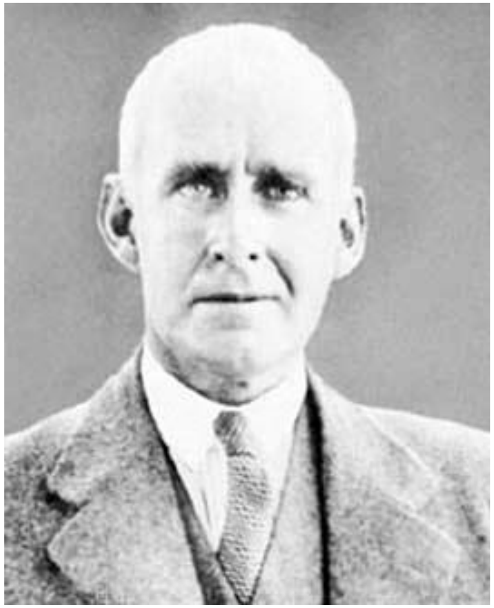
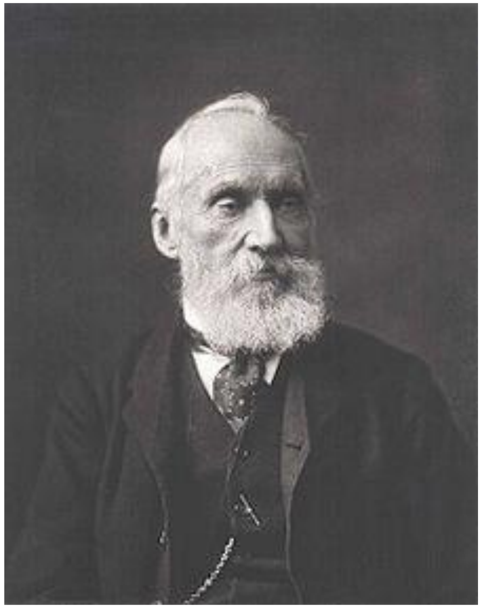
The equations of stellar structure. Energy generation and transport.

THE LIKELY FORM OF ENERGY GENERATION
THE EQUATION OF CONSERVATION OF ENERGY
HOW ENERGY IS TRANSPORTED IN THE SUN:
THE CRITERION FOR CONVECTION



Sir Arthur Stanley Eddington
(1882-1944)

Karl Schwarzschild
(1873-1916)



Lord Kelvin (William Thomson)
(1824-1907)

Energy generation in stars

So far, we have only considered the dynamical properties of the star, and the state of the stellar material. We need to consider **the source of the stellar energy**.

Let's consider the origin of the energy, i.e. the conversion of energy from some form in which it is not immediately available into some form that it can radiate.

How much energy does the Sun need to generate in order to shine with it's measured luminosity ?

$L_\odot = 4\times10^{26}$ W $= 4\times10^{33}$ erg s^{-1}

Sun has not changed flux in 10^9 yr (3×10^{16} s) $\Rightarrow$ Sun has radiated at least 1.2×10^{50} erg

The rest mass energy: $E = mc^2$ $\Rightarrow$ $m_{lost} = 10^{29}$ g $\approx 10^{-4} M_\odot$

What is the source of this energy ?

- Accretion (Mayer)
- Cooling
- Contraction (gravitational; Helmholtz, Lord Kelvin)
- Chemical reactions
- Nuclear reactions

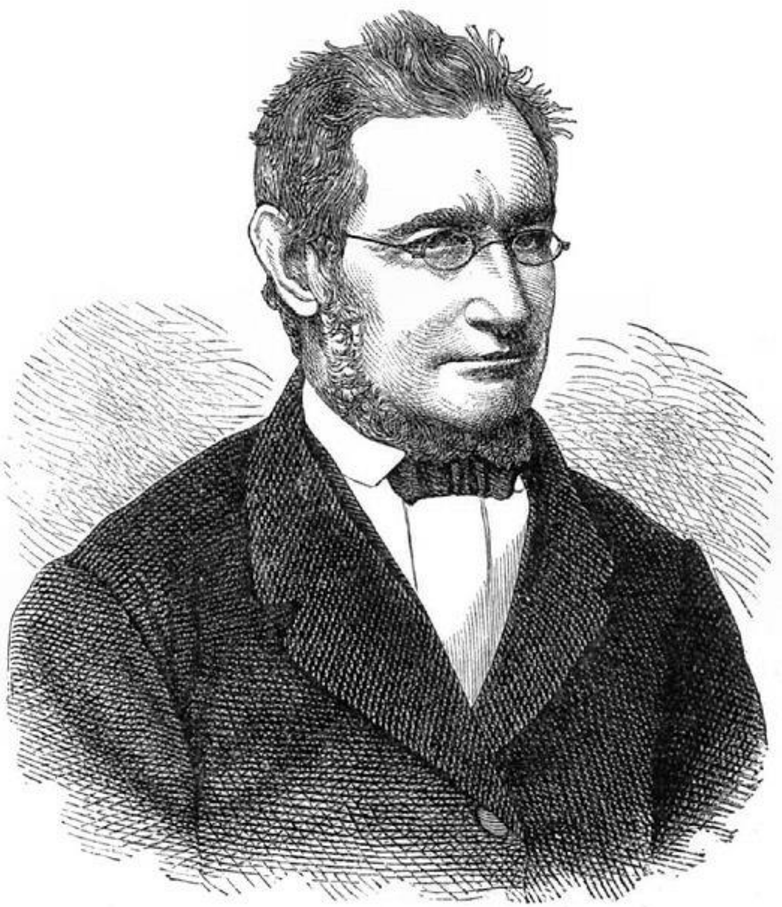
Source of energy generation

Accretion

- From the very beginning it was obvious that one of the energy source of the Sun could be gravitation. Mayer, one of the fathers of the energy conservation law, suggested that kinetic energy of meteorites would keep the Sun hot.

- This idea was not considered seriously. Now, however, we know that the most compact objects in the Galaxy, neutron stars in X-ray binaries, produce their large luminosity in a similar way, by accreting matter from a companion star.

- But for the Sun it cannot be important.



Julius Robert von Mayer
(1814-1878)

Source of energy generation

Cooling and contraction

These are closely related, so we consider them together. Cooling is simplest idea of all. Suppose the radiative energy of the Sun is due to the Sun being much hotter when it was formed and has since been cooling down. We can test how plausible this is.

Or is the Sun slowly contracting with consequent release of gravitational potential energy, which is converted to radiation?

Recently, assuming that stellar material is ideal monatomic gas (negligible P_{rad}), we obtained this form of the Virial theorem: $2E_T + E_G = 0$

> The negative gravitational energy of a star is equal to twice its thermal energy. This means that the time for which the present thermal energy of the Sun can supply its radiation and the time for which the past release of gravitational potential energy could have supplied its present rate of radiation differ by only a factor two.

Total release of gravitational potential energy would have been sufficient to provide radiant energy at a rate given by the luminosity of the star L, for a time equal approximately to the thermal timescale (the time required for a star to radiate all its reservoir of thermal energy):

$$t_K \equiv \frac{E_T}{L} = \frac{GM^2}{2RL}$$

Cooling and contraction (1)

$t_K = \frac{GM^2}{2RL} \approx 1.5 \times 10^7$ yr (for the Sun).

This limit of the age of the Sun obtained in 1862 by Lord Kelvin (William Thomson) was used as an argument against Charles Darwin's (1859) theory of evolution. Darwin's theory required geological time to be much larger, so as to account for the slow evolution of species of plants and animals by natural selection.

In 1899 geologists Thomas C. Chamberlin noted:

"Is present knowledge relative to the behaviour of matter under extraordinary conditions as obtain in the interior of the Sun sufficiently exhaustive to warrant assertion that no unrecognized sources of heat reside there? What is the internal constituents of atoms may be is yet an open question. Is it not improbable that they are complex organizations and the seats of enormous energy?"

Later, Arthur Eddington in *Observatory* (1920) predicts that stellar energy is subatomic: "Only the inertia of tradition keeps the contraction hypothesis alive - or rather, not alive, but an unburied corpse… A star is drawing on some vast reservoir of energy by means unknown to us. This reservoir can scarcely be other than the subatomic energy which, it is known, exists abundantly in all matter… There is sufficient in the Sun to maintain its output for 15 billion years… If, indeed, the subatomic energy in the star is being freely used to maintain their great furnaces, it seems to bring a little nearer to fulfilment our dream of controlling this latent power for the well-being of the human race - or for its suicide."

Cooling and contraction (2)

Now the Earth age 4.6×10^9 years is determined using nuclear physics (from halftime decay of some radioactive elements). However, already in XVIII century Edmond Halley proposed a method of measuring the age from the speed the salt increases in the ocean. In XIX century, geological method were used that measure the time for sedimentation of various materials.

Let us now consider gravitational contraction in other stars.

$$\frac{R}{R_\odot} \propto \left(\frac{M}{M_\odot}\right)^\beta, \qquad \frac{L}{L_\odot} \propto \left(\frac{M}{M_\odot}\right)^l$$

$$t_{th} \sim \frac{GM^2}{RL}$$

$\beta=1, \quad l=4$ for $M<M_\odot$, $E_G \propto M \quad \Rightarrow t_{th} \propto M^{-3}$

$\beta={}^3/_4, l=3{}^1/_4$ for $M>M_\odot$, $E_G \propto M^{5/4} \quad \Rightarrow t_{th} \propto M^{-2}$

Nuclear energy $E_N \propto M$

$E_N / E_G \approx 100$, and varies only by a factor 2 - 3

Cooling and contraction (3)

If the Sun was powered by either contraction or cooling, it would have changed substantially in the last 10 million years. A factor of ~100 too short to account for the constraints on age of the Sun imposed by fossil and geological records.

The Virial theorem: $2E_T + E_G = 0$

The total energy is $E = E_T + E_G = -E_T = E_G / 2 < 0$

If no energy source exist in a star, then its luminosity comes from the decreasing total energy:

$$L = -\dot{E} \quad \Rightarrow \quad L = \dot{E}_T \qquad\qquad as\; L > 0 \quad \Rightarrow \quad \dot{E}_T > 0 \quad \Rightarrow \quad \dot{E}_T \uparrow$$

Thus, a star is heating up when it loses its energy!

Another important consequence: $L = -\dot{E}_G/2$

a star can radiate only 1/2 of its gravitational energy released during contraction.

Source of energy generation

Chemical Reactions

Can quickly rule these out as possible energy sources for the Sun. We calculated above that we need to find a process that can produce at least 10^{-4} of the rest mass energy of the Sun. Chemical reactions such as the combustion of fossil fuels release $\sim 5\times 10^{-10}$ of the rest mass energy of the fuel.

Nuclear Reactions

Hence the only known way of producing sufficiently large amounts of energy is through nuclear reactions. There are two types of nuclear reactions, fission and fusion.

Fission reactions, such as those that occur in nuclear reactors, or atomic weapons can release $\sim 5\times 10^{-4}$ of the rest mass energy through fission of heavy nuclei (uranium or plutonium).

Class task: Can you show that the fusion reactions can release enough energy to feasibly power a star? Assume atomic weight of H=1.008172 and He^4=4.003875, atomic mass unit =$1.66054\mathrm{x}10^{-24}$ g

$L_{\odot} = 4 \times 10^{33}$ erg s^{-1}

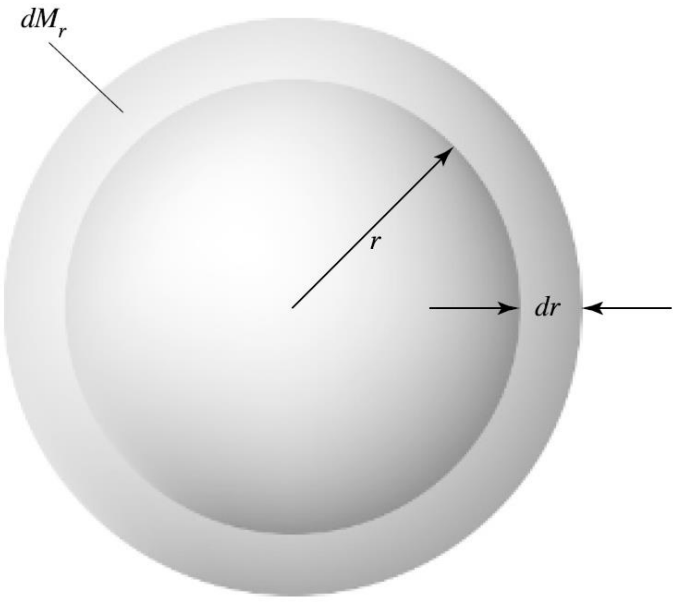
Yes, energy release from fusing one gram of hydrogen to helium is $\sim 6 \times 10^{18}$ erg

Hence, we can see that both fusion and fission could in principle power the Sun. Which is the more likely? As light elements are much more abundant in the solar system than heavy ones, we would expect nuclear fusion to be the dominant source. Given the limits on P and T that we have obtained - are the central conditions suitable for fusion ? We will return to this later.

Equation of energy production (1)

The third equation of stellar structure: relation between energy release and the rate of energy transport.

Consider a spherically symmetric star in which energy transport is radial and in which time variations are unimportant.



$L(r)$=rate of energy flow across sphere of radius r.

$L(r+\mathrm{d}r)$=rate of energy flow across sphere of radius $r+\mathrm{d}r$.

Because shell is thin:

$$dV(r) = 4\pi r^2 dr$$

and

$$dm = dV\rho(r) = 4\pi r^2 \rho(r) dr$$

Equation of energy production (2)

We define ε= energy release per unit mass per unit time (erg s^{-1} g^{-1})
Hence energy release rate in shell is written: $4\pi r^2\rho(r)\varepsilon dr$
Conservation of energy leads us to

$$L(r+dr) = L(r) + 4\pi r^2\rho(r)\varepsilon dr$$

$\Rightarrow$

$$\frac{L(r+dr) - L(r)}{dr} = 4\pi r^2\rho(r)\varepsilon$$

And for $dr \rightarrow 0$:

$$\frac{dL(r)}{dr} = 4\pi r^2\rho(r)\varepsilon$$

This is the **equation of energy production**.

We now have three of the equations of stellar structure. However, we have five unknowns: $P(r), m(r), L(r), \rho(r)$, $\varepsilon(r)$.

In order to make further progress we need to consider energy transport in stars.

Method of energy transport

There are three ways energy can be transported in stars:

- **Conduction** – by exchange of energy during collisions of gas particles (usually e^-)
- **Radiation** – energy transport by the emission and absorption of photons
- **Convection** – energy transport by mass motions of the gas

Conduction and radiation are similar processes – they both involve transfer of energy by direct interaction, either between particles or between photons and particles.

Which is the more dominant in stars ?

Energy carried by a typical particle $\sim 3kT/2$ is comparable to energy carried by typical photon $\sim h\nu$

But number density of particles is much greater than that of photons. This would imply conduction is more important than radiation, but…

Mean free path of photon $l_{ph} \sim 1/(n\sigma_T) \sim 1$ cm

Mean free path of particle $l_p \sim 1/(n\sigma_o) \sim 10^{-8}$ cm

n is particle concentration in cm^{-3}, $\sigma_T \sim (8\pi/3) r_e^2 \sim 7\text{x}10^{-25}$ cm^2, $\sigma_o \sim \pi a_0^2 \sim 10^{-16}$ cm^2.

$a_0 = 5\text{x}10^{-9}$ cm is Bohr radius, $r_e = 3\text{x}10^{-13}$ cm is the classical electron radius.

Photons can move across temperature gradients more easily, hence larger transport of energy.

Conduction is negligible, radiation transport in dominant.

The equation of radiative transport

We will derive an expression relating the change in temperature with radius in a star assuming all energy is transported by radiation. Hence, for now, we ignore the effects of convection and conduction which we will discuss later.

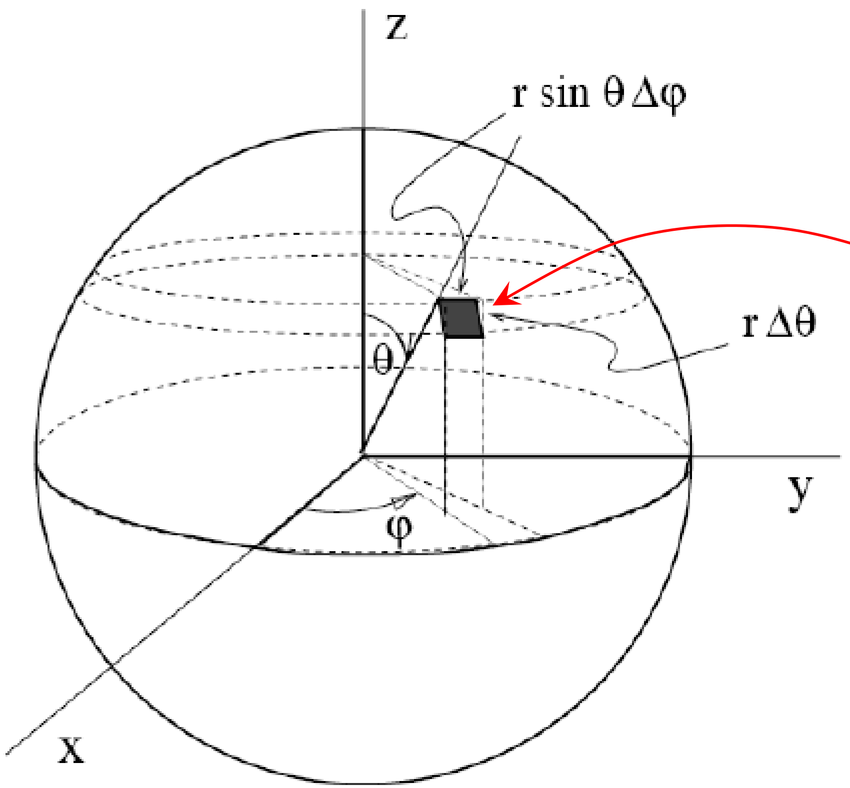
The radiative transfer equation describes how the physical properties of the material are coupled to the radiation spectrum.

First of all, we must have a carefully defined terminology to properly describe light and its interaction with the material in stellar interiors and atmospheres.

Basics about radiative transfer

RADIATION TERMS
SPECIFIC INTENSITY
CHANGE OF INTENSITY ALONG PATH ELEMENT
ABSORPTION AND EMISSION COEFFICIENTS
OPTICAL DEPTH, SOURCE FUNCTION
RADIATIVE TRANSFER EQUATION

Definition of solid angle and steradian



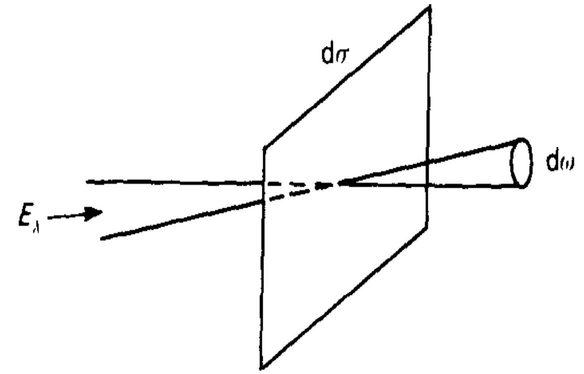
Put the polar axis along the star's radius.
Area of a patch dS on a sphere limited by (θ, $\theta+\Delta\theta$) and (φ, $\varphi+\Delta\varphi$) is
dS = $r \sin\theta \; r\, \Delta\theta\, \Delta\varphi \equiv r^2\, d\omega$

dω is the solid angle subtended by the area dS at the center of the sphere.
For a unit radius sphere:
dω = $\sin\theta \, d\theta \, d\varphi$

Unit of solid angle is the steradian. 4π steradians cover whole sphere.

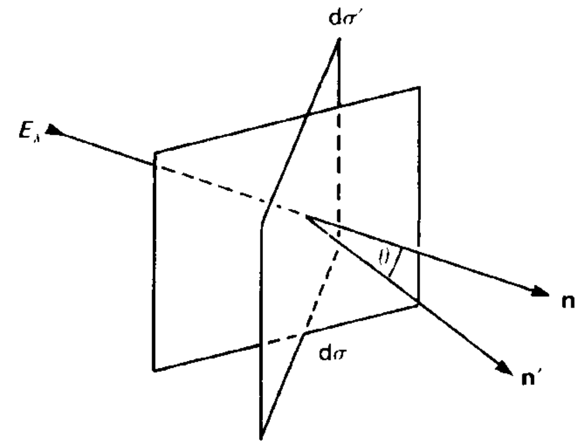
Specific Intensity (1)

Consider light passing through a *perpendicular* surface area dσ in a narrow cone of opening solid angle dω. The amount of energy E_λ passing through this area per second is given by



$$E_\lambda = I_\lambda \, d\lambda \, d\sigma \, d\omega \, dt$$

Now consider the energy passing through a surface area dσ at an angle θ with respect to the normal of this surface area, the effective beam width is reduced by cos(θ):



$$E_\lambda = I_\lambda \cos\theta \, d\lambda \, d\sigma \, d\omega \, dt$$

Specific intensity of the radiation

Specific Intensity (2)

Another, more intuitive name for the specific intensity is brightness.

$$I_\lambda = \frac{E_\lambda}{\cos\theta \, d\lambda \, d\sigma \, d\omega \, dt}$$

The (specific) intensity $\boldsymbol{I_\lambda}$ is then a measure of brightness with units of erg s^{-1} cm^{-2} $steradian^{-1}$ $Å^{-1}$

(or erg s^{-1} cm^{-2} $steradian^{-1}$ Hz^{-1}).

Bolometric Intensity

- One can alternatively define intensity in frequency units such that

$$I_\lambda d\lambda = I_\nu d\nu$$

- Note that $I_\lambda \neq I_\nu$! The two spectral distributions have different shapes for the same spectrum. The Solar spectrum has a maximum in the green in I_λ (5175Å), but for I_ν the maximum is in the far-red (8800Å).

Why?

$$c=\lambda\nu, \; d\nu/d\lambda=-c/\lambda^2$$

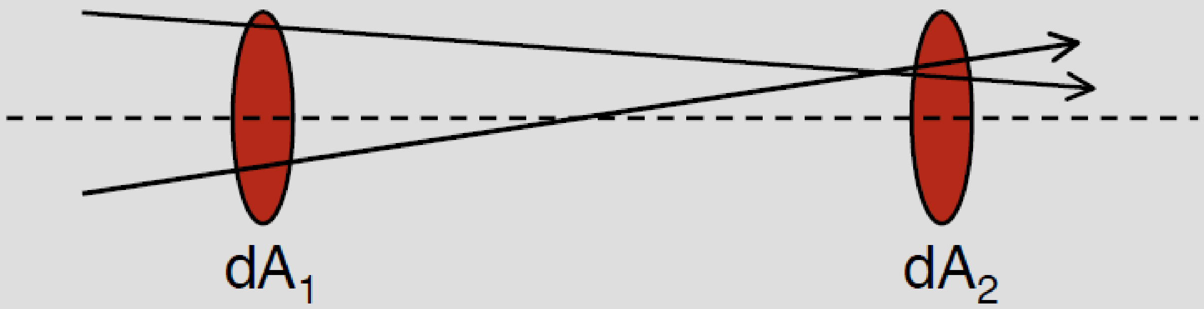
so equal intervals of λ correspond to different intervals of ν across the spectrum.

- Integrated (bolometric) intensity is

$$I = \int_0^\infty I_\lambda \, \mathrm{d}\lambda = \int_0^\infty I_\nu \, \mathrm{d}\nu$$

How does specific intensity change along a ray?

If there is no emission or absorption, specific intensity is just constant along the path of a light ray. Consider any two points along a ray, and construct areas dA_1 and dA_2 normal to the ray at those points. How much energy is carried by those rays that pass through both dA_1 and dA_2?



$$\left.\begin{aligned} E_1 &= I_{\lambda,1} d\lambda_1 dA_1\, d\omega_1\, dt \\ E_2 &= I_{\lambda,2} d\lambda_2 dA_2\, d\omega_2\, dt \end{aligned}\right\}$$ where $d\omega_1$ is the solid angle subtended by dA_2 at dA_1 etc

How does specific intensity change along a ray?

The same photons pass through both dA_1 and dA_2, without change in their frequency. Conservation of energy gives:

$E_1 = E_2$ - equal energy

$d\lambda_1 = d\lambda_2$ - same wavelength interval

Using definition of solid angle, if dA_1 is separated from dA_2 by distance r:

$$d\omega_1 = \frac{dA_2}{r^2}, \qquad d\omega_2 = \frac{dA_1}{r^2}$$

Substitute:

$$\left.\begin{aligned} I_{\lambda,1} d\lambda_1 dA_1 dt\, d\omega_1 &= I_{\lambda,2} d\lambda_2 dA_2 dt\, d\omega_2 \\ I_{\lambda,1} d\lambda_1 dA_1 dt \frac{dA_2}{r^2} &= I_{\lambda,2} d\lambda_2 dA_2 dt \frac{dA_1}{r^2} \end{aligned}\right\} \quad \begin{aligned} E_1 &= E_2 \\ d\lambda_1 &= d\lambda_2 \end{aligned}$$

$$\boldsymbol{I_{\lambda,1} = I_{\lambda,2}}$$

Specific Intensity (3)

- Thus, specific intensity I_λ is ***independent of distance*** from the source, it remains the same as radiation propagates through **free** space.
- Justifies use of alternative term '**brightness**' - e.g. brightness of the disk of a star remains same no matter the distance - flux goes down but this is compensated by the light coming from a smaller area.
- If we measure the distance along a ray by variable s, then we can express result equivalently in differential form:

$$\frac{dI_\lambda}{ds} = 0$$

- Specific intensity can only be measured **directly** if we **resolve** the radiating surface (e.g. Sun, nebulae, planets).