The equations of stellar structure

THE EQUATIONS OF STELLAR STRUCTURE AND POSSIBLE WAYS TO SOLVE THEM.

BOUNDARY CONDITIONS.

CONVECTION AND CONDITIONS FOR ITS OCCURRENCE

Solving the equations of stellar structure

Now we have all four differential equations, which govern the structure of stars
(Note! in the absence of convection)

- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$
- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$
- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$
- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma r^2}\frac{\rho(r)\kappa_R(r)}{T^3(r)}L(r)$

Where

- r = radius
- P = pressure at r
- m = mass of material within r
- ρ = density at r
- L = luminosity at r (rate of energy flow across sphere of radius r)
- T = temperature at r
- κ_R = Rosseland mean opacity at r
- ε = energy release per unit mass per unit time

We will consider the quantities:

- $P = P(\rho, T$, chemical composition)
- $\kappa_R = \kappa_R(\rho, T$, chemical composition)
- $\varepsilon = \varepsilon(\rho, T$, chemical composition)

$\longrightarrow$ **The equation of state**

Boundary conditions

- Two of the boundary conditions are fairly obvious, at r=0, the centre of the star, m=0, L=0.

- At the surface of the star its not so clear, but we use approximations to allow solution.
  - There is no sharp edge to the star, but for the Sun ρ(surface)~10^{-7} g cm^{-3}. It is much smaller than mean density $\bar{\rho}$~1.4 g cm^{-3}.
  - We also know the surface temperature (T_{eff}=5780K) is much smaller than its minimum mean temperature (2x10^6 K).
- Thus, we make two approximations for the surface boundary conditions: ρ=0, T=0 at r=R, i.e. that the star does have a sharp boundary with the surrounding vacuum.

Use of mass as the independent variable

The above formulae would (in principle) allow theoretical models of stars with a given radius. However, from a theoretical point of view it is the mass of the star which is chosen, the stellar structure equations solved, then the radius (and other parameters) are determined. We observe stellar radii to change by orders of magnitude during stellar evolution, whereas mass appears to remain constant. Hence it is much more useful to rewrite the equations in terms of m rather than r.
We did it before: divide the equations by the equation of mass conservation:

- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$
- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$
- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$
- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma_{SB} r^2}\frac{\rho(r)\kappa_R(r)}{T^3(r)}L(r)$

- $\frac{dr}{dm} = \frac{1}{4\pi r^2 \rho}$
- $\frac{dP}{dm} = -\frac{Gm}{4\pi r^4}$
- $\frac{dL}{dm} = \varepsilon$
- $\frac{dT}{dm} = -\frac{3\kappa_R L}{256 \times \pi^2 \sigma_{SB} r^4 T^3}$

We specify m and the chemical composition and now have a well-defined set of relations to solve. It is possible to do this analytically if simplifying assumptions are made, but in general these need to be solved numerically on a computer.

Stellar evolution (1)

- We have a set of equations that will allow the complete structure of a star to be determined, given a specified mass and chemical composition.
  However, what do these equations not provide us with?

- In deriving the equation for hydrostatic support, we have seen that provided the evolution of star is occurring slowly compared to the dynamical time t_d, we can ignore temporal changes (e.g. pulsations). Indeed, for the Sun $t_d \sim 30$ min, hence this is certainly true.

- And we have also assumed that time dependence can be omitted from the equation of energy generation, if the nuclear timescale (the time for which nuclear reactions can supply the stars energy) is greatly in excess of t_{th}.

Stellar evolution (2)

- If there are no bulk motions in the interior of the star, then any changes of chemical composition are localized in the element of material in which the nuclear reactions occurred. So, the star would have a chemical composition which is a function of mass $m(r)$.
- In the case of no bulk motions – the set of equations we derived must be supplemented by equations describing the rate of change of abundances of the different chemical elements. Let $C_{X,Y,Z}$ be the chemical composition of stellar material in terms of mass fractions of hydrogen (X), helium (Y), and metals (Z)
[e.g., for the Solar system X=0.7, Y=0.28, Z=0.02].

$$\frac{\partial (C_{X,Y,Z})_m}{\partial t} = f(\rho, T, C_{X,Y,Z})$$

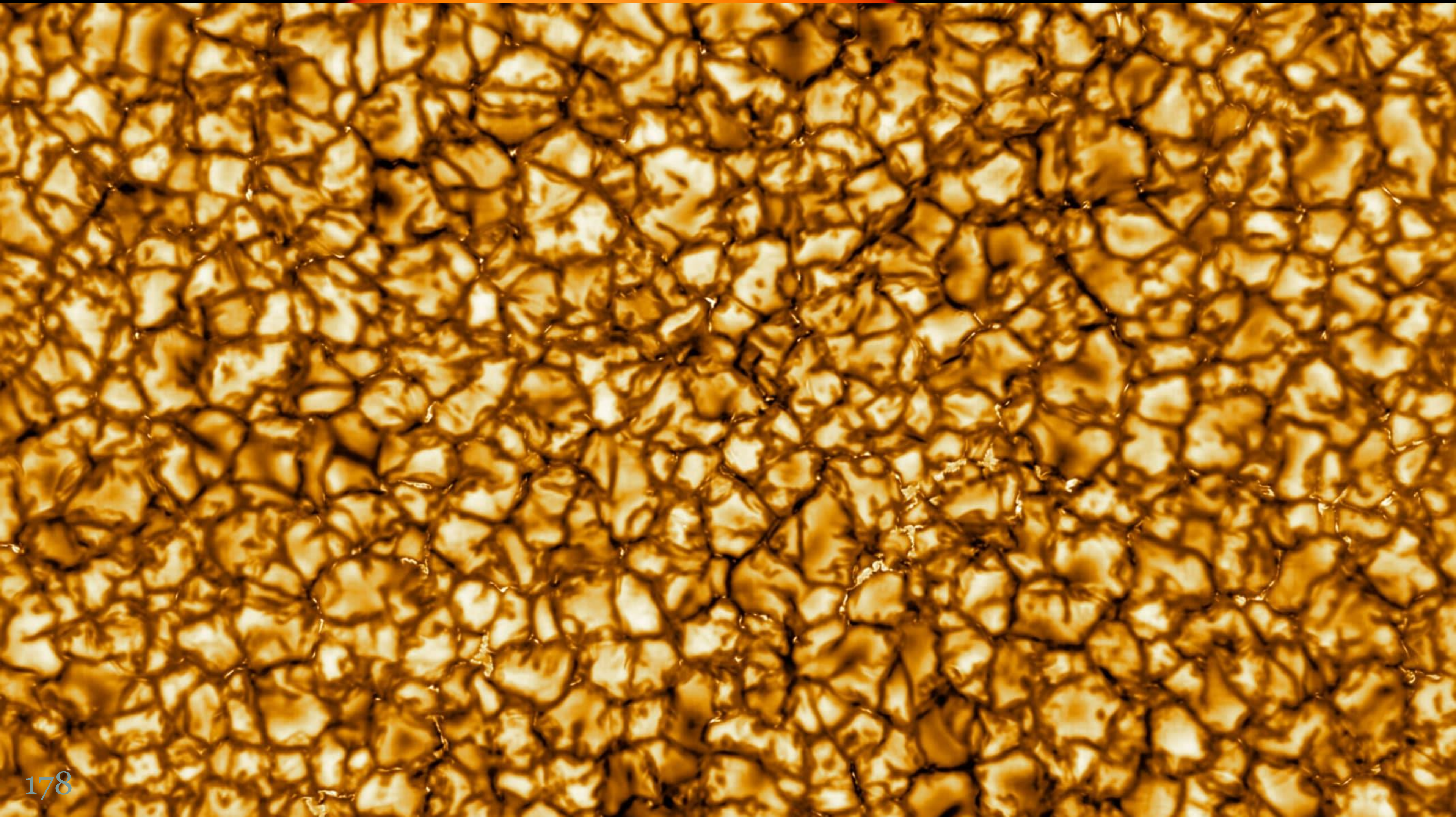
- Now let's consider how we could evolve a model

$$(C_{X,Y,Z})_{m,t_0+\delta t} = (C_{X,Y,Z})_{m,t_0} + \frac{\partial (C_{X,Y,Z})_m}{\partial t}$$

However ...

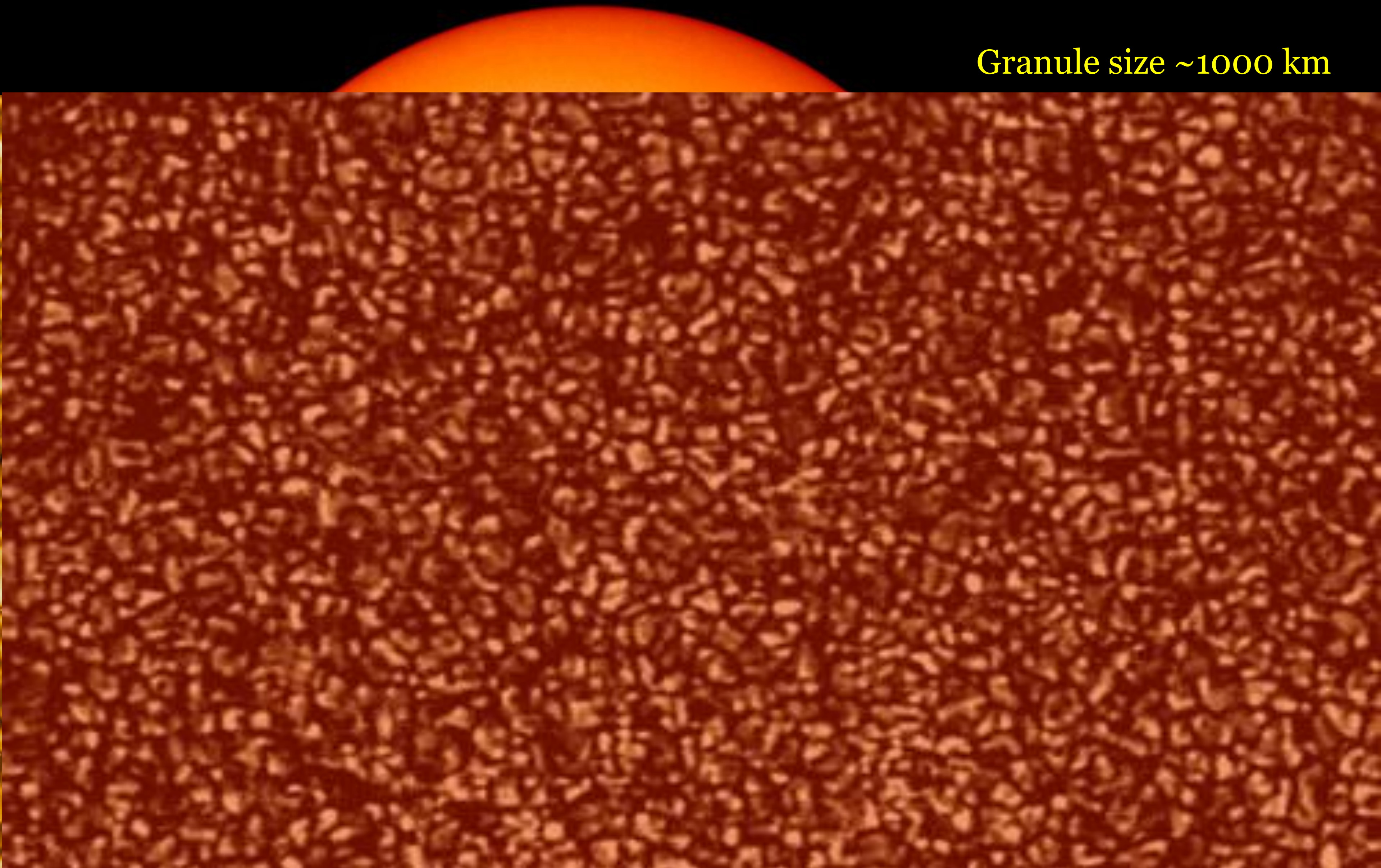
Solar surface

Granule size ~1000 km



Solar surface

Granule size ~1000 km



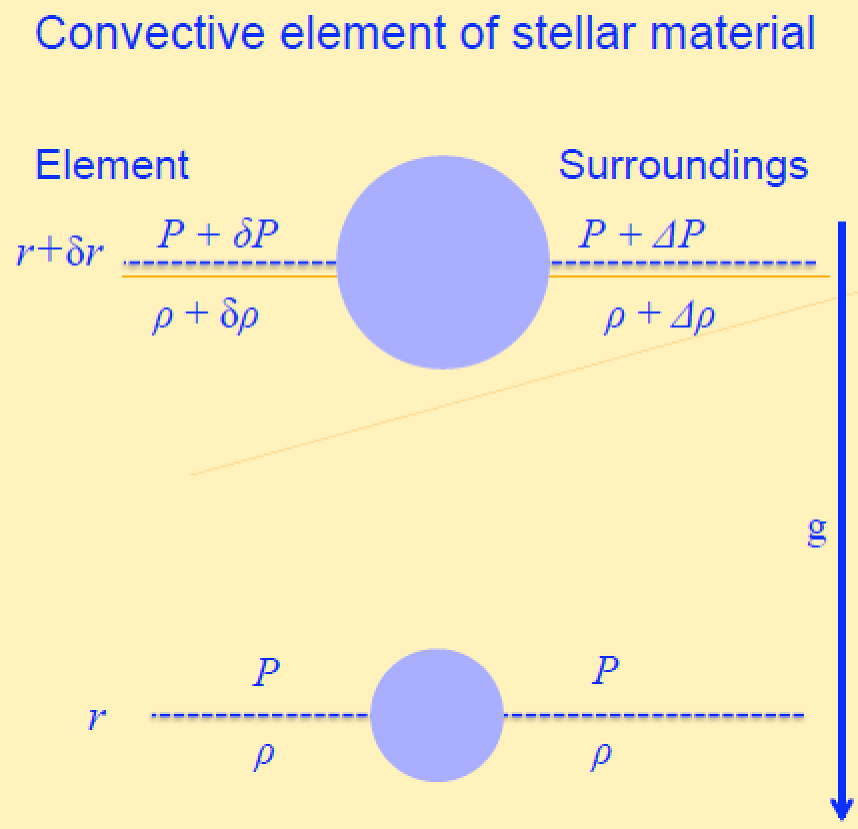
Convection (1)

Convection is the mass motion of gas elements – only occurs when temperature gradient exceeds some critical value.
We can derive an expression for this.

Consider a convective element at distance r from the centre of star. Element is in equilibrium with the surrounding.

Now let's suppose it rises to $r+\delta r$. Element expands to stay in pressure balance with the new environment, $P(r)$ and $\rho(r)$ are reduced to $P + \delta P$ and $\rho + \delta\rho$.

But these may not generally equal the new surrounding gas conditions.
Define those as $P + \Delta P$ and $\rho + \Delta\rho$.



If gas element is denser than surroundings at $r + \delta r$ then will sink (i.e. stable).
If it is less dense then it will keep on rising – ***convectively unstable***.

Convection (2)

The condition for instability is therefore

$$\boxed{\rho + \delta\rho < \rho + \Delta\rho}$$

Whether or not this condition is satisfied depends on two things:

- The rate at which the element expands due to decreasing pressure
- The rate at which the density of the surroundings decreases with height

Let's make two assumptions

1. The element rises adiabatically, i.e. no heat is exchanged with the surrounding;
2. The element rises at a speed much less than the sound speed.
   During motion, sound waves have time to smooth out the pressure differences between the element and the surroundings. Hence $\delta P = \Delta P$ at all times.

The first assumption means that the element must obey the adiabatic relation between pressure and volume

$$PV^{\gamma} = \text{constant}$$

where $\gamma = c_p / c_V$ is the adiabatic index or heat capacity ratio defined as specific heat (i.e. the energy to raise temperature of 1 g of material by 1K) at constant pressure, divided by specific heat at constant volume.

Convection (3)

Given that V is inversely proportional to ρ, we can write

$$\frac{P}{\rho^{\gamma}} = \text{constant}$$

Hence equating the term at r and $r+\delta r$:

$$\frac{P + \delta P}{(\rho + \delta\rho)^{\gamma}} = \frac{P}{\rho^{\gamma}}$$

If $\delta\rho$ is small, we can expand $(\rho + \delta\rho)^{\gamma}$ using the binomial theorem as follows $(\rho + \delta\rho)^{\gamma} \sim \rho^{\gamma} + \gamma\,\delta\rho\,\rho^{\gamma-1}$. Combining last two expressions we obtain

$$\delta\rho = \frac{\rho}{\gamma P}\delta P$$

Now we need to evaluate the change in density of the surroundings, $\Delta\rho$.

Let's consider a very small rise of δr

$$\Delta\rho = \frac{d\rho}{dr}\delta r$$

Convection (4)

$\rho + \delta\rho < \rho + \Delta\rho$

And substituting these expressions for $\delta\rho$ and $\Delta\rho$ into the condition for convective instability derived above:

$$\delta\rho = \frac{\rho}{\gamma P}\delta P \qquad \frac{\rho}{\gamma P}\delta P < \frac{d\rho}{dr}\delta r \qquad \Delta\rho = \frac{d\rho}{dr}\delta r$$

And this can be rewritten by recalling our 2nd assumption that element will remain at the same pressure as its surroundings, so that in the limit

$$\delta r \to 0, \qquad \frac{\delta P}{\delta r} = \frac{dP}{dr}$$

$$\frac{\rho}{\gamma P}\frac{dP}{dr} < \frac{d\rho}{dr}$$

The LHS is the density gradient that would exist in the surroundings if they had an adiabatic relation between density and pressure. RHS is the actual density in the surroundings.

We can convert this to a more useful expression, by first dividing both sides by $\mathrm{d}P/\mathrm{d}r$.
Note that $\mathrm{d}P/\mathrm{d}r$ is negative, hence the inequality sign must change.

Convection (5)

$$\frac{\rho}{\gamma P} > \frac{d\rho}{dr} \Big/ \frac{dP}{dr} \quad \Rightarrow \quad \frac{\rho}{\gamma P} > \frac{d\rho}{dP}$$

$$\left(\frac{P}{\rho}\right)\frac{d\rho}{dP} < \frac{1}{\gamma} \quad \text{or} \quad \frac{d\ln\rho}{d\ln P} < \frac{1}{\gamma}$$

For an ideal gas in which radiation pressure is negligible (where μ is the mean molecular weight of particles in the stellar material in unit of proton mass m_p)

$$P = \frac{\rho k T}{\mu m_p} \quad \Rightarrow \quad \ln P = \ln\rho + \ln T + \text{const}$$

And can differentiate to give

$$\frac{dP}{P} = \frac{d\rho}{\rho} + \frac{dT}{T} \quad \text{or} \quad 1 = \frac{d\ln\rho}{d\ln P} + \frac{d\ln T}{d\ln P}$$

And combining this with the equation above gives ....

Schwarzschild condition for occurrence of convection

$$\frac{P}{T}\frac{dT}{dP} = \frac{d\ln T}{d\ln P} > \frac{\gamma - 1}{\gamma}$$

which is the **Schwarzschild** condition for the occurrence of convection (in terms of the temperature gradient).

A gas is convectively unstable if the actual temperature gradient is steeper than the adiabatic gradient. If the condition is satisfied, then large scale rising and falling motions transport energy upwards.

Condition for occurrence of convection

A gas is convectively unstable if the actual temperature gradient is steeper than the adiabatic gradient. The criterion can be satisfied in two ways:

1. The temperature gradient is very steep

   For example, if a large amount of energy is released at the centre of a star, it may require a large temperature gradient to carry the energy away. Hence where nuclear energy is being released, convection may occur.

2. The ratio of specific heats γ is close to unity

   Alternatively, in the cool outer layers of a star, gas may only be **partially ionized**, hence much of the heat used to raise the temperature of the gas goes into ionization and hence the specific heat of the gas at constant V is nearly the same as the specific heat at constant P (because T~const), and γ~1.

   In such a case, a star can have a **cool outer convective layer**. We will come back to the issues of convective cores and convective outer envelopes later.

Condition for occurrence of convection

Convection is an extremely complicated subject, and it is true to say that the lack of a good theory of convection is one of the worst defects in our present studies of stellar structure and evolution.

We know the conditions under which convection is likely to occur but don't know how much energy is carried by convection.

Fortunately, we will see that we can often find occasions where we can manage without this knowledge.

Influence of convection

Let's back to the equations of stellar structure.

Ideally, we would like to know exactly **how much energy is transported by convection** – but lack of a good theory makes it difficult to predict exactly. Fully self-consistent models of stellar convection are an active area of research and require considerable computational resources to accurately capture the three-dimensional fluid dynamics.

However, it can be shown that **even a very small difference between the actual temperature gradient and adiabatic gradient is sufficient to carry all energy**. This suggests that the actual gradient is not greatly in excess of the adiabatic gradient. We can assume that the temperature gradient has exactly the adiabatic value in a convective region in the interior of a star and hence can rewrite the condition of occurrence of convection in the form

$$\frac{d\ln T}{d\ln P} = \frac{\gamma - 1}{\gamma}$$

Thus, the simplest model of convection is to assume that the process is highly efficient – so much so that it drives the system to saturate the Schwarzschild criterion.

Equations of stellar structure in a convective region

Thus, **in a convective region**, we must solve the four differential equations, together with equations for ε and P:

- $\dfrac{dr}{dm} = \dfrac{1}{4\pi r^2 \rho}$
- $\dfrac{dP}{dm} = -\dfrac{Gm}{4\pi r^4}$
- $\dfrac{dL}{dm} = \varepsilon$
- $\dfrac{P}{T}\dfrac{dT}{dP} = \dfrac{\gamma - 1}{\gamma}$

The equation for luminosity due to radiative transport is still true:

$$L_{rad} = -\frac{256 \times \pi^2\, \sigma_{SB}\, r^4\, T^3}{3\kappa_R}\frac{dT}{dm}$$

And once the other equations have been solved, $L_{\rm rad}$ can be calculated. This can be compared with L (from $dL/dm = \varepsilon$) and the difference gives the value of luminosity due to convective transport $L_{\rm conv} = L - L_{\rm rad}$

In solving the equations of stellar structure, the equations appropriate to a convective region must be switched on whenever the temperature gradient reaches the adiabatic value, and switched off when all energy can be transported by radiation.

Summary

- We have derived the 4th equation to describe the stellar structure and explored the ways to solve these equations.
- As they are not time dependent, we must iterate with the calculation of changing chemical composition to determine short steps in the lifetime of stars. The crucial changing parameter is the H/He content of the stellar core (and afterwards, He burning will become important – to be explored in next lectures).
- We have discussed the boundary conditions applicable to the solution of the equations and made approximations, that do work with real models.
- We have also derived the condition for convection and explored the influence of convection on energy transport within stars. We have shown that it must be considered, but only in areas where the temperature gradient approaches the adiabatic value. In other areas, the energy can be transported by radiation alone and convection is not required. We saw that convection may be important in hot stellar cores and cool outer envelopes, but that a good quantitative theory is lacking.
- The next lectures will explore stellar interiors and the nuclear reactions.

The equations of stellar structure - II

EQUATION OF STATE (EOS)
STELLAR OPACITY

Introduction

- We have 4 differential equations of stellar structure.
- Accurate expressions for pressure, opacity and energy generation are extremely complicated, but we can find simple approximate forms.
- Equations of stellar structure too complicated to find exact analytical solution, hence must be solved with computer.
- Sometimes simplifications can be made to find analytical solutions that still have most of the physics.

The equations of stellar structure

- $$\frac{dm}{dr} = 4\pi r^2 \rho(r)$$
- $$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$
- $$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r)$$
- $$\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma r^2}\frac{\rho(r)\kappa_R(r)}{T^3(r)}L(r)$$
- $$\frac{P}{T}\frac{dT}{dP} = \frac{\gamma - 1}{\gamma}$$

- r = radius
- P = pressure at r
- m = mass of material within r
- ρ = density at r
- L = luminosity at r (rate of energy flow across sphere of radius r)
- T = temperature at r
- κ_R = Rosseland mean opacity at r
- ε = energy release per unit mass per unit time

To these four differential equations we need to add three equations connecting the pressure, the opacity, and the energy production rate of the gas with its density, temperature, and composition:

$P = P$ (ρ, T, chemical composition) $\longrightarrow$ **usually called the equation of state (EOS)**

$\kappa_R = \kappa_R$ (ρ, T, chemical composition)

$\varepsilon = \varepsilon$ (ρ, T, chemical composition)

The equation of state (EOS)

- The equation of state (EOS) describes the microscopic properties of stellar matter for given density ρ, temperature T and composition X_i.
- It is usually expressed as the function that relates the pressure P to ρ, T, and mean molecular weight μ at any place in the star.
- Since it is a solely an internal property of the gas, it can, in principle, be computed once externally, and used via a lookup table, i.e., $P_{\text{gas}} = P(\rho, \mu, T)$.

EOS in stars

- We have seen that stellar gas is ionized plasma, and although density is so high that typical inter-particle spacing is of the order of an atomic radius, the effective particle size is more like a nuclear radius (10^5) times smaller.
- Thus, interior of a star contains a mixture of ions, electrons, and radiation (photons). For most stars (except for very low mass stars and stellar remnants), the ions and electrons can be treated as **an ideal gas** and quantum effects can be neglected.

- The net pressure can be divided into three components, pressure from ions, pressure from electrons, and pressure from radiation.

Total pressure: $P = P_\mathrm{i} + P_\mathrm{e} + P_\mathrm{rad} = P_\mathrm{gas} + P_\mathrm{rad}$

P_i is the pressure of the ions
P_e is the electron pressure
P_rad is the radiation pressure

However, P_gas may not obey the ideal gas law due to the effects of degeneracy.

EOS of an ideal gas

The equation of state for an ideal gas is:

$$P_{\text{gas}} = nkT$$

where n is concentration (number of particles per cm^3 = $n_\text{I} + n_\text{e}$, where n_I and n_e are the number densities of ions and electrons respectively), T is the temperature, k is Boltzmann's constant.

But we want this equation in the form: $P = P\,(\rho,\ T,$ chemical composition$)$

This can be written as:

$$P_{\text{gas}} = \frac{\rho k T}{\mu m_p} = \frac{\Re \rho T}{\mu} \qquad \text{where} \qquad \Re = \frac{k}{m_p}$$ is the gas constant, and

μ = mean molecular weight,
i.e. the average mass of particles in unit of proton mass m_p.

Mean molecular weight (1)

The mean molecular weight μ (the average mass of particles in unit of proton mass m_p) depends upon the composition of the gas and the state of ionization. For example:

- Neutral hydrogen: $\mu = 1$
- Fully ionized hydrogen: $\mu = 0.5$

An exact solution is complex, depending on fractional ionization of all the elements in all parts of the star.

For simplicity, let's now assume that all of the material in the star is fully ionized. This is justified as hydrogen and helium are most abundant and they are certainly fully ionized in stellar interiors (however, this assumption will break down near stellar surface).

Mean molecular weight (2)

Denote abundances of different elements per unit mass by:

X = fraction of material by mass of H

Y = fraction of material by mass of He

Z = fraction of material by mass of all heavier elements ("metals")

$$X + Y + Z = 1$$

Hence in 1 cm^3 of stellar gas of density ρ, there is mass $X\times(\rho$ of H), $Y\times(\rho$ of He), $Z\times(\rho$ of metals). In a fully ionized gas,

H gives 2 particles per m_H

He gives 3/4 particles per m_H (α particle, plus two e^-)

Metals, average mass Am_H, give ~1/2 particles per m_H

(^{12}C has nucleus plus $6e^-$ = 7/12)

(^{16}O has nucleus plus $8e^-$ = 9/16)

where A is the atomic weight of the species.

Mean molecular weight (3)

If the density of the plasma is ρ, then add up number densities of hydrogen, helium, and metal nuclei, plus electrons from each species:

	H	He	metals
Number density of nuclei	$\frac{X\rho}{m_H}$	$\frac{Y\rho}{4m_H}$	$\frac{Z\rho}{Am_H}$
Number density of electrons	$\frac{X\rho}{m_H}$	$\frac{2Y\rho}{4\,m_H}$	$\frac{A}{2} \times \frac{Z\rho}{Am_H}$

The total number of particles per cm^3 is then given by the sum:

$$n = 2\frac{X\rho}{m_H} + \frac{3}{4}\frac{Y\rho}{m_H} + \frac{2Z\rho + \boldsymbol{A}Z\rho}{2\boldsymbol{A}m_H} \approx \frac{\rho}{m_H}\left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right] = \frac{\rho}{\mu\, m_H}$$

...assuming that $A \gg 1$

Thus,

$$\mu = \left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]^{-1}$$

Mean molecular weight (4)

$$\mu = \left[2X + \frac{3}{4}Y + \frac{1}{2}Z\right]^{-1}$$

This is a good approximation to μ except in cool, outer stellar regions.

For solar abundances, X = 0.73, Y = 0.25, Z = 0.02, and therefore $\mu = 0.60$, i.e. the mean mass of particles in a star of solar composition is a little over half the mass of the proton.

In the central regions of the Sun, about half of the hydrogen has already been converted into helium by nuclear reactions, and as a result X = 0.34, Y = 0.64, and Z = 0.02, giving $\mu = 0.85$.

When Z is negligible: Y = 1 − X; $\mu = 4/(3 + 5X)$

The electron number density n_e plays a considerable role for the properties of the gas.
It is convenient to introduce the mean molecular weight per electron, μ_e, such that

$$n_e = \frac{\rho}{\mu_e m_H} \quad \Longrightarrow \quad \mu_e \approx \frac{2}{1 + X}$$

Prove it!

The Ionization Fraction

- The accurate calculation of mean molecular weight μ requires knowledge of the chemical composition of the material and the ionization fraction. To calculate ionization fraction, one needs **the Saha equation**, which we will derive later, in the Stellar atmospheres part of this course :

$$\frac{N_1^+}{N_1} = \frac{2g_1^+}{N_e g_1} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{ion}/kT}$$

where m_e is the mass of the electron, χ_{ion} is the ionization energy, N_1^+ and N_1 are the number density of ions and neutral atoms in their ground state, N_e is the electron number density , g_1^+ and g_1 are the statistical weight of the ground state of the ion and neutral atom.

- In general, the Saha equation can be used to compute ionization fractions over most of the star. It does, however, require that the gas be in the thermodynamic equilibrium. This is true throughout almost the whole star, as at high densities, collisions will control the level populations. This approximation only breaks down in the solar corona, where the densities become very low.
- However, the Saha equation also breaks down in the centers of stars, where high densities cause the ionization energies of atoms to be reduced. Indeed, if the mean distance between atoms is d, then there can be no bound states with radii greater than $\sim d/2$). In practice, the Saha equation begins to break down at nuclear distances of $\sim 10a_0$ ($\sim$10 Bohr radii).
- To correct for this effect, the Saha equation is normally used until it begins to show decreasing ionization fractions toward the center of the star. When this happens, complete ionization is assumed.