Astrophysics

Compulsory Home Exercises.
Solutions to Problem Set 5.

1. In terms of the specific intensity I, which of the listed below is the correct expression for the amount of radiation flowing per unit time, per unit solid angle through a unit area at an angle to the normal?

   (a) $I \sin\theta$
   (b) $I\ \theta$
   (c) $I \cos\theta$ [Correct answer]
   (d) $I \cos\theta\ \sin\theta$
   (e) I

2. Which 2 opacity sources do dominate in a stellar atmosphere (T_{eff}=8064 K, P_e=30 dyn/cm^2) at 5000 Å and 18000 Å? Is the contribution of the second one is negligible enough to be not taken into account?

   Solution:
   From lecture 21 (e.g., slides 15 and 16) we can see that at 5000 Å the two dominant comparable sources of opacity are H (bound-free) and the H^- ion (bound-free). At 18000 Å the dominant source is H (bound-free), while the H^- ion (free-free) seems to be much weaker.

   To confirm it, we should compare the corresponded absorption coefficients α_λ. For this, we have to multiply the cross-section σ by the number of atoms N in the corresponded state: $\boldsymbol{\alpha_\lambda = \sigma_\lambda N}$.

   H (bound-free):
   From lecture 20, Kramers approximation for continuous cross-section for level n for Hydrogen is $\sigma_{bf}(\mathrm{H}) = a_0 \frac{\lambda^3}{n^5} G_{bf}$ cm^2 per neutral H atom ($G_{bf} \approx 1$, a_0=1.04×10^{-26} for λ in angstroms).
   5000 Å corresponds to the Paschen continuum (n=3);
   18000 Å to the Pfund continuum (n=5).
   Thus, $\sigma_{bf,5000}$=5.3×10^{-18} and $\sigma_{bf,18000}$=1.9×10^{-17} per neutral H atom in the state n that contribute at this wavelength.
   We can use the Saha equation to derive the population of H in the neutral state (similarly to problem 8 from the Set 3). We find that ~83% of the total number of H atoms is in the neutral state.
   How many of them are at the levels n=3 and 5? For this we need to use the Boltzmann formula:

   $$\log N(H_{n=3})/N(H_{n=1}) = \log 2(3)^2/2(1)^2 - 5040/8064 \times 12.089 = -6.60$$

   N_H(n=3)/N_H(n=1)=2.5×10^{-7} or N_H(n=3)/N_H(total)=0.83×2.5×10^{-7} = 2.1×10^{-7}

   $$\log N(H_{n=5})/N(H_{n=1}) = \log 2(5)^2/2(1)^2 - 5040/8064 \times 13.056 = -6.76$$

   N_H(n=5)/N_H(n=1)=1.7×10^{-7} or N_H(n=5)/N_H(total)=0.83×1.7×10^{-7} = 1.4×10^{-7}

H^- ion:
We didn't derive formulae for the cross-sections of H^- but we can estimate them from slide 32 of lecture 20.
Roughly, $\sigma_{bf,5000}(H^-) \approx 3\times10^{-17}$ and $\sigma_{ff,18000}(H^-) \approx 2\times10^{-26}$ per H^- ion.
Let's now use the Saha equation to derive the relative population of $N(H^-)$ – see an example in slide 30 of lecture 20:
$\log N(H^-)/N(H^0) = -7.94$, so only 1.1 out of 10^8 hydrogen atoms is in the form of H^-.
$N(H^-)/N(H^0) = 1.1\times10^{-8}$

We can now compare the absorption coefficients:
5000 Å:

$$\frac{\alpha_{bf}(H)}{\alpha_{bf}(H^-)} = \frac{\sigma_{bf}(H)\times N_{bf}(H)}{\sigma_{bf}(H^-)\times N(H^-)} = \frac{5.3\times10^{-18}\ \times 2.1\times10^{-7}}{3\times10^{-17}\times 1.1\times10^{-8}} = 3.3$$

18000 Å:

$$\frac{\alpha_{bf}(H)}{\alpha_{ff}(H^-)} = \frac{\sigma_{bf}(H)\times N_{bf}(H)}{\sigma_{ff}(H^-)\times N(H^-)} = \frac{1.9\times10^{-17}\ \times 1.4\times10^{-7}}{2\times10^{-26}\times 1.1\times10^{-8}} = 1.2\times10^{10}$$

Thus, the calculations confirm that at 5000 Å the two dominant sources of opacity, bound-free H and bound-free H^- are comparable. At 18000 Å, bound-free H is the dominant source of opacity, the H^- ion can be neglected.

3. Calculate the ratio of the absorption coefficients due to bound-free absorption above and below the Balmer edge (Balmer jump) for a hydrogen atmosphere with T_{eff}=9520K.

Solution:
We solved a similar problem in lecture 20 (slide 27), see also lecture 21.
We have to compare the value of the H absorption coefficient $\boldsymbol{\alpha}$ (per atom) in the Balmer (n=2) to Paschen (n=3) continua at 3646 Å:

$$\frac{\alpha(Balmer)}{\alpha(Paschen)} = \frac{\sigma_{i2}}{\sigma_{i3}}\frac{N_2}{N_3} = \frac{\sigma_{i2}g_2}{\sigma_{i3}g_3}e^{(1.89eV/kT)} = \frac{\sigma_{i2}g_2}{\sigma_{i3}g_3}10^{(1.89\times5040/9520)}$$

Pay attention that in contrast to the lectures, we now have the longer wavelength in the denominator and shorter in the numerator.
$\sigma_n \propto n^{-5}$ and $g_n = 2n^2$ so

$$\frac{\alpha(Balmer)}{\alpha(Paschen)} = \frac{2^{-5}\times 8}{3^{-5}\times 18}\times 10.0 = 33.8$$

Thus, the Balmer jump is not as huge as the Lyman jump. Still, it is very strong.

4. Balmer hydrogen lines are not seen in the spectra of either O stars or K stars. Why not?

Answer: In O stars, the temperature is so high that all of the hydrogen is ionized; therefore, there are no neutral atoms to absorb radiation. In K stars, in contrast, the atmospheric temperature is too low for the hydrogen atoms to be in the n=2 state. Thus, no Balmer absorption lines are possible.

5. An F star has a temperature T_{eff}=7000 K. Microturbulence in the atmosphere has RMS velocity ξ_t=3 km/s. Determine the FWHM of an optically thin line of iron with wavelength 4000 Å.

Answer: The line is broadened due to microturbulence in the atmosphere AND a thermal motion of particles. They both have the Gaussian line profiles. Convolution of two Gaussian profiles gives the Gaussian profile with quadratic summation of half-widths: $v_{total} = \sqrt{v_{th}^2 + \xi_{turb}^2}$.

$$v_{th} = \sqrt{\frac{2kT}{m}} = \sqrt{\frac{2 \times 1.38 \times 10^{-16} \times 7000}{56 \times 1.66 \times 10^{-24}}} \approx 1.4\ \mathrm{km/s}$$

$$v_{total} = \sqrt{v_{th}^2 + \xi_{turb}^2} = 3.3\ \mathrm{km/s}$$

$$\Delta\lambda_{1/2} = \frac{\lambda_0^2}{c}\Delta\nu_{1/2} = \frac{\lambda_0^2}{c} \times 1.67\frac{\nu_0}{c}v_{total} = 1.67 \times \frac{\lambda_0}{c}v_{total} = 0.07\text{Å}$$

6. Determine the FWHM of an optically thin line which is broadened due to both the quadratic Stark effect with the FWHM of $\Delta\lambda_{1/2}$=3 Å, and other pressure effects with the FWHM of $\Delta\lambda_{1/2}$=0.5 Å.

Answer: The lines broadened due to pressure effects have the Lorentzian profiles. Convolution of two Lorentzian profiles gives the Lorentzian profile with sum of half-widths. FWHM=3.5 Å.