive the temperature structure in the photosphere.

- The radiation we see from the Sun comes from a layer of geometrical height of a few hundred km.

- In a column of 100 km height and 1 cm^2 cross-section there are 10^{24} particles (since $n \sim 10^{17}/cm^3$ in Sun), each of which has a thermal energy of $3kT/2$ (10^{-12} erg). The total thermal energy of this column is therefore 10^{12} erg/cm^2. The observed radiative energy loss (per cm^2) of the solar surface is $F_\odot$=6.3x10^{10} erg cm^{-2} s^{-1}.

- If the Sun shines at a constant rate, the energy content of the solar photosphere can only last for 15 seconds without being replenished from below.

- Exactly the same amount of energy must be supplied or else the photosphere would quickly change temperature.

First equation of radiative equilibrium

- Since this does **not** happen, $\mathrm{d}F/\mathrm{d}t = 0$ or $\mathrm{d}F/\mathrm{d}x = 0$ or $\mathrm{d}F/\mathrm{d}\tau = 0$, i.e. the total flux must be constant at all depths of the photosphere (**conservation of energy**) – the **1st equation of radiative equilibrium**

$$F(x) = F(0) = const = \sigma T_{eff}^4$$

- When all the energy is carried by radiation, we have

$$F(x) = \int_0^\infty F_\lambda(\tau_\lambda) d\lambda = F(0)$$

Although the shape of F_λ can be expected to change very significantly with depth, its integral remains invariant.

- If other sources of energy transport are significant, then a more general expression of flux constancy must be applied:

$$\Phi(x) + \int_0^\infty F_\lambda(\tau_\lambda) d\lambda = F(0)$$

$\Phi(x)$ is, for example, the convective flux

Radiative equilibrium

- We may integrate the plane-parallel transfer equation over solid angle ω.

$$\int \cos\theta \frac{dI_\lambda(\tau_\lambda,\theta)}{d\tau_\lambda}\,d\omega = \int I_\lambda(\tau_\lambda,\theta)\,d\omega - \int S_\lambda(\tau_\lambda)\,d\omega$$

$$\frac{d}{d\tau_\lambda}[F_\lambda(\tau_\lambda)] = 4\pi[J_\lambda(\tau_\lambda)] - \int S_\lambda(\tau_\lambda)\,d\omega$$

Based on the definition of mean intensity and flux:

$$J_\lambda = \frac{1}{4\pi}\oint I_\lambda d\omega \quad \text{and} \quad F_\lambda = \oint I_\lambda \cos\theta\, d\omega$$

- Finally, assuming S_λ to be isotropic we obtain,

$$\frac{1}{4\pi}\frac{d}{d\tau_\lambda}[F_\lambda(\tau_\lambda)] = J_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

Second equation of radiative equilibrium

- In the **grey** case, for which the opacity κ is independent of wavelength

$$\frac{1}{4\pi}\frac{d}{d\tau}F(\tau) = -S(\tau) + J(\tau) = 0$$

Since $dF/d\tau=0$, the Source function must be equal the mean intensity J.

- If the atmosphere is **not** grey, which is the situation for most stars, let's incorporate the opacity κ into the RHS, and integrating over wavelength

$$\frac{1}{4\pi}\frac{d}{ds}\left[\int_0^\infty F(\tau_\lambda)d\lambda\right] = \int_0^\infty(-\kappa_\lambda S_\lambda + \kappa_\lambda J_\lambda)d\lambda = 0$$

$$\tau_\lambda = \int_0^S \kappa_\lambda \rho ds$$

Since dF/ds =0, we get the **radiative balance equation (energy conservation)**

$$\int_0^\infty \kappa_\lambda S_\lambda d\lambda = \int_0^\infty \kappa_\lambda J_\lambda d\lambda$$

- This is **the second equation of radiative equilibrium** and can be understood as the total energy absorbed (RHS) must equal the total energy re-emitted (LHS) if no heating or cooling is taking place.

Third equation of radiative equilibrium

The third radiative equilibrium condition is obtained by multiplying the transfer equation by $\cos\theta$ and integrating over solid angle and then wavelength

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

$$\oint \cos^2\theta \frac{dI_\lambda(\tau_\lambda,\theta)}{d\tau_\lambda} d\omega = \oint \cos\theta \, dI_\lambda(\tau_\lambda,\theta)\omega - \oint \cos\theta \, S_\lambda(\tau_\lambda,\theta) d\omega$$

$$K_\lambda(\tau_\lambda) = \frac{1}{4\pi}\oint I_\lambda \cos^2\theta \, d\omega \qquad F_\lambda = \oint I_\lambda \cos\theta \, d\omega$$

0 (S_λ is isotropic)

$$4\pi \int \frac{dK_\lambda}{d\tau_\lambda} d\lambda = \int F_\lambda \, d\lambda = F(\tau)$$

The third radiative equilibrium condition:

$$\boxed{\int_0^\infty \frac{dK_\lambda}{d\tau_\lambda} d\lambda = \frac{F(\tau)}{4\pi}}$$

Equations of radiative equilibrium

- All the three radiative equilibrium conditions are not independent. S_λ that is a solution of one will be the solution of all three.

- The flux constant $F(0)$ is often expressed in terms of an effective temperature $F(0) = \sigma T_{eff}^4$.

- When model photospheres are constructed using flux constancy as a condition to be fulfilled by the model, the effective temperature becomes one of the fundamental parameters characterizing the model.

- In real stars, energy is created or lost from the radiation field through e.g. convection, magnetic fields,
  plus in supernovae atmospheres energy conservation is not valid (radioactive decay of Ni to Fe),
  so the energy constraints are more complicated in reality.