Nuclear Energy Production

BASICS ON NUCLEAR REACTIONS
THE BINDING ENERGY
QUANTUM TUNNELLING
REACTION CROSS-SECTION
THE GAMOW PEAK
NUCLEAR REACTION RATES
ELECTRON SHIELDING

Introduction

- We have seen that the 4 equations of stellar structure must be supplemented with expressions for P, κ, ε.
- We have discussed that
  - P is given by the equation of state of the stellar matter.
  - κ is determined by the atomic physics of the stellar material.
- The last function we still need to describe is the power density $\varepsilon = \varepsilon(\rho, T,$ chemical composition).
- In Lecture 4 we have concluded that the energy source behind ε must be nuclear burning.
- Thus, ε is defined by the nuclear energy source in the interiors. We need to develop a theory and understanding of nuclear physics and reactions.

We now move from the studies of a stellar body to studies of its "soul" (V.V. Ivanov).

How to compute ε?

- Our goal is to compute the rate of energy generation ε per unit mass per unit time.
- The computation can be separated into three parts:
  - the cross section for a reaction between a pair of nuclei, which is determined predominantly by the properties of the nuclei;
  - the amount of energy generated per reaction, which again is a property of the nuclei;
  - the total reaction rate which, beside the cross section, also depends on the statistics of the motion of the nuclei.

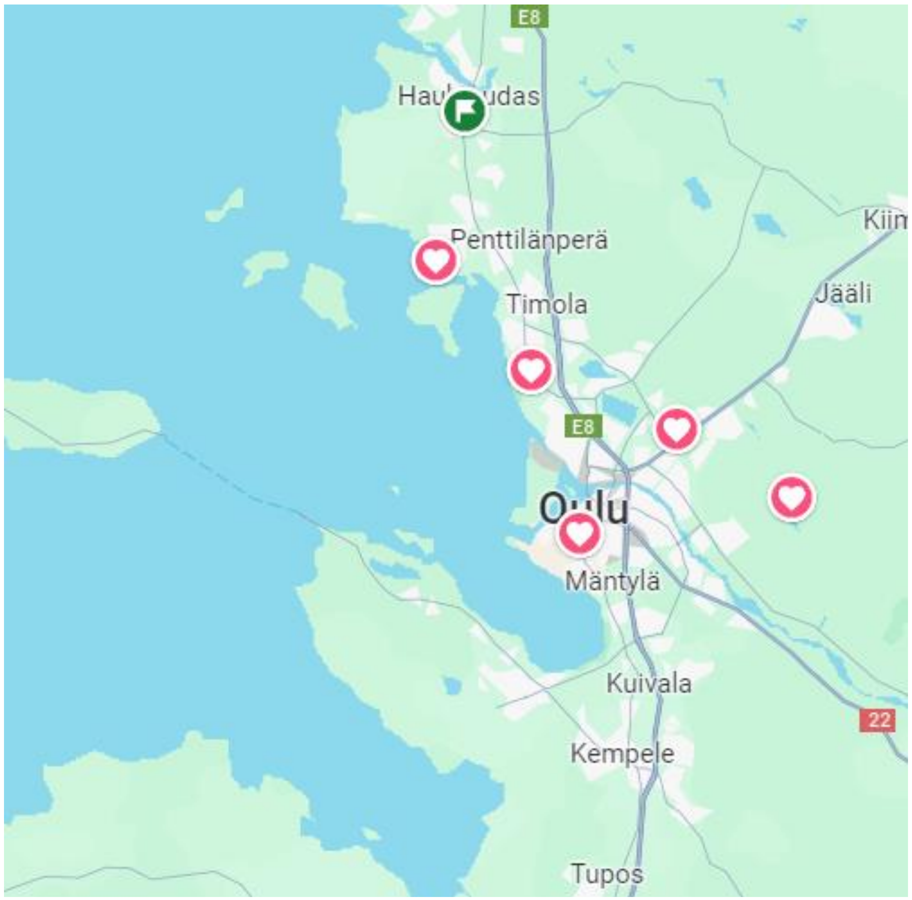
- An additional consequence of the nuclear processes is a gradual change of the chemical composition, which controls the evolution of the star. Hence, we must determine the rate of change of the abundances.

Nuclear dimensions

- A convenient unit for measuring the radius of an atom is the Bohr radius, $\sim 0.5\times 10^{-10}$ m $= 0.5$ Å.
- Atomic radii are typically a few to a few tens of Bohr radii, or of order 10^{-10} m, or a few Å.
- Proton radius $= 0.83 \times 10^{-15}$ m $\approx 10^{-13}$ cm $= 10^{-5}$ Å.
- The radius R of a nucleus containing A nucleons, i.e. having an atomic mass number of A, is $R \approx r_0 A^{1/3}$, where $r_0 = 1.2\times 10^{-15}$ m.
- Nuclei are therefore of order a few $\times 10^{4}$ times smaller in radius than an atom.

- How compact the nucleus compared to the atom within which it resides?

Nuclear dimensions



~40 km

~1 m

Thus, atoms are very empty, but nuclei are very compact and incompressible dense.

Nuclear processes in stars

- The most important series of fusion reactions are those converting hydrogen into helium (H-burning). As we will see later, this dominates ~90% of lifetime of nearly all stars.
- Mass of nuclei with several protons and/or neutrons does not exactly equal mass of the constituents - slightly smaller because of the binding energy of the nucleus.
- Hence, there is a decrease of mass, and from the Einstein mass-energy relation $E=mc^2$ the mass deficit is released as energy.

Example: $4\times{}^1H \rightarrow {}^4He$

4 protons × each of mass 1.0078 atomic mass units: 4.0313 amu

mass of helium nucleus: 4.0026 amu

Mass difference: 0.0287 amu = 4.76×10^{-26} g

$$\Delta E = \Delta Mc^2 = 4.278\times10^{-5}\ \text{erg} = 26.7\ \text{MeV}$$

- Since binding energy differs for different nuclei, it can release or absorb energy when nuclei either fuse or fission.

The binding energy

- General calculation: define the binding energy of a nucleus as the energy required to break it up into constituent protons and neutrons.
- Suppose nucleus has:
  - Proton number Z
  - N neutrons
  - Atomic mass number A (number of protons + neutrons, $Z+N$), or the baryon number, or nucleon number, or nuclear mass
  - Nucleus mass $m_{nuc} = m(Z,N)$
- Binding energy of the nucleus (also known as the difference between the masses of nucleons and nucleus) is:

$$Q(Z,N) \equiv [Zm_p + Nm_n - m(Z,N)]c^2$$

- Some useful units:
  - 1 eV $=1.602\times10^{-12}$ erg $=11.65\times10^3$ K
  - 1amu $=1/12\ m(^{12}C) = 931.49$ MeV$/c^2 = 1.660\times10^{-24}$ g
  - $m_e c^2 = 0.511$ MeV
  - $m_p = 1.007825$ amu $\quad m_p c^2 = 938.72$ MeV
  - $m_n = 1.008665$ amu $\quad m_n c^2 = 939.56$ MeV
  - $m(^4He) = 4.002603$ amu

The binding energy per nucleon (1)

- For our purposes, a more useful quantity for considering which nuclear reactions yield energy is the total binding energy per nucleon:

$$q = \frac{Q(Z,N)}{A} \qquad {}^{A}_{Z}X$$

- We can then consider this number relative to the hydrogen nucleus.
- For reaction $4\times{}^{1}\mathrm{H} \rightarrow {}^{4}\mathrm{He}$

$$q = \frac{28.30}{4} = 7.07\ \mathrm{MeV}$$

- This corresponds to 7.07 MeV / 931 MeV = 0.7% of the rest mass converted to energy when hydrogen burns. The time for the Sun to radiate away just 10% of the energy available from this source is

$$\tau_{nuc} = \frac{0.1 \times 0.007 \times M_{\odot} \times c^2}{L_{\odot}} \approx 10^{10}\ \mathrm{yr}$$

  In terms of energy budget, hydrogen fusion can easily produce the solar luminosity over the age of the Solar System.

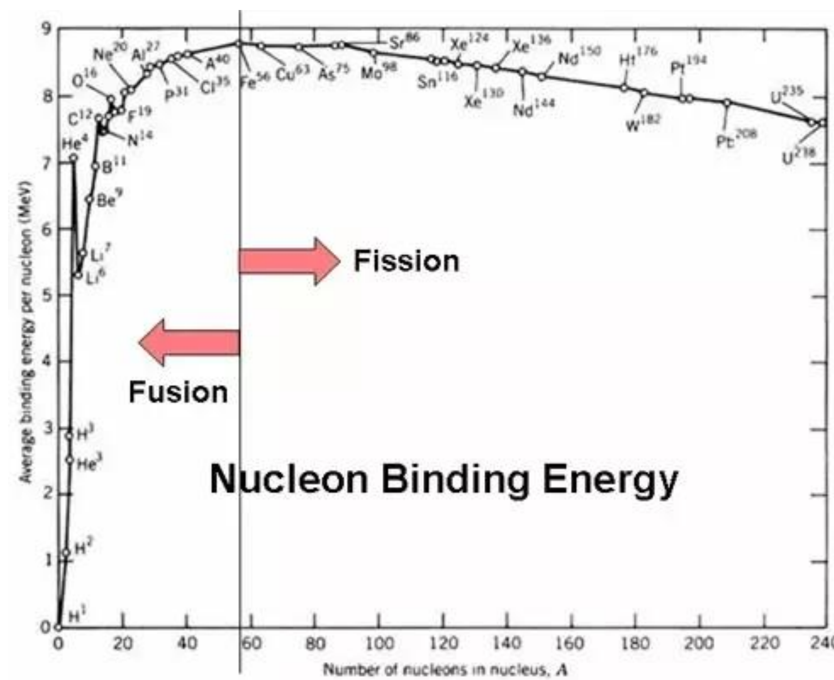
- Compare with fusion of hydrogen to iron ${}^{56}\mathrm{Fe}$: q=8.8 MeV per nucleon. Most of this is already obtained in forming helium (7.07 MeV).

The binding energy per nucleon (2)

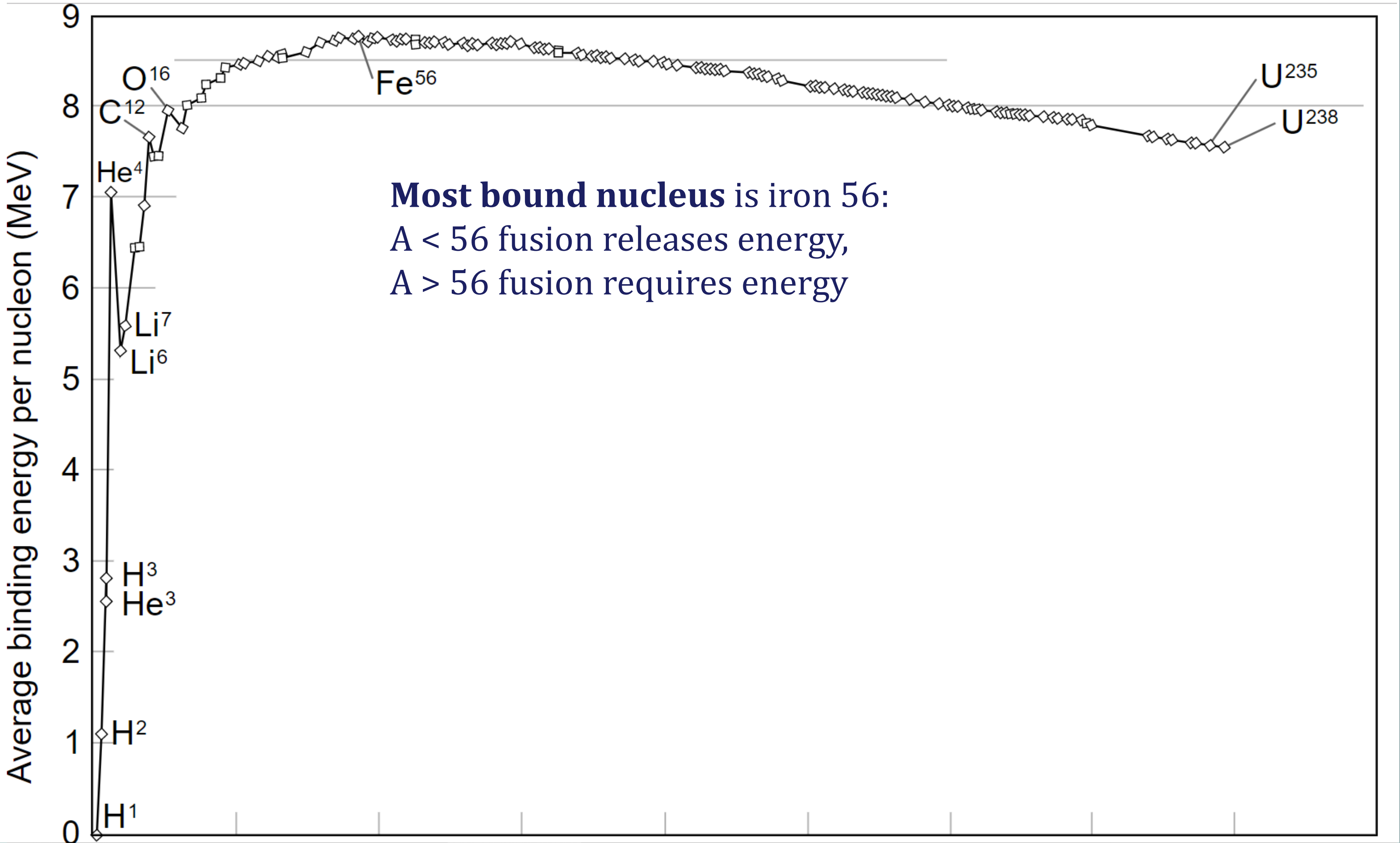
On average, the binding energy per nucleon is about $q \approx 8$ MeV/nucleon

More accurately it depends on the baryon number A.

- General trend is an increase of q with atomic mass up to $A = 56$ (Fe). Then slow monotonic decline.
- There is steep rise from H through ^{2}H, ^{3}He, to ^{4}He $\rightarrow$ fusion of H to He should release larger amount of energy *per unit mass* than say fusion of He to C.
- Energy may be gained by *fusion* of light elements to heavier, up to iron
- Or from *fission* of heavy nuclei into lighter ones down to iron.



The binding energy per nucleon (3)



Most bound nucleus is iron 56:
A < 56 fusion releases energy,
A > 56 fusion requires energy

Occurrence of fusion reactions (1)

Let's now discuss the conditions under which these fusion reactions can occur and whether such conditions exist in stellar interiors.

Consider two nuclei with proton numbers Z_1 and Z_2. Nuclei interact through four forces of physics, but only electromagnetic and strong nuclear important here:

- Two positively charged nuclei repel each other and must overcome a repulsive Coulomb potential

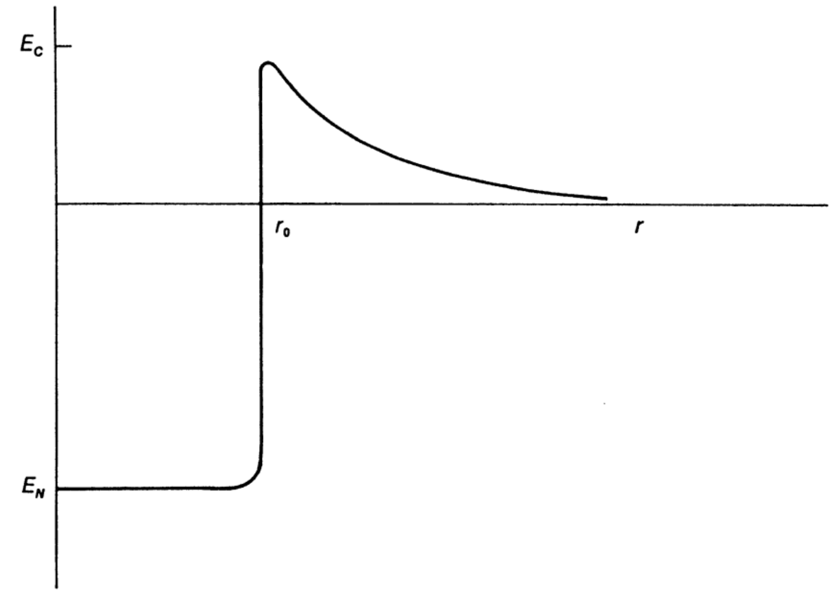
$$V_C = \frac{Z_1 Z_2 e^2}{r} = 1.44 \frac{Z_1 Z_2}{r[\mathrm{fm}]}\ \mathrm{MeV}$$

$r[\mathrm{fm}]$ – r in fermis ($\times 10^{-13}$ cm)

- to reach separation distances where strong force dominates (typical size of nucleus)

$$r_0 \approx 1.44 \times 10^{-13} A^{1/3}\ \mathrm{cm}$$

A – Atomic mass number



Schematic potential energy between two nuclei. For $r < r_0$ the attractive nuclear forces dominate; for $r > r_0$ Coulomb repulsion dominates.

At $r = r_0$, height of the Coulomb barrier is:

$$V_0 = \frac{Z_1 Z_2 e^2}{r_0[fm]} = Z_1 Z_2\ \mathrm{MeV}$$

i.e. of the order of 1 MeV for two protons…

Femtometer [**fm**] (derived from the Danish and Norwegian word *femten* 'fifteen' equal to 10^{-15} m. This distance is also called **fermi**.

Classical approach

At $r = r_0$, height of the Coulomb barrier is $V_0 = Z_1 Z_2$ MeV, of the order of 1 MeV for two protons.

Class task:

Compare this to the average kinetic energy of a particle ($3kT/2$). What T do we require for fusion? How does this compare with the minimum mean T of the Sun? Any comments on these two temperatures?

- At a typical internal stellar temperature of 10^7 K, the kinetic energy of a nucleus is $3kT/2 \sim 1$ keV. The characteristic kinetic energy is thus of order 10^{-3} of the energy required to overcome the Coulomb barrier.
- Typical nuclei will approach each other only to a separation $r \sim 10^{-10}$ cm, 1000 times larger than the distance at which the strong nuclear binding force operates.
- Can nuclei that are in the high-energy tail of the Maxwell–Boltzmann distribution overcome the barrier? The fraction of nuclei with such energies is

$$e^{E/kT} \approx e^{-1000} \approx 10^{-434}$$

- Even considering the high-energy tail of the Maxwell-Boltzmann distribution, the fraction of particles with $E > E_C$ is **vanishingly small**.
- With purely classical considerations nuclear reactions have **no chance of happening** at such temperatures.

Boltzmann's constant $k = 1.38 \times 10^{-16}$ cm^2 g s^{-2} K^{-1} = 8.62×10^{-5} eV K^{-1}

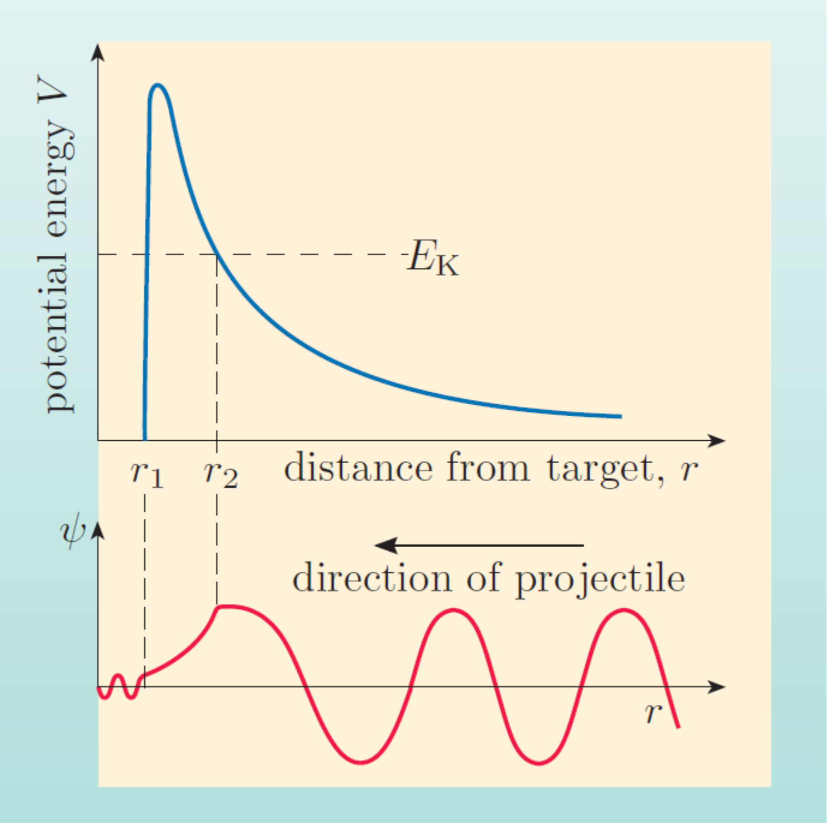
Occurrence of fusion reactions (2)

At $r = r_0$, height of the Coulomb barrier is $V_0 = Z_1 Z_2$ MeV, i.e. of the order of 1 MeV for two protons…

If we imagine what will happen as the nuclei approach one another with a certain kinetic energy E_K then, ignoring quantum mechanics, they will simply come to rest temporarily when their kinetic energy has been converted into electrical potential energy, before 'bouncing' apart again.

The distance of least separation is given by

$$r_2 = \frac{Z_1 Z_2 e^2}{E_K}$$



Thus, **classically**, a particle with kinetic energy E_K cannot get closer to the origin than $r = r_2$ because of the potential barrier of height $V > E_K$. **Quantum-mechanically**, however, the wave function shows that there is a **non-zero probability** of finding the particle beyond the barrier, i.e. at $r < r_1$.

Quantum tunnelling

- We need to turn to quantum mechanics to see how nuclear reactions are possible at stellar temperatures.
- As was discovered by George Gamow (1928), there is a finite non-zero probability for a particle to penetrate the repulsive Coulomb barrier even if $E_K \ll V_C$ as if "tunnel" existed.
- For the Coulomb barrier, the penetration probability can be expressed in terms of the particle energy $E=E_K$, and the Gamow energy E_G which depends on the atomic number (and therefore charge) of the interacting nuclei, and hence the size of the Coulomb barrier:

$\boxed{P_{pen} = g(E) \approx e^{-\sqrt{E_G/E}} \text{ with the Gamow energy } E_G = 2m_r c^2(\pi\alpha Z_1 Z_2)^2}$, where $m_r = m_1 m_2/(m_1+m_2)$ is the reduced mass of particles, α is the fine structure constant ≈ 1/137.

  - P_{pen} which is also called the Gamow factor $g(E)$ increases rapidly with E
  - P_{pen} decreases with Z_1Z_2 - lightest nuclei can fuse more easily than heavy ones
  - Higher energies / temperatures needed to fuse heavier nuclei, so different nuclei burn in well-separated phases during stellar evolution.

Quantum tunnelling for two protons

$$E_G = 2m_r c^2 (\pi \alpha Z_1 Z_2)^2$$

$$P_{pen} \approx e^{-\sqrt{E_G/E}}$$

$m_r = m_1 m_2/(m_1+m_2)$ is the reduced mass of particles, α is the fine structure constant ≈ 1/137.

- **Class task:** calculate E_G for two protons and find P_{pen} for the typical kinetic energy of particles in the Sun's core, E ~ 1 keV.
  It is convenient to remember that the rest energy of a proton, $m_p c^2$, is 938 MeV.

  - For two protons, $E_G \approx 500$ keV
  - $P_{pen} \approx e^{-22} \approx 2 \times 10^{-10}$. It is very small, but considerably larger than the classical probability.

Reaction cross-section

- Even if tunneling occurs, and two nuclei are within the strong force's interaction range, the probability of a nuclear reaction will still depend on a nuclear cross section, which will generally depend inversely on the kinetic energy. Thus, the total cross-section for a nuclear reaction can be written as

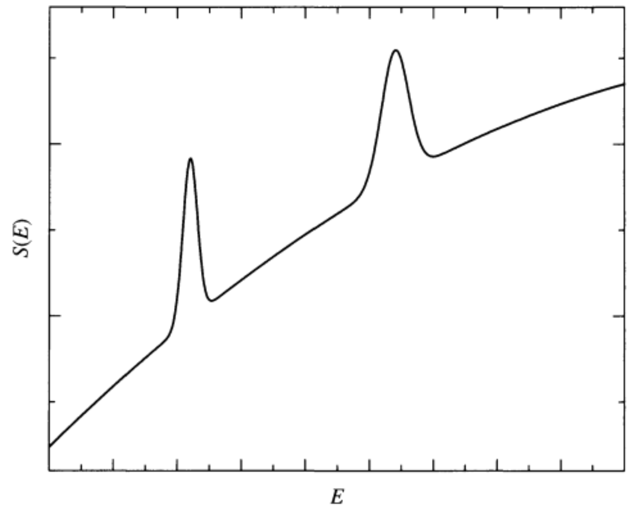
The cross-section is defined as

$$\sigma = \frac{\text{number of reactions per sec}}{\text{number of incident particles per sec per cm}^2}$$

$$\sigma = \frac{S(E)}{E} g(E) = \frac{S(E)}{E} e^{-\sqrt{E_G/E}}$$

where $S(E)$ is a **slowly varying** S-factor:

- $S(E)$ contains all remaining effects, i.e. the intrinsic nuclear properties of the reaction including possible resonances.
- Its evaluation requires laboratory data.
- Generally, it is insensitive to particle energy or velocity.
- In some cases, however, $S(E)$ can vary quite rapidly, peaking at specific energies. These energies correspond to energy levels within the nucleus, analogous to the orbital energy levels of electrons. It is a resonance between the energy of the incoming particle and differences in energy levels within the nucleus that accounts for these strong peaks.



The speed-averaged cross-section

- The number of reactions per second in a unit volume is

$$r_{ij} = \frac{n_i n_j}{1+\delta_{ij}} \sigma v$$

where n_i and n_j are number densities of particles, and v is a relative velocity.
If a reaction between identical particles is considered (i.e., protons on protons) then r needs to be divided by 2, to avoid double counting , i.e. $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$

- In reality, the nuclei in a gas will have a distribution of velocities (a range of kinetic energies), so every velocity has some probability of occurring. Hence (after some algebra, which we will not describe)

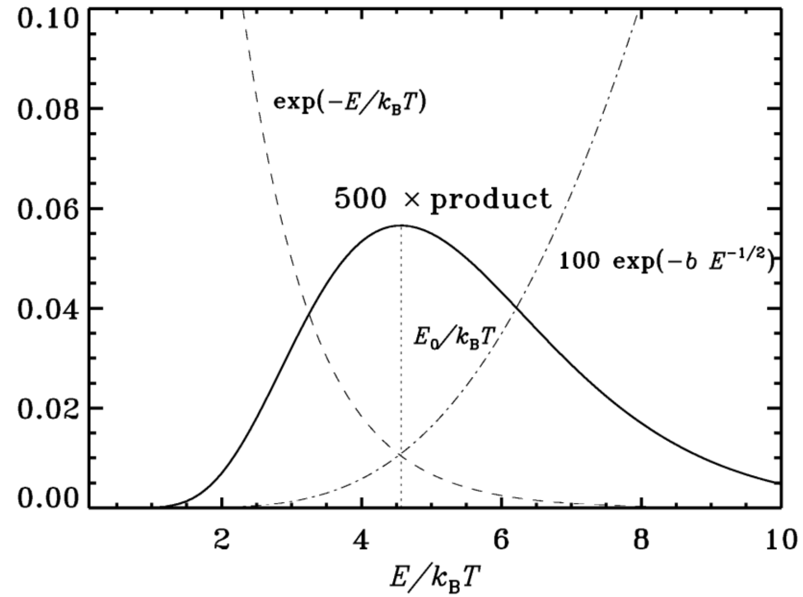
$$\langle \sigma v \rangle = \left(\frac{8}{\pi m_r}\right)^{1/2} \frac{S(E_0)}{(kT)^{3/2}} \int_0^\infty e^{-E/kT} e^{-\sqrt{E_G/E}} dE$$

- The integrand in this expression, $f(E) = e^{-E/kT} e^{-\sqrt{E_G/E}}$, is composed of the product of two exponential functions, one (from the Boltzmann distribution) falling with energy, and the other (due to the Gamow factor embodying the Coulomb repulsion) rising with energy. Obviously, $f(E)$ will have a narrow maximum at some energy E_0, at which most of the reactions take place:

$$E_0 = \left(\frac{kT}{2}\right)^{2/3} E_G^{1/3}$$

The Gamow peak

- The Gamow peak is the product of the Maxwellian distribution and tunnelling probability. The area under the Gamow peak determines the reaction rate.
- The higher the electric charges of the interacting nuclei, the greater the repulsive force, hence the higher the E_k and T are needed for reaction to occur.
- Highly charged nuclei are obviously the more massive, so reactions between light elements occur at lower T than reactions between heavy elements.



The Boltzmann probability distribution for kT = 1 keV, the Gamow factor for the case of two protons, with E_G = 500 keV, and their product. Scaled up by large factors for display purposes.

Nuclear Reaction Rates

$Q\equiv [Zm_p+Nm_n-m(Z,N)]c^2$

- Each reaction releases an amount of energy Q_{ij}, so then $Q_{ij}r_{ij}$ is the energy generated per unit volume and per second. The energy generation rate per unit mass from the reaction between nuclei of type i and j is then

$n_i = \frac{X_i \rho}{A m_H}$

$$\varepsilon_{ij} = \frac{Q_{ij} r_{ij}}{\rho} = \frac{Q_{ij}}{(1+\delta_{ij}) A_i A_j m_H^2} \rho X_i X_j \langle \sigma v \rangle_{ij}$$

- The final expression for the power density due to a given nuclear reaction:

$$\varepsilon_{ij} = \frac{2^{\frac{5}{3}}\sqrt{2}\, Q_{ij}}{(1+\delta_{ij})\sqrt{3}} \frac{\rho X_i X_j}{m_H^2 A_i A_j \sqrt{m_r}} S_0 \frac{E_G^{1/6}}{(kT)^{2/3}} \exp\left[-3\left(\frac{E_G}{4kT}\right)^{\frac{1}{3}}\right]$$

where $S_0=S(E_0)$.

- The total power density at a point in a star with a given temperature, density, and abundance will be the sum of the power densities due to all the possible nuclear reactions, each described by this equation.
- Because of the exponential term in the Eqn, there will be a strong preference for reactions between species with low atomic number, and hence small E_G.
- Furthermore, the higher the Gamow energy, the more strongly will the reaction rate depend on temperature.

Timescale of a nuclear reaction

- The typical timescale of a nuclear reaction is inversely proportional to the reaction rate r_{ij}. The mean time it takes for a particular nucleus of type i to undergo fusion with a nucleus of type j is
$$\tau_i = \frac{n_i}{(1+\delta_{ij})\, r_{ij}}$$
- The extremely high sensitivity of nuclear reaction rates to temperature leads to the concept of "ignition" of a nuclear fuel: each reaction (or nuclear process) has a typical narrow temperature range over which its rate increases by orders of magnitudes, from negligible values to very significant ones.
- Around this range, the temperature dependence of the reaction rate may be well approximated by a power law (with a high power) and an ignition or threshold temperature may be defined. Hence, the equation of the power density ε should characteristically have
$$\varepsilon \approx \varepsilon_0 (T/T_0)^n$$
H-burning: n = 5÷15; He-burning n = 40.
- The process of creation of new nuclear species by fusion reactions is called nucleosynthesis. Since the kinetic energy of particles is that of their thermal motion, the reactions between them are called thermonuclear.

Electron shielding

- We found that the repulsive Coulomb force between nuclei plays a crucial role in determining the rate of a thermonuclear reaction. In our derivation of the cross-section, we have ignored the influence of the surrounding free electrons, which provide overall charge neutrality in the gas.
- In a dense medium, the attractive Coulomb interactions between atomic nuclei and free electrons cause each nucleus to be effectively surrounded by a cloud of electrons. This electron cloud reduces the Coulomb repulsion between the nuclei at large distances, and may thus increase the probability of tunneling through the Coulomb barrier. This effect is known as **electron screening** or **electron shielding**.
- According to the weak screening approximation, which applies to relatively low densities and high temperatures such as found in the centre of the Sun and other main-sequence stars, clouds of negatively charged electrons can increase r_{jk} by about 10%.
- The description of electron screening becomes complicated at high densities and relatively low temperatures, where the weak screening approximation is no longer valid. A general result is that with increasing strength of electron screening, the temperature sensitivity of the reaction rate diminishes, and the density dependence becomes stronger. At very high densities, $\rho > 10^6$ g/cm^3, the screening effect is so large that it becomes the dominating factor in the reaction rate.