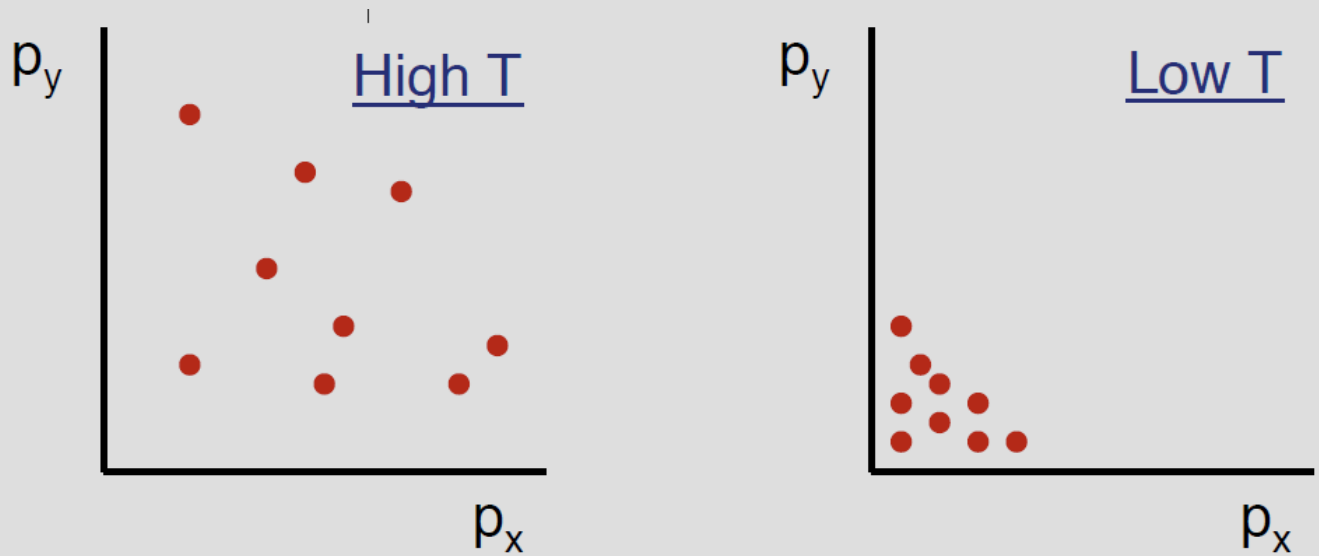
Degeneracy Pressure (1)

- An ordinary classical gas: $P_{\text{gas}} \propto T \to 0$ as $T \to 0$
- Simultaneously, the mean speed of particles in the gas also goes to zero:

$$v = \sqrt{2kT/m}$$

- The momenta are given by: $p_x = mv_x$; $p_y = mv_y$; $p_z = mv_z$
- … if we plot the momenta of particles in a 3D space of p_x, p_y, and p_z then as T decreases the particles become concentrated near the origin:



Degeneracy Pressure (2)

At low enough temperatures / high enough densities, the concentration of particles with similar (low) momenta would violate the Pauli exclusion principle:

No two electrons can occupy the same quantum state

i.e. have the same momentum, spin, and location.

To avoid violating the exclusion principle, electrons in a dense, cold gas **must have larger** momenta than we would predict classically.

Since the pressure P is mean rate of transport of momentum across unit area

$$P = \frac{1}{3}\int_0^\infty vpn(p)dp$$

...where $n(p)dp$ is the number of particles with momentum between p and $p+dp$

... larger momentum means higher pressure.

This quantum mechanical source of pressure is **degeneracy pressure**.

Degeneracy Pressure (3)

This quantum mechanical source of pressure is **degeneracy pressure**.
We will discuss it later.

$$P_{ideal} = \frac{k}{\mu m_p}\rho T$$

- **Non-relativistic degeneracy pressure** (speeds v ≪ c) :

$$P_{\text{deg}} = K_1\rho^{5/3} = K_1\rho^{1+\frac{1}{n}}$$

A polytrope of index n=1.5

  - K_1 is constant
  - Does not depend upon temperature for low enough T
  - Depends upon composition via the relation between N_e and ρ

- **Relativistic degeneracy pressure :**

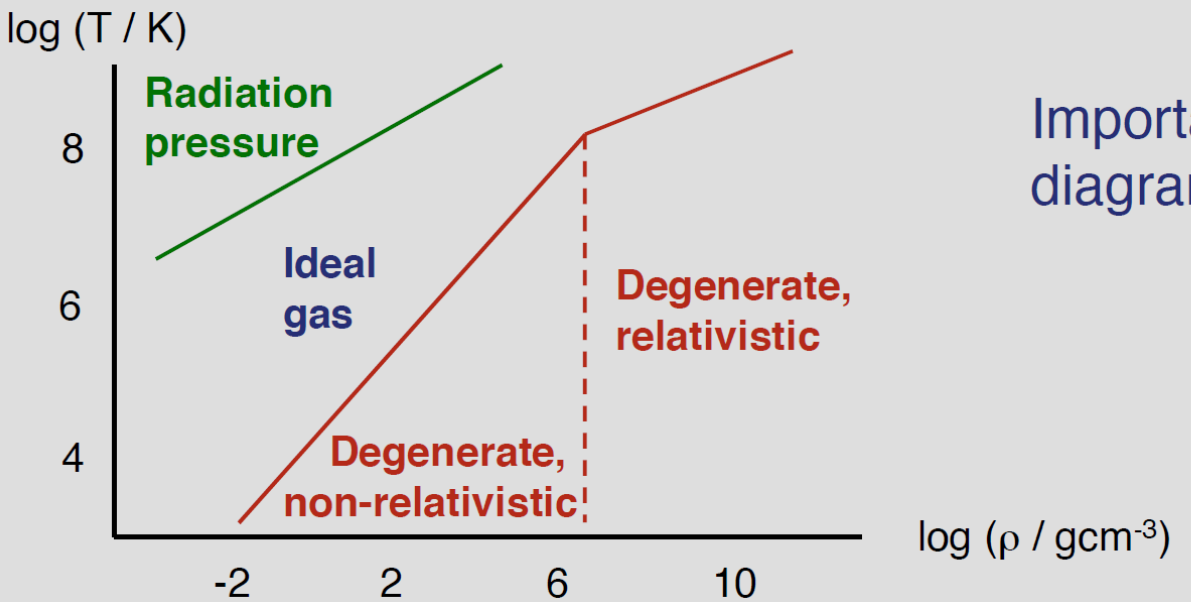
$$P_{\text{deg}} = K_2\rho^{4/3}$$

A polytrope of index n=3

  - K_2 is another constant
  - Equation of state for relativistic degenerate matter, which applies at high density. This is a "softer" equation of state, since P rises more slowly with increasing density than for the non-relativistic case.

A relation of the form $P = K\rho^{1+\frac{1}{n}}$ where K and n are constants is called a polytropic relation, and n is the polytropic index.

When do the different pressures matter?



Important diagram!

Different types of star occupy different portions of the plane:

- Solar-type stars - ideal gas throughout
- Massive stars - radiation pressure
- White (and brown) dwarfs - non-relativistic degeneracy pressure

Relativistic degeneracy implies an **unstable** equation of state, so **no stable stars** in that part of the plane.

Radiation Pressure

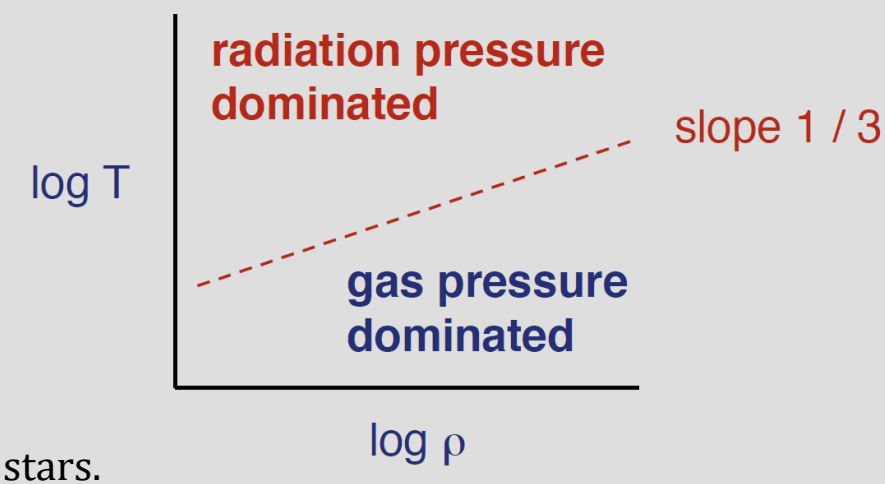
We have already showed before that radiation pressure can be neglected for Solar-type stars:

$$\frac{P_{rad}}{P_g} = \frac{aT^4}{3} \Big/ \frac{kT\rho}{\mu m_p} = \frac{\mu a}{3\Re}\frac{T^3}{\rho} \approx 10^{-4} \text{ (for the Sun)}$$

But becomes **very important** for early-type stars due to the T^4 sensitivity.

In which stars are gas and radiation pressure important?

$$\left.\begin{aligned} P_{rad} &= \frac{aT^4}{3} \\ P_g &= \frac{\Re T\rho}{\mu} \end{aligned}\right\} \text{ equal when } T^3 = \frac{3\Re}{a\mu}\rho$$



From the virial theorem (see Lecture 3):

$$\bar{T} \propto \frac{M}{R} \Rightarrow \frac{P_{rad}}{P_g} \propto M^2$$

i.e. $P_{\rm rad}$ becomes more significant in higher mass stars.

Effect of radiation pressure

For stars in which radiation pressure plays a non-negligible role we can write the generalized form of the equation of hydrostatic support (Lecture 2):

$$\frac{dP(r)}{dr} + g\rho(r) = a\rho(r)$$

Then

$$\frac{dP(r)}{dr} = -g\rho(r) - \frac{dP_{rad}}{dr} = -g_{\text{eff}}(r)\rho(r)$$

From Lecture 6 (slide 168):

$$\frac{dP_{rad}}{dr} = -\frac{\rho\kappa_R}{c}F \quad \Longrightarrow \quad g_{\text{eff}}(r) = g - \frac{\kappa_R}{c}F$$

Consider relative contributions of radiation and (ideal) gas pressures:

$$P_g = \beta P = \frac{\Re T\rho}{\mu}, \quad P_{rad} = (1-\beta)P = \frac{aT^4}{3}$$

Exclude temperature: $P = \left[\left(\frac{\Re}{\mu}\right)^4 \frac{3}{a}\frac{1-\beta}{\beta^4}\right]^{1/3} \rho^{4/3} \quad \Longrightarrow \quad P = K\rho^{1+\frac{1}{n}}$ ⟵ A polytrope of index n=3

Opacity (1)

Concept of opacity was introduced when deriving the equation of radiation transport.
It will be discussed extensively in the Stellar atmospheres part of this course.
Opacity is the resistance of material to the flow of radiation through it. In most stellar interiors it is determined by all the processes which scatter and absorb photons.

Four main processes:

- Bound-bound absorption:
  - is related to photon-induced transitions of a (bound) electron in atoms or ions to a higher energy state by the absorption of a photon. The atom is then de-excited either spontaneously or by collision with another particle, whereby a photon is emitted. Although this is limited to certain transition frequencies, the process can be efficient because the absorption lines are strongly broadened by collisions.
- Bound-free absorption:
  - which is another name for photoionization - the removal of an electron from an atom (ion) caused by the absorption of a photon. The inverse process is radiative recombination.
- Free-free absorption:
  - the absorption of a photon by a free electron, which makes a transition to a higher energy state by briefly interacting with a nucleus or an ion. The inverse process, leading to the emission of a photon, is known as bremsstrahlung.
- [Electron] Scattering:
  - the scattering of a photon by a free electron, the photon's energy remaining unchanged (known as Thomson scattering).

Opacity (2)

Which process is most important in the deep stellar interiors?

- Temperatures are very high there, thus the last two processes are dominant, simply because the material is almost completely ionized, there are very few bound electrons.
- Furthermore, the energy of most photons in the Planck distribution is of the order of keV, whereas the separation energy of atomic levels is only a few tens eV. Hence most photons interacting with bound electrons would set them free. Thus bound-bound (and even bound-free) transitions have extremely low probabilities, interactions occurring predominantly between photons and free electrons.
- Thus, the most dominant process in the deep stellar interiors is electron scattering.

The opacity per unit mass of material in this case is $\kappa_e = \frac{n_e \sigma_T}{\rho}$,

where $\sigma_T = 6.65 \times 10^{-25}$ cm^2 is the Thomson cross section. The opacity is therefore:

$$\kappa_e \cong 0.2(1 + \mathrm{X})\ \mathrm{cm}^2 \mathrm{g}^{-1}$$

$$n_e = \frac{\rho}{\mu_e m_H}$$

The opacity resulting from electron scattering is temperature and density **independent**!

Opacity (3)

We need an expression for opacity to solve the equations of stellar structure. For stars in thermodynamic equilibrium with only a slow outward flow of energy, the opacity should have the form $\kappa = \kappa(\rho, T, \text{chemical composition})$

Opacity coefficients may be calculated, taking into account all possible interactions between the elements and photons of different frequencies. This requires an enormous amount of calculation and is beyond the scope of this course. Such calculation are done e.g. by the OPAL opacity project at Lawrence Livermore National Laboratory.

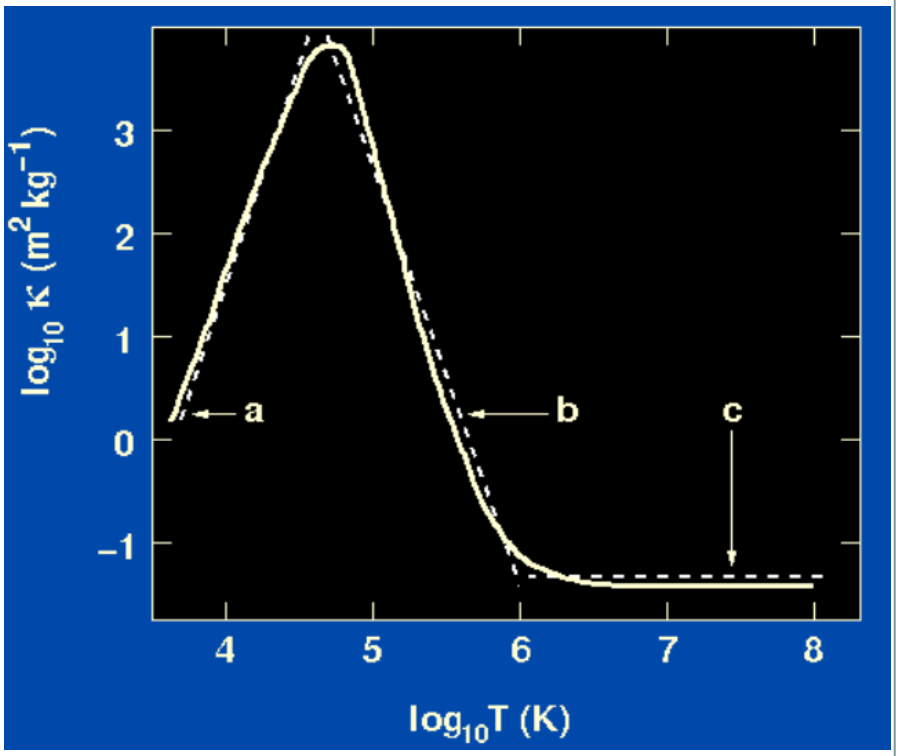
When it is done, the results are usually approximated by the relatively simple formula:

$$\kappa = \kappa_0 \rho^{\alpha} T^{\beta}$$

where α, β are slowly varying functions of density and temperature and κ_0 is a constant for a given chemical composition.

Opacity (4)

Figure shows opacity as a function of temperature for a star of given ρ (10^{-4} g cm^{-3}). Solid curve is from detailed opacity calculations. Dotted lines are approximate power-law forms.



- At high T: κ is low and remains constant. Most atoms are fully ionized, high photon energy, hence free-free absorption unlikely. Dominant mechanism is electron scattering, independent of T, $\alpha=\beta=0$:

$$\kappa = \kappa_1 = \sigma_T / m_H \mu_e \quad \text{(curve c)}$$

- Opacity is low a low T, and increases towards higher T. Most atoms are not ionized, few electrons available to scatter photons or for free-free absorption. Approximate analytical form is $\alpha=1/2$, $\beta=4$:

$$\kappa = \kappa_2 \rho^{0.5} T^4 \quad \text{(curve a)}$$

- At intermediate T, κ peaks, when bound-free and free-free absorption are very important, then decreases with T (Kramers opacity law):

$$\kappa = \kappa_3 T^{-7/2} \quad \text{(curve b)}$$

- κ_1, κ_2, κ_3 are constants for stars of given chemical composition but all depend on composition.

Summary and conclusions

- We have learned the approximate forms of the equation of state and the opacity.
- Next lecture: A method of simplifying the solution of the stellar structure equations.
- After that we will discuss nuclear reactions and move on to discussing the output of full numerical solutions of the equations and realistic predictions of modern theory