The Virial theorem (1)

Let's again take the hydrostatic equilibrium equation, in which enclosed mass m is used as the dependent variable (or combine the equation of hydrostatic equilibrium with the equation of mass conservation):

$$\frac{dP(r)}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dP(r)}{dr} \times \frac{dr}{dm} = -\frac{Gm}{r^2}\rho \times \frac{1}{4\pi r^2 \rho}$$

$$\frac{dmM}{dr} = 4\pi r^2 \rho(r)$$

Now multiply both sides by volume V=(4/3)πr^3:

$$3V(r)dP = -\frac{Gm}{r}dm$$

And integrate over the whole star:

$$3\int_{P_c}^{P_s} V(r)dP = -\int_0^M \frac{Gm}{r}dm$$

integrating by parts

$$3[PV]_c^s - 3\int_{V_c}^{V_s} PdV = -\int_0^M \frac{Gm}{r}dm$$

At centre, V_c=0 and at surface P_s=0

The Virial theorem (2)

Hence, we have

$$3\int_0^V P\,dV = \int_0^M \frac{Gm}{r}\,dm = -E_G$$

Now the right-hand term = – total gravitational binding energy of the star,
or it is the energy needed to spread the star to infinity,
or to assemble the star by bringing gas from infinity.

$$3\int_0^V P\,dV = -E_G$$

version of the virial theorem

The left-hand side contains pressure integral. With some assumptions about the pressure, we can progress further.

The Virial theorem (3)

$$3\int_0^V P dV = -E_G$$

For **ideal** gas, $P = NkT$,

where N is concentration, T is the temperature, k is Boltzmann's constant,

while the thermal (kinetic) energy per particle is $e_{kin} = \frac{3}{2}NkT$

Thus, the LHS is

$$3\int_0^V P dV = 2\int_0^V e_{kin} dV = 2E_T$$

where E_T is the thermal energy of the star.

Thus, we can write the Virial Theorem:

or for the total energy $E = E_T + E_G$:

$$2E_T + E_G = 0$$

$$E = -E_T$$

This is of great importance in astrophysics and has many applications.

Timescales of stellar evolution (1)

1. Dynamical time scale

Measure of the time scale on which a star would expand or contract if the balance between pressure gradients and gravity was suddenly disrupted (same as free-fall time scale).

Previously we obtained a generalized form of the equation of hydrostatic support, and then assuming a non-zero inward acceleration to be $a = \beta g$, we obtained the spatial displacement d after a time t: $d = \frac{1}{2}\beta g t^2$

$$\frac{dP(r)}{dr} + g\rho(r) = a\rho(r)$$

$$t = \left(\frac{2d}{\beta g}\right)^{1/2}$$

Now, if we allow the star to collapse, i.e. set $\beta \approx 1$ and $d = R$ and substitute $g = GM/r^2$

$$t_{dyn} = \left(\frac{2R^3}{GM}\right)^{1/2}$$

t_{dyn} is known as the dynamical time.

The dynamical timescale (1)

We can get a better estimation if assume that the whole mass is concentrated in the centre. The equation of motion is

$$\ddot{r} + \frac{GM}{r^2} = 0$$

At home you will show that the time for collapse from radius R to 0 is

$$t_{dyn} = \frac{\pi}{2\sqrt{2}}\left(\frac{R^3}{GM}\right)^{1/2}$$

One can express that through mean density as: $t_{dyn} = \left(\frac{3\pi}{32}\right)^{1/2} \frac{1}{\sqrt{G\overline{\rho}}}$

For different radii, we get

$$t_{dyn} = \frac{\pi}{2\sqrt{2}}\left(\frac{r^3}{Gm}\right)^{1/2} = \left(\frac{3\pi}{32}\right)^{1/2} \frac{1}{\sqrt{G\overline{\rho_r}}}$$

where $m = m(r)$ is the mass interior to r, $\overline{\rho_r}$ is the mean density in sphere of radius r.

The dynamical timescale (2)

$$t_{dyn} = \frac{\pi}{2\sqrt{2}}\left(\frac{r^3}{Gm}\right)^{1/2} = \left(\frac{3\pi}{32}\right)^{1/2}\frac{1}{\sqrt{G\overline{\rho_r}}}$$

$\overline{\rho_r}$ is the mean density in sphere of radius r.

We see that density decreasing with radius, $\overline{\rho_r}$ also decreases and time-scale grows.

Thus, shell at larger radii falls down longer.

This also confirmed our assumption of the whole mass concentrated inside.

For the Sun $R_\odot = 6.96\text{x}10^{10}$ cm, $M_\odot = 1.99\text{x}10^{33}$ g: $t_{\text{dyn}} = 1770 \text{ sec} \approx 0.5 \text{ hour}$

$\bar{\rho} = 1.4 \text{ g cm}^{-3}$

Timescales of stellar evolution (2)

2. Thermal time scale (Kelvin-Helmholtz time scale)

Suppose nuclear reaction were suddenly cut off in the Sun.

Thermal time scale is the time required for the Sun to radiate all its reservoir of thermal energy:

$$t_K \equiv \frac{E_T}{L} = \frac{GM^2}{2RL} \approx 1.5 \times 10^7 \text{ yr (for the Sun)}$$

Virial theorem: the thermal energy E_T is roughly equal to half the gravitational potential energy

Important timescale: determines how quickly a star contracts before nuclear fusion starts - i.e. sets roughly the pre-main sequence lifetime.

Timescales of stellar evolution (3)

3. Nuclear time scale

Time scale on which the star will exhaust its supply of nuclear fuel if it keeps burning it at the current rate.

Energy release from fusing one gram of hydrogen to helium is ~6 x 10^{18} erg, so:

$$t_n = \frac{qXM \times 6 \times 10^{18}}{L} \text{erg g}^{-1}$$

where

- X is the mass fraction of hydrogen initially present (X=0.7)
- q is the fraction of fuel available to burn in the core (q=0.1)

$$t_n \approx 7 \times 10^9 \text{ yr (for the Sun)}$$

Reasonable estimate of the main-sequence lifetime of the Sun.

Stellar timescales

Ordering time scales:

$$t_{dyn} \ll t_K \ll t_n$$

For the Sun:

$t_{\text{dyn}} = 30$ min
$t_{\text{K}} = 15$ million years
$t_{\text{n}} = 7$ billion years

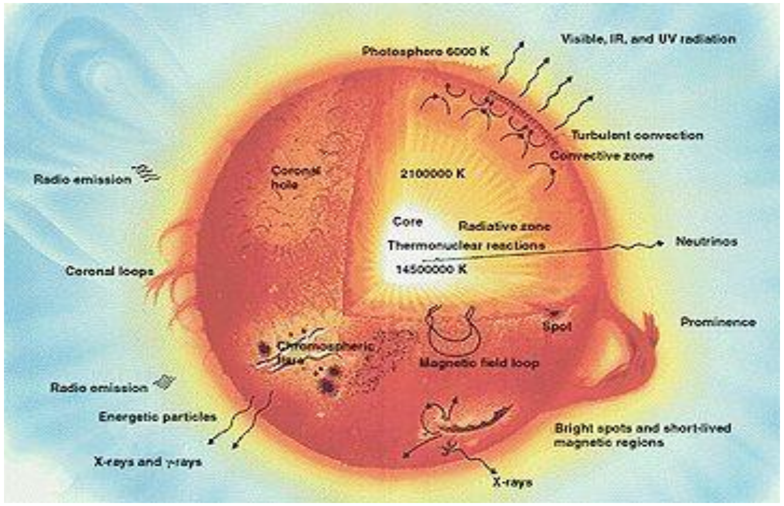
Most stars, most of the time, are in hydrostatic and thermal equilibrium, with slow changes in structure and composition occurring on the (long) time scale t_{n} as fusion occurs.

Do observe evolution on the shorter time scales also:

- Dynamical - stellar collapse / supernova
- Thermal / Kelvin-Helmholtz - pre-main-sequence

Conditions in stellar interiors

MINIMUM VALUE FOR CENTRAL PRESSURE OF A STAR
MINIMUM MEAN TEMPERATURE OF A STAR
STATE OF STELLAR MATERIAL



Content

Let us consider several applications of our current knowledge. We will derive mathematical formulae for the following

1. Minimum value for central pressure of a star
2. Minimum mean temperature of a star
3. State of stellar material

In doing this you will learn important assumptions and approximations that allow the values for minimum central pressure, mean temperature and the physical state of stellar material to be derived.

Minimum value for central pressure of star (1)

We have only 2 of the 4 equations, and no knowledge yet of material composition or physical state. But we can deduce a minimum central pressure.

Given what we know, **what is this likely to depend upon?**

Let's again take the hydrostatic equilibrium equation, in which enclosed mass m is used as the dependent variable (or combine the equation of hydrostatic equilibrium with the equation of mass conservation):

$$\frac{dP(r)}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dP(r)}{dr} \times \frac{dr}{dm} = -\frac{Gm}{r^2}\rho \times \frac{1}{4\pi r^2 \rho}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

Can integrate this over the whole star to give:

$$P_c - P_s = P_c = \int_0^M \frac{Gm(r)}{4\pi r^4}\, dm(r)$$

The integration requires functional forms of *m(r)*. Unfortunately, such explicit expression is not available.

However, replacing r by the stellar radius $R \geq r$, we obtain a lower limit for the central pressure:

$$P_c \geq \int_0^M \frac{Gm}{4\pi R^4}\, dm = \frac{GM^2}{8\pi R^4}$$

$$\boxed{P_c \geq \frac{GM^2}{8\pi R^4}}$$

Minimum value for central pressure of star (2)

We can improve the lower limit making a natural assumption that density does not increase towards the surface.

Define the average density within sphere of radius r as $\overline{\rho_r} \equiv m/\left(\frac{4\pi}{3} r^3\right)$

$$P_c = \int_0^M \frac{Gm(r)}{4\pi r^4} dm(r) = \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} G \int_0^M \overline{\rho_r}^{\,4/3} m^{-1/3}\, dm$$

For density **not increasing outwards** $\overline{\rho_r} \geq \bar{\rho} \equiv \overline{\rho_R}$, we get

$$P_c \geq \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} \bar{\rho}^{\,4/3} G \int_0^M m^{-1/3}\, dm = \left(\frac{\pi}{6}\right)^{1/3} G\bar{\rho}^{\,4/3} M^{2/3}$$

$$\boxed{P_c \geq \frac{3}{8\pi}\frac{GM^2}{R^4}}$$

Maximum value for central pressure of star

An upper limit on central pressure can be obtained just assuming $\rho_c \geq \overline{\rho_r}$ which is e.g., valid when density is largest in the centre:

$$P_c = \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} G \int_0^M \overline{\rho_r}^{\,4/3} m^{-1/3}\, dm \leq \frac{1}{3}\left(\frac{4\pi}{3}\right)^{1/3} \rho_c^{4/3} G \int_0^M m^{-1/3}\, dm$$

We can write it as

$$P_c \leq \frac{3}{8\pi}\frac{GM^2}{R_c^4}$$

where R_c is the radius of the star with mass M and density ρ_c defined by $\frac{4\pi}{3} R_c^3 \rho_c = M$

Thus, we get

$$\left(\frac{\pi}{6}\right)^{1/3} GM^{2/3}\bar{\rho}^{\,4/3} \leq P_c \leq \left(\frac{\pi}{6}\right)^{1/3} GM^{2/3}\rho_c^{4/3}$$

which is valid for $\rho_c \geq \overline{\rho_r} \geq \bar{\rho}$

Central pressure in stars

$$P_c \geq \frac{3}{8\pi}\frac{GM^2}{R^4}$$

$$P_c \geq \frac{3}{8\pi}\frac{GM^2}{R^4} = 1.35 \times 10^{15} \frac{(M/M_\odot)^2}{(R/R_\odot)^4}\frac{\text{dyn}}{\text{cm}^2} = 1.33 \times 10^9 \frac{(M/M_\odot)^2}{(R/R_\odot)^4} \text{atm}$$

1 dyn=1 g·cm/s^2 1 dyn/cm^2 = 0.1 Pa = 9.8692×10^{-7} atm

For stars at main sequence (MS): $R \propto M^\beta$, where $\beta = (0.5 \div 1)$

- for low-mass stars: $\beta \approx 1$
- for masses above solar: $\beta \approx 2/3$
- for very high masses: $\beta \approx 0.5$

$$P_c \geq 1.3 \times 10^9 (M/M_\odot)^{2-4\beta} \text{ atm}$$

P_c strongly depends on the stellar structure and in reality, always much larger than simple estimates above. The central pressure in MS stars is about $10^{10} \div 10^{11}$ atm.

This seems rather large for gaseous material – we shall see that this is not an ordinary gas. This is huge pressure by Earth standards where experiments have reached only about 10^6 atm. But even those pressures are small compared to that inside white dwarfs (10^{19} atm), or neutron stars ($10^{29} \div 10^{30}$ atm).

Calculate P_c for the Sun

Minimum mean temperature of a star (1)

We have seen that pressure, P, is an important term in the equation of hydrostatic equilibrium and the virial theorem. We have derived a minimum value for the central pressure ($P_c > 10^9$ atmospheres).

What physical processes give rise to this pressure – which are the most important?

- Gas pressure P_g
- Radiation pressure P_{rad}
- We shall show that P_{rad} is negligible in majority of stellar interiors and pressure is dominated by P_g

To do this we first need to estimate the minimum mean temperature of a star.

Consider the gravitational binding energy:

$$-E_G = \int_0^M \frac{Gm}{r} dm = e_G \frac{GM^2}{R} \quad \text{where } e_G = \int_0^1 \frac{q}{x} dq$$

The dimensionless gravitational energy

and $q=m/M$, $x=r/R$

Minimum mean temperature of a star (2)

We can obtain a lower bound on e_G by noting: at all points inside the star x<1 and hence

The dimensionless gravitational energy $$e_G = \int_0^1 \frac{q}{x} dq \geq \int_0^1 q\, dq = 1/2$$

For a constant density sphere one can get: $q = m/M = (r/R)^3 = x^3 \Rightarrow e_G = 3/5$

And for density decreasing outwards $e_G \geq 3/5$ because one needs to move some mass towards centre which releases gravitational energy. For the Sun $e_G = 1.62$. Now, $dm = \rho dV$ and the *virial theorem* can be written

$$-E_G = 3\int_0^V P dV = 3\int_0^M \frac{P}{\rho} dm$$

Pressure is sum of radiation pressure and gas pressure: $P = P_g + P_r$

Assume, for now, that stars are composed of ideal gas with negligible P_r

Then, the equation of state of ideal gas:

$$P = NkT = \frac{\rho}{\mu m_p} kT$$

where N is concentration (number of particles per cm^3), T is the temperature, k is Boltzmann's constant, μ = mean molecular weight, i.e. the average mass of particles in unit of proton mass m_p.

Minimum mean temperature of a star (3)

Hence, we have

$$-E_G = 3\int_0^M \frac{P}{\rho}dm = 3\int_0^M \frac{kT}{\mu m_p}dm = e_G\frac{GM^2}{R}$$

Assuming chemically homogeneous star, μ=const,
and defining the average over mass temperature

$$\bar{T} \equiv \frac{1}{M}\int_0^M T dm$$

we get the mean temperature of the star

$$\bar{T} = \frac{e_G}{3}\frac{\mu m_p}{k}\frac{GM}{R}$$

This temperature is called virial temperature for obvious reason.
For density not increasing outwards, $e_G \geq 3/5$, and therefore

$$\bar{T} \geq \frac{1}{5}\frac{\mu m_p}{k}\frac{GM}{R}$$

Minimum mean temperature: Example

- As an example, for a chemically homogeneous star, we have

$$\bar{T} = 7.7 \times 10^6 e_G \mu \frac{M/M_\odot}{R/R_\odot}\ \mathrm{K} \qquad \overline{kT} = 660 e_G \mu \frac{M/M_\odot}{R/R_\odot}\ \mathrm{eV}$$

- We know that H is the most abundant element in stars and for a fully ionized hydrogen star $\mu=1/2$ (as there are two particles, p + e$^-$, for each H atom). And for any other element μ is greater => $T_\odot > 2.3\times10^6$ K.
- Also, the average kinetic energy of particles at $\overline{T_\odot}$ is much higher than the ionization potential of H (13.6 eV) or for double ionization of He (13.6x2^2=54 eV). Thus, the gas must be highly ionized, i.e. is a plasma. As e_G is actually larger than 1, temperatures in the stellar interiors are $(1\div3)\times10^7$ K, which corresponds to the energies about 1-3 keV.

- Mean density of the Sun is higher than water and other ordinary liquids.
However, at such a temperature the gas is ionized. An ideal gas demands that the distances between the particles are much greater than their sizes, and nuclear dimension is 10^{-13} cm compared to atomic dimension of 10^{-8} cm (which would be of interest in neutral gas). It can thus withstand greater compression without deviating from an ideal gas.

Temperature in stellar interiors

- The dimensionless gravitational energy e_G is not very sensitive to the stellar structure and for MS stars it is about unity. The chemical composition is about the same for all MS stars and $\bar{T} \propto {}^{M}/_{R}$
- Because $R \propto M^{\beta}$, where $\beta = (0.5 \div 1)$
  the mean temperatures should decrease for smaller masses and change only by a factor of a few between the lower and upper end of stellar masses.
  However, variation in luminosities are much larger than variation in mass: $10^{10} \div 10^{11}$ and 10^3 times, respectively. As temperature varies only very slightly, we have to conclude that the energy production rate is a very strong function of temperature.
  This conclusion is supported by studies of nuclear reactions.

- The masses of red giants (RG) are about solar, but luminosities are large, therefore their temperatures should be at least as high as for the Sun. But their radii are 100 x solar R. We are forces to conclude that $e_G > 100$, which can be achieved by strong concentration of matter towards the centre. Thus, the RG structure should be very much different from MS stars. A large fraction of RG mass should be contained in its core (a white dwarf “under construction”).

Physical state of stellar material

Let us revisit the issue of radiation vs gas pressure. We assumed that the radiation pressure was negligible. The pressure exerted by photons on the particles in a gas is:

$$P_{rad} = \frac{aT^4}{3}$$

where $a = \frac{4\sigma_{SB}}{c} = 7.56 \times 10^{-15}$ erg cm^{-3} K^{-4} = radiation density constant

Now compare gas and radiation pressure at a typical point in the Sun:

$$\frac{P_{rad}}{P_g} = \frac{aT^4}{3} \Big/ \frac{kT\rho}{\mu m_p} = \frac{\mu m_p a T^3}{3k\rho}$$

Taking $T \sim \bar{T} = 2 \times 10^6$ K, $\rho \sim \bar{\rho} = 1.4$ g cm^{-3}, and $\mu m_P = \frac{1}{2} 1.67 \times 10^{-24}$ g gives

$$\frac{P_{rad}}{P_g} = 10^{-4}$$

Hence radiation pressure appears to be negligible at a typical (average) point in the Sun. In summary, with no knowledge of how energy is generated in stars we have been able to derive a value for the Sun's internal temperature and deduce that it is composed of a near ideal gas plasma with negligible radiation pressure.

Mass dependency of radiation to gas pressure

However, we shall later see that P_{rad} becomes significant in higher mass stars. To give a basic idea of this dependency: replace ρ in the ratio equation above:

$$\frac{P_{rad}}{P_g} = \frac{\mu m_p a T^3}{3k\rho} = \frac{\mu m_p a T^3}{3k\left(\frac{3M}{4\pi R^3}\right)} = \frac{4\pi\mu m_p a}{9k}\frac{R^3 T^3}{M}$$

And from the virial theorem: $\bar{T} \propto \frac{M}{R}$

$$\Rightarrow \qquad \frac{P_{rad}}{P_g} \propto M^2$$

i.e. P_{rad} becomes more significant in higher mass stars.

This is one of the reasons why there are no stars of very high masses >100M$_\odot$

Summary

For our stars – which are isolated, static, and spherically symmetric –there are four basic equations to describe structure. All physical quantities depend on the distance from the centre of the star alone:

1. Equation of hydrostatic equilibrium: at each radius, forces due to pressure differences balance gravity
2. Conservation of mass
3. Conservation of energy : at each radius, the change in the energy flux = local rate of energy release
4. Equation of energy transport : relation between the energy flux and the local gradient of temperature

These basic equations supplemented with

- Equation of state (pressure of a gas as a function of its density and temperature)
- Opacity (how opaque the gas is to the radiation field)
- Core nuclear energy generation rate

With only two of the four equations of stellar structure, we have derived important relations for P_c and mean T.

We have used the *Virial theorem* – this is an important formula and concept in this course and astrophysics in general. You should be comfortable with the derivation and application of this theorem.

We were able to make interesting conclusions about the energy dissipation rate dependence on temperature, structure of red giants, and the role of radiation pressure.