Solution of the Equations of Stellar Structure

THE STELLAR STRUCTURE EQUATIONS AND HOW TO SOLVE THEM?

SIMPLE STELLAR MODELS

POLYTROPIC MODELS

LANE-EMDEN EQUATION

RELATIONSHIPS FOR POLYTROPIC STARS

CHANDRASEKHAR MASS

DYNAMICAL STABILITY OF STARS

Introduction

We have now derived all the four differential equations and the three additional functions that, together with boundary conditions, define uniquely the equilibrium properties of a star of a given mass and composition.

- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$
- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$
- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$
- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma r^2}\frac{\rho(r)\kappa_R(r)}{T^3(r)}L(r)$ — by radiation
- $\frac{P}{T}\frac{dT}{dP} = \frac{\gamma - 1}{\gamma}$ — by convection

Energy transport: by radiation / by convection

- r = radius
- P = pressure at r
- m = mass of material within r
- ρ = density at r
- L = luminosity at r
- T = temperature at r
- κ_R = Rosseland mean opacity at r
- ε = energy release

Three supplement equations:

$P = P\,(\rho, T,$ chemical composition) — **EOS**

$\kappa_R = \kappa_R\,(\rho, T,$ chemical composition)

$\varepsilon = \varepsilon\,(\rho, T,$ chemical composition)

Plus, the equation of composition changes:

$$\frac{dX_i}{dt} = A_i \frac{m_H}{\rho}\left(-\sum_j (1 + \delta_{ij}) r_{ij} + \sum_{k,l} r_{kl,i}\right)$$

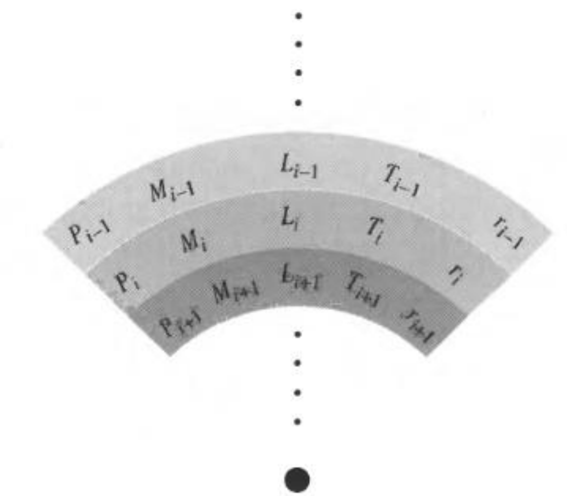
How to solve the Stellar Structure equations?

- "Solving" this system of coupled equations means finding the functions $P(r)$, $T(r)$, and $\rho(r)$, which are the ones that are usually considered to describe the structure of the star.
- **The Vogt-Russell theorem:**
  "The mass and the composition structure throughout a star **uniquely** determine its radius, luminosity, and internal structure, as well as its subsequent evolution."
  - This "theorem" has not been proven and is not even rigorously true; there are known exceptions. However, an actual star would probably adopt one unique structure as a consequence of its evolutionary history. In this sense, the Vogt-Russell "theorem" should be considered a general rule rather than a rigorous law.
- Unfortunately, unless some unrealistic assumptions are made, there is no analytic solution to the equations, given the complicated nature of the functions P, κ, and ε when all relevant processes are included. Because the complete set of equations with two-point boundary values is highly non-linear and time-dependent, their full solution requires a complicated numerical procedure. This is what is done in detailed stellar evolution codes, the results of which we will discuss in the following few lectures.

Numerical modeling of the equations

We will not go into any detail about the numerical methods commonly used in such codes, but shortly they can be described as the following:

- In a numerical solution, the differentials in the equations are replaced by differences. For instance, by replacing dP/dr by $\Delta P/\Delta r$. The star is then imagined to be constructed of spherically symmetric shells.
- The numerical integration of the stellar structure equations may be carried out shell by shell from the surface toward the center, from the center toward the surface, or, as is often done, in both directions simultaneously.
- If the integration is carried out in both directions, the solutions will meet at some fitting point where the variables must vary smoothly from one solution to the other.
- Simultaneously matching the surface and central boundary conditions for a desired stellar model usually requires several iterations before a satisfactory solution is obtained. If the surface-to-center and center-to-surface integrations do not agree at the fitting point, the starting conditions must be changed. The initial conditions of the next integration are estimated from the outcome of the previous integration.



Simple stellar models

- Although today the standard approach is to hand the problem to a computer, insight into the structure of stars may be gained both by analyzing the equations, without actually solving them, and by seeking simple solutions based on additional simplifying assumptions.
- The main purpose of this lecture is to briefly analyze the differential equations of stellar evolution and their boundary conditions, and to see how the full set of equations can be simplified in some cases to allow simple or approximate solutions – so-called simple stellar models. We will concentrate on polytropic models.
- Although nowadays their practical use has mostly been superseded by more realistic stellar models, due to their simplicity polytropic models still give useful insight into several important properties of stars. Moreover, in some cases the polytropic relation is a good approximation to the real equation of state
- As the very first simplification, we assume that a star is in both **hydrostatic** and **thermal** equilibrium. In this case, the four partial differential equations for stellar structure reduce to ordinary, time-independent differential equations.

Simplifying assumptions

- The four equations of stellar structure divide naturally into two groups:
  - one describing the mechanical structure of the star
  - and the other giving the thermal structure.

$$\frac{dm}{dr} = 4\pi r^2 \rho(r); \frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r); \frac{P}{T}\frac{dT}{dP} = \frac{\gamma-1}{\gamma}$$

- However, the only contact between the mechanical variables and thermal equations is through the temperature dependence of the equation of state.
- If we can write the pressure in terms of the density alone, without reference to the temperature, then we can separate these two equations from the others and solve them by themselves. Solving two differential equations (plus one algebraic equation relating P and ρ is much easier than solving seven equations.
- We have already seen, that under certain circumstances, the pressure can indeed become independent of temperature, and only depend on density, i.e., degeneracy pressure, or the case where pressure and density are related adiabatically (convection).
- In the above examples we derived a relation of the form $P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}}$ where K and γ are constants; this is called a polytropic relation, and the resulting models are called polytropic models.

Polytropic models

When the equation of state can be written in this form, the temperature does not enter at all into the equations and the calculations of stellar structure simplify enormously. There are even analytical solutions for certain values of γ.

- If we then take the equation for hydrostatic support, multiply it by r^2/ρ, differentiate with respect to r, and then divide by r^2, we get:

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho \,;\, \frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{r^2}{\rho}\frac{dP}{dr} = -Gm \quad \rightarrow \quad \frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr}$$

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

$$\frac{1}{r^2}\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -4\pi G\rho$$

What we have done is exact. Now we make our approximation. We approximate that the pressure and density are related by a power-law $P = K\rho^{\gamma} = K\rho^{1+\frac{1}{n}}$ (it customary to adopt γ=1+1/n, or n=1/(1- γ), where n is the polytropic index):

$$\frac{K(n+1)}{r^2 n}\frac{d}{dr}\left(\frac{r^2}{\rho}\rho^{1/n}\frac{d\rho}{dr}\right) = -4\pi G\rho$$

Lane-Emden equation

$$\frac{K(n+1)}{4\pi G n}\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho^{1/n-1}\frac{d\rho}{dr}\right) = -\rho$$

The solution $\rho(r)$ for $0 \le r \le R$ is called a polytrope and requires two boundary conditions.

Hence a polytrope is uniquely defined by three parameters: K, n, and R. This enables calculation of additional quantities as a function of radius, such as pressure, mass or gravitational acceleration.

Let's define a dimensionless variable θ in the range $0 \le \theta \le 1$ by $\rho=\rho_c\theta^n$, where ρ_c is the central density. Then the equation becomes

$$\frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\theta}{dr}\right) = -\theta^n$$

To simplify the equation further, we introduce the dimensionless radius $\xi=r/\alpha$, where

$$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}$$

constant having the dimension of length squared!

The equation finally becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

This equation is called the *Lane-Emden equation*, and the solution $\theta= \theta_n(\xi)$ is called the *Lane-Emden function*.

Solving the Lane-Emden equation

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n$$

Since it is a second order differential equation, we need two boundary conditions.
The 1st is at the center: from spherical symmetry, the pressure gradient at the center ($\theta = 1$) must be zero.
The 2nd condition comes from the surface, $\xi = \xi_1$, where the density should go to zero.
So, our boundary conditions are $\frac{d\theta}{d\xi} = 0,\ \theta = 1$ at $\xi=0$ (the center), and $\theta = 0$ at $\xi = \xi_1$ (the surface).

Solving the equation for the dimensionless function $\theta_n(\xi)$ in terms of ξ for a specific polytropic index n leads directly to the profile of density with radius $\rho_n(r)$. The polytropic equation of state provides the pressure profile. In addition, if the ideal gas law and radiation pressure are assumed for constant composition, then the temperature profile, $T(r)$, is also obtained.

$$P = \frac{\Re T\rho}{\mu} + \frac{aT^4}{3}$$

Unfortunately, the Lane-Emden equation does not have an analytic solution for arbitrary values of n.
In fact, there are only three analytic solutions, namely n= 0, 1, and 5:

$$n = 0, \theta = 1 - \frac{\xi^2}{6} \qquad \xi_1 = \sqrt{6}$$

$$n = 1, \theta = \frac{\sin\xi}{\xi} \qquad \xi_1 = \pi$$

$$n = 5, \theta = \left(1 + \frac{\xi^2}{3}\right)^{-1/2} \qquad \xi_1 \to \infty$$

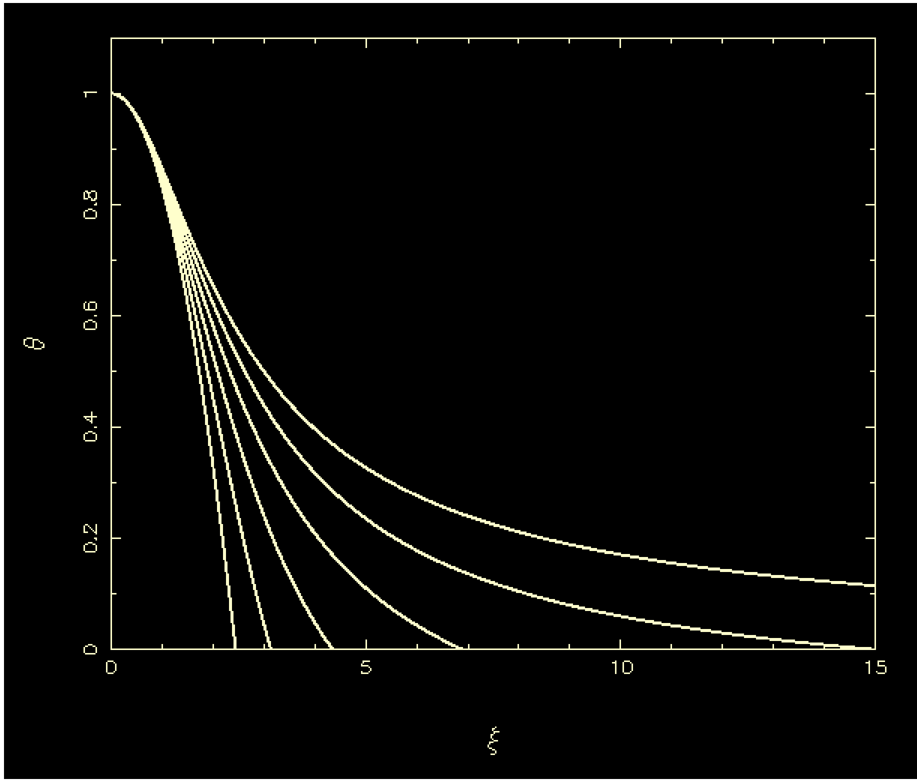
Solutions for all other values of n must be solved numerically.

1. Solve the Eqn for n=0

2. Find the dimensionless radius of these polytropic stars

Solutions of the Lane-Emden equation

Numerical solutions to the Lane-Emden equation for (left-to-right) n = 0,1,2,3,4,5. Some key values resulting from the integration are shown in the table.



n	ξ_1	$-\xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}$	$\rho_c/\bar{\rho}$
0.0	2.4494	4.8988	1.0000
0.5	2.7528	3.7871	1.8361
1.0	3.14159	3.14159	3.28987
1.5	3.65375	2.71406	5.99071
2.0	4.35287	2.41105	11.40254
2.5	5.35528	2.18720	23.40646
3.0	6.89685	2.01824	54.1825
3.25	8.01894	1.94980	88.153
3.5	9.53581	1.89056	152.884
4.0	14.97155	1.79723	622.408
4.5	31.83646	1.73780	6189.47
4.9	169.47	1.7355	934800.
5.0	∞	1.73205	∞

Solutions decrease monotonically and have θ=0 at $\xi = \xi_1$ (i.e. the stellar radius).
With increasing polytropic index, the star becomes more centrally condensed.

Polytropic stars

$\xi = r/\alpha$

$\rho = \rho_c \theta^n$

$$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}$$

The surface radius of the polytropic model is

$$R = \left(\frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}\right)^{1/2} \xi_1$$

The total mass M of a polytropic star is given by

$$M = \int_0^R 4\pi r^2 \rho dr = 4\pi\alpha^3\rho_c \int_0^{\xi_1} \xi^2\theta^n d\xi = -4\pi\alpha^3\rho_c \int_0^{\xi_1} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi = -4\pi\alpha^3\rho_c \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1}$$

From a polytropic model, we can derive other useful numbers and relationships.
As one example, it is often convenient to know how centrally concentrated a star is, i.e. how much larger its central density is than its mean density. We define this quantity as

$$D_N \equiv \frac{\rho_c}{\bar{\rho}} = \frac{\rho_c 4\pi R^3}{3M} = \frac{4\pi}{3}\rho_c(\alpha\xi_1)^3 \left[-4\pi\alpha^3\rho_c\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{-1} = \left[-\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\right]^{-1}$$

Values in Table in slide 263

Mass-Radius relationship for polytropic stars

Another useful relationship is between mass and radius. We start by expressing the central density ρ_c in terms of the other constants and our length scale α:

$$\rho_c = \left[\frac{K(n+1)}{4\pi G\alpha^2}\right]^{n/(n-1)}$$

$$\alpha^2 = \frac{K(n+1)\rho_c^{1/n-1}}{4\pi G}$$

Substitute this into the equation for the mass:

$$M = -4\pi\alpha^3\rho_c\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1} = -4\pi\alpha^3\left[\frac{K(n+1)}{4\pi G\alpha^2}\right]^{n/(n-1)}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}$$

Making the substitution $\alpha = R/\xi_1$ and re-arranging, we arrive at

$$\left[\frac{GM}{-\xi_1^2(d\theta/d\xi)_{\xi_1}}\right]^{(n-1)}\left(\frac{R}{\xi_1}\right)^{3-n} = \frac{[K(n+1)]^n}{4\pi G}$$

n = 1 is a special case, for which the radius is independent of mass and is uniquely determined by K:

$$R = \xi_1\left(\frac{K}{2\pi G}\right)^{1/2}$$

Another important polytropic index is n=3, for which the R dependence disappears.
We find that

$$M = -\frac{4}{\sqrt{\pi}}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\left(\frac{K}{G}\right)^{3/2}$$

M-R relationship for polytropic stars

For other n, mass and radius are related by $M \sim R^{(n\text{-}3)/(n\text{-}1)}$.

Note what this means: for a polytropic index of n=1.5 (the $\gamma = 5/3$ case), $R \sim M^{-1/3}$. Thus, for a set of stars with the same K and n (i.e., white dwarfs), the stellar radius is inversely proportional to the mass. We will use it a few slides later.

Eddington standard model

Another important polytropic index is n=3, "Eddington standard model" associated with a star in radiative equilibrium. The contribution to the total pressure at a certain location in the star due to an ideal gas is given by

$$P = \frac{\Re T\rho}{\mu} + \frac{aT^4}{3}$$

$$P_g = \frac{\Re T\rho}{\mu} = \beta P$$

$$\Re = \frac{k}{m_p}$$

Then the contribution due to radiation pressure is

$$P_r = \frac{aT^4}{3} = (1-\beta)P$$

Combining both equations to eliminate T we get:

$$P^3 = \frac{3(1-\beta)}{a}\left(\frac{\Re\rho}{\mu\beta}\right)^4$$

This leads immediately to an expression for the total pressure in terms of the density, namely

$$P = K\rho^{\gamma} \quad \text{where} \quad K \equiv \left[\frac{3(1-\beta)}{a}\right]^{1/3}\left(\frac{\Re}{\mu\beta}\right)^{4/3}, \qquad \gamma = 4/3, \qquad \text{and} \quad n = 3$$

Thus, we have obtained a polytropic equation of state of index 3, which implies a unique relation between K and M. The *Eddington quartic equation* is

$$1-\beta = 0.003\left(\frac{M}{M_{sun}}\right)^2 \mu^4\beta^4$$

The central pressure in polytropic stars

Another important relation is obtained between the central pressure and the central density.
Substitute K from the mass-radius relation:

$$P_c = K\rho_c^{1+\frac{1}{n}} \quad \rightarrow \quad \left[\frac{GM}{-\xi_1^2(d\theta/d\xi)_{\xi_1}}\right]^{(n-1)} \left(\frac{R}{\xi_1}\right)^{3-n} = \frac{[K(n+1)]^n}{4\pi G}$$

We obtain

$$P_c = \frac{(4\pi G)^{1/n}}{(n+1)} \left[\frac{GM}{M_n}\right]^{\frac{n-1}{n}} \left(\frac{R}{R_n}\right)^{\frac{3-n}{n}} \rho_c^{\frac{n+1}{n}}$$

where $M_n = -\xi_1^2(d\theta/d\xi)_{\xi_1}$ and $R_n = \xi_1$.

Now eliminating R, using $D_N \equiv \frac{\rho_c}{\bar{\rho}} = \frac{\rho_c 4\pi R^3}{3M}$, and assembling all n-dependent coefficients into one constant B_n, we get

$$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3}$$

n	D_n	M_n	R_n	B_n
1.0	3.290	3.14	3.14	0.233
1.5	5.991	2.71	3.65	0.206
2.0	11.40	2.41	4.35	0.185
2.5	23.41	2.19	5.36	0.170
3.0	54.18	2.02	6.90	0.157
3.5	152.9	1.89	9.54	0.145

The remarkable property of this relation is that it depends on the polytropic equation of state only through the value of B_n, which varies very slowly with n.

It therefore constitutes an almost universal relation!

The degeneracy pressure in polytropic stars

As a star contracts, the density may become so high that the electrons become degenerate and exert a (much) higher pressure than they would if they behaved classically. Stars that are so compact and dense that their interior pressure is dominated by degenerate electrons are known observationally as white dwarfs. They are the remnants of stellar cores in which hydrogen has been completely converted into helium. In most cases, also helium has been fused into carbon and oxygen.

We discussed the degeneracy pressure in Lecture 8. Let's now add a bit more detail.

The pressure of a completely degenerate electron gas in the non-relativistic limit is

$$P_e = K_{NR}\left(\frac{\rho}{\mu_e}\right)^{5/3} \quad \text{with} \quad K_{NR} = \frac{h^2}{20 m_e m_H^{5/3}}\left(\frac{3}{\pi}\right)^{2/3} = 1.0036 \times 10^{13}\ [\text{cgs}]$$

This corresponds to a polytropic relation with n=1.5 (the $\gamma = 5/3$ case). Since in the limit of strong degeneracy the pressure no longer depends on the **temperature**, this degeneracy pressure can hold the star up against gravity, regardless of the temperature. Therefore, a degenerate star does **not** have to be **hot** to be in hydrostatic equilibrium, and it can remain in this state forever even when it cools down. This is the situation in white dwarfs.

A few slides ago we obtained that for n=1.5, $R \sim M^{-1/3}$, i.e. the stellar radius is inversely proportional to the mass.

The relativistic degeneracy in polytropic stars

More massive white dwarfs are thus more compact, and therefore have a higher density. Above a certain density the electrons will become relativistic as they are pushed up to higher momenta by the Pauli exclusion principle. The degree of relativity increases with density, and therefore with the mass of the white dwarf, until at a certain mass all the electrons become extremely relativistic, i.e., their speed $v_e \rightarrow c$. In this limit the equation of state has changed to (the pressure increases less steeply with density)

$$P_e = K_{ER}\left(\frac{\rho}{\mu_e}\right)^{4/3} \quad \text{with} \quad K_{ER} = \frac{hc}{8m_H^{4/3}}\left(\frac{3}{\pi}\right)^{1/3} = 1.2435 \times 10^{15}\ [\text{cgs}]$$

which is also a polytropic relation but with $n = 3$.

We have already seen above that an $n=3$ polytrope is special in the sense that it has a unique mass, which is determined by K and is independent of the radius:

$$M = -\frac{4}{\sqrt{\pi}}\xi_1^2\left(\frac{d\theta}{d\xi}\right)_{\xi_1}\left(\frac{K}{G}\right)^{3/2}$$

This value corresponds to an upper limit to the mass of a gas sphere in hydrostatic equilibrium that can be supported by degenerate electrons, and thus to the **maximum possible mass** for a white dwarf. Its existence was first found by Subrahmanyan Chandrasekhar in 1931, after whom this limiting mass was named.

Chandrasekhar mass

2.01824

$K_{ER} = 1.2435 \times 10^{15}$ [cgs]

A relativistic electron gas has $K = K_{ER}/\mu_e^{4/3}$

Substituting it and other proper numerical values into

$$M = -\frac{4}{\sqrt{\pi}} \xi_1^2 \left(\frac{d\theta}{d\xi}\right)_{\xi_1} \left(\frac{K}{G}\right)^{3/2}$$

we obtain the Chandrasekhar mass

$$M = M_{Ch} = \frac{5.826}{\mu_e^2} M_\odot$$

Thus, for a highly relativistic electron gas, there is only a **single** possible mass which can be in hydrostatic equilibrium.

White dwarfs are typically formed of helium, carbon or oxygen, for which $\mu_e = 2$ and therefore $M_{Ch} = 1.456\ M_\odot$.

This quantity is called the Chandresekhar mass, after Subrahmanyan Chandresekhar, who first derived it. He did the calculation while on his first trip out of India, to start graduate school at Cambridge at age 20… This work earned Chandrasekhar the 1983 Nobel Prize for Physics (which he shared with Fowler for their contributions to the understanding of stellar evolution).

A further increase of the mass (e.g., due to accretion from a companion star) leads to the loss of stability and collapse. This is the cause of supernovae type Ia explosions.

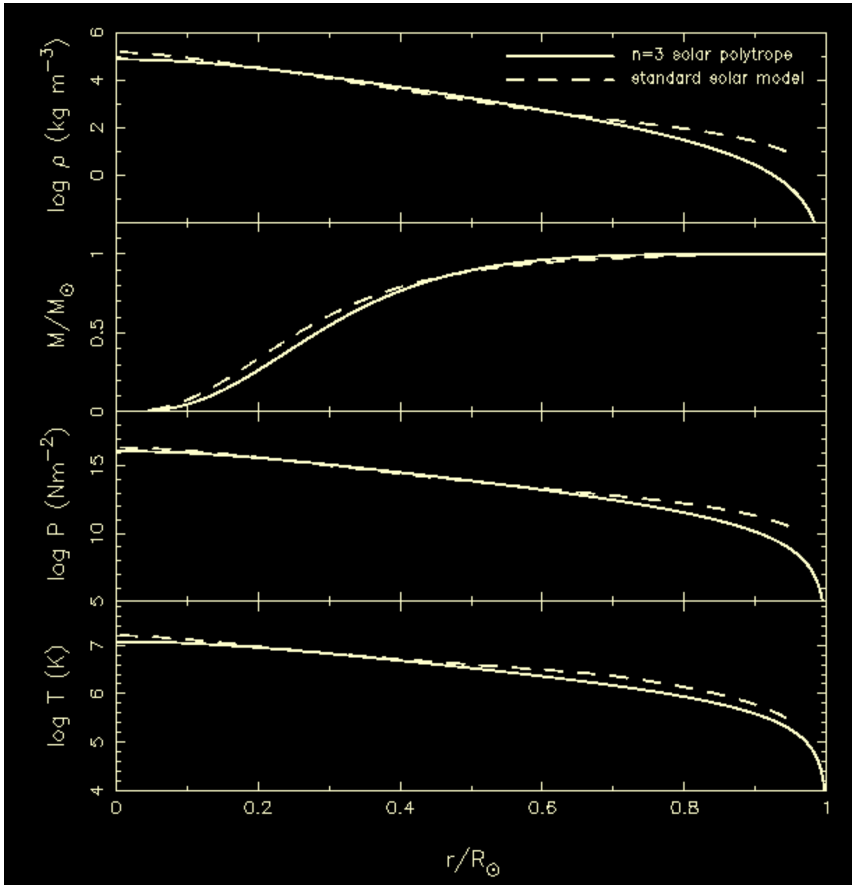
Find the value

Comparison with real models

- How do these polytropic models, compare to the results of a detailed solution of the equations of stellar structure? To make this comparison we will take an $n=3$ polytropic model of the Sun (often known as the Eddington Standard Model, a model with the constant fraction of radiation pressure and μ=const), with the co-called Standard Solar Model (SSM - Bahcall 1998, Physics Letters B, 433, 1).
- For this, we need to convert the dimensionless radius ξ and density θ to actual radius (in cm) and density (in g cm^{-3}).
- Polytrope does remarkably well (particularly at the core) considering how simple the physics is.

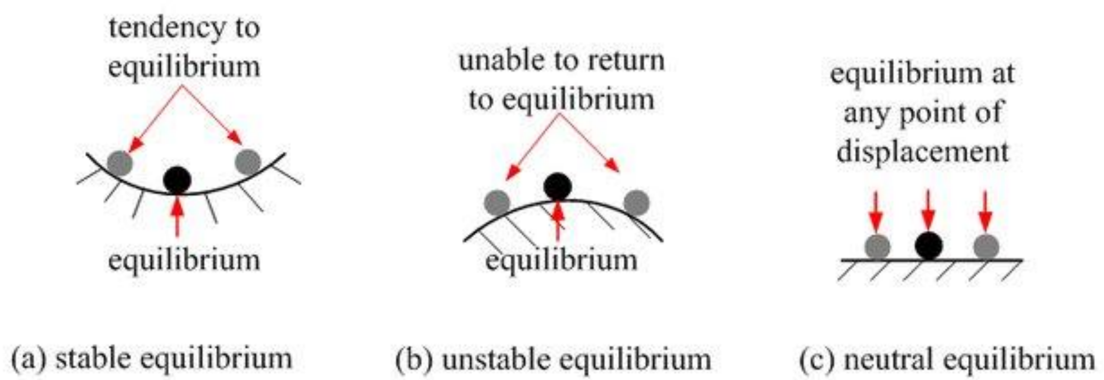
Property	n=3 polytrope	SSM
ρ_c	7.65×10^1 g cm^{-3}	1.52×10^2 g cm^{-3}
P_c	1.25×10^{17} dyn cm^{-2}	2.34×10^{17} dyn cm^{-2}
T_c	1.18×10^7 K	1.57×10^7 K

Comparison of numerical solution for an $n = 3$ polytrope of the Sun versus the Standard Solar Model.



The stability of stars

We have so far considered stars in both hydrostatic and thermal equilibrium (HE & TE). But an important question is whether these equilibria are stable?



A rigorous treatment of this problem is very complicated, so we will only look at a very simplified example to illustrate the principles.

Dynamical stability of stars

- Suppose a star in hydrostatic equilibrium is compressed on a short timescale, $\tau << \tau_{KH}$, so that the compression can be considered as adiabatic.
- It can be shown that a star that has $\gamma_{ad}>4/3$ everywhere is dynamically stable, and if $\gamma_{ad}=4/3$, it is neutrally stable.
  However, the situation when $\gamma_{ad}<4/3$ in some part of the star requires further investigation.
- If $\gamma_{ad}<4/3$ in a sufficiently large core, where P/ρ is high, the star becomes unstable. However if $\gamma_{ad}<4/3$ in the outer layers where P/ρ is small, the star as a whole need **not** become unstable.
- Stars dominated by an ideal gas or by non-relativistic degenerate electrons have $\gamma_{ad}=5/3$ and are therefore dynamically stable. However, we have seen that for relativistic particles $\gamma_{ad}\to 4/3$ and stars dominated by such particles tend towards a neutrally stable state.
- A small disturbance of such a star could either lead to a collapse or an explosion. This is the case if radiation pressure dominates (at high T and low ρ), or the pressure of relativistically degenerate electrons (at very high ρ).

Overall, if the configuration of a star is to be approximately described by a polytrope (in which case γ and γ_{ad} are identical), the index n may only vary between 1.5 and 3, or

$$\frac{4}{3} < \gamma_{ad} \leq \frac{5}{3}$$

Cases of dynamical instability

- We have seen earlier (Lectures 3 and 8) that the contribution of radiation pressure increases with mass and becomes dominant for $M \gtrsim 100\ M_\odot$. A gas dominated by radiation pressure has an adiabatic index $\gamma_{ad}=4/3$, or $n=3$, which means that hydrostatic equilibrium in such stars becomes marginally unstable. Therefore, stars much more massive than 100 $M_\odot$ should be very unstable, and indeed none are known to exist (while those with $M > 50\ M_\odot$ indeed show signs of being close to instability, e.g. they lose mass very readily).
- A process that can lead to $\gamma_{ad}<4/3$ is partial ionization (e.g. $H \leftrightarrow H^+ + e^-$). Since this normally occurs in the very outer layers, where P/ρ is small, it does not lead to overall dynamical instability of the star. However, partial ionization is connected to driving oscillations in some kinds of star.
- At very high temperatures two other processes can occur that have a similar effect to ionization:
  - These are pair creation ($\gamma + \gamma \leftrightarrow e^+ + e^-$) and photo-disintegration of nuclei (e.g. γ + Fe $\leftrightarrow \alpha$). These processes, that may occur in massive stars in late stages of evolution, also lead to $\gamma_{ad}<4/3$ but now in the core of the star. These processes can lead to a stellar explosion or collapse.

Summary

- We discussed methods of finding the solution of the equations of stellar structure.
- We have defined a method to relate the internal pressure and density as a function of radius – the polytropic equation of state.
- We derived the Lane-Emden equation.
- We saw how this equation could be numerically integrated in general.
- We derived a number useful relations between stellar parameters.
- There is a theoretical upper limit to the mass of a white dwarf (Chandrasekhar limit). It is confirmed by observations, we do not see WDs with masses $>1.4M_{\odot}$.
- Further increase of the WD mass e.g. as a result of accretion from the companion, will lead to the loss of stability and collapse, causing supernovae type Ia explosions.
- We compared the n=3 polytrope with the Standard Solar model, finding quite good agreement considering how simple the input physics was.
- Finally, we discussed cases of dynamical instability of stars.