Astrophysics

Compulsory Home Exercises.
Solutions of Problem Set 4.

1. Let's approximate the directional (μ) dependence of the specific intensity using the first-order Taylor expansion,

$$I(\tau,\mu) = I_0(\tau) + I_1(\tau)\,\mu$$

where I_0 and I_1 depend on the vertical optical depth τ but not on μ.
Assuming a gray atmosphere,

a. Derive expressions for the flux F, mean intensity J, and radiation pressure P_{rad}, in terms of I_0 and I_1.

Solution:
The mean intensity:

$$J_\lambda = \frac{1}{4\pi}\oint I_\lambda d\omega = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_0^{\pi} I_\lambda \, sin\theta \, d\theta = \frac{1}{4\pi}\int_0^{2\pi} d\varphi \int_{-1}^{1} (I_0 + I_1\,\mu) d\mu = I_0$$

The flux:

$$F_\lambda = \oint I_\lambda \cos\theta \, d\omega = \int_0^{2\pi} d\varphi \int_{-1}^{1} (I_0 + I_1\mu)\mu d\mu = \frac{4\pi}{3} I_1$$

The radiation pressure:

$$P_{rad} = \frac{1}{c}\int_0^{2\pi} d\varphi \int_{-1}^{1} (I_0 + I_1\mu)\mu^2 d\mu = \frac{4\pi}{3c} I_0$$

b. Show that the radiation field obeys the Eddington approximation and

$$P_{rad} = \frac{4\pi}{3c} J$$

Solution:
From the above, $I_0 = J = \frac{3cP_{rad}}{4\pi}$ → $P_{rad} = \frac{4\pi}{3c} J$

c. Within the Eddington approximation, the solution of the radiative transfer equation is

$$S = \frac{3}{4\pi}\left(\tau + \frac{2}{3}\right) F$$

Use this solution to find an expression for I_0 as a function of I_1 and τ.

Solution:
In the grey case, $S = J$.
Then, from the above expressions for J and F:

$$S = J = I_0 \;=\; \frac{3}{4\pi}\left(\tau + \frac{2}{3}\right)\frac{4\pi}{3} I_1 = \left(\tau + \frac{2}{3}\right) I_1$$

$$I_0 = \left(\tau + \frac{2}{3}\right) I_1$$

d. Show that if $\tau \gg 1$ then $I_0 \gg I_1$ (this result justifies the use of the first-order Taylor expansion at large optical depths, showing that the radiation field becomes isotropic).

Solution:
if $\tau \gg 1$ then from $I_0 = \left(\tau + \frac{2}{3}\right) I_1 \rightarrow I_0 \approx \tau\, I_1 \rightarrow I_0 \gg I_1$

2. Assuming a grey atmosphere and using the Eddington approximation and an expression for the source function S as a function of vertical optical depth τ and flux F (Lecture 18),

a. Calculate the upward specific intensity $I(\tau, \mu)$ as a function of F, τ_ν, and direction $\mu>0$, using the formal solution to RTE (Lecture 18)

$$I_\lambda(\tau_{\lambda,\mathrm{v}}, \mu > 0) = \int_{\tau_{\lambda,\mathrm{v}}}^{\infty} S_\lambda \mathrm{e}^{(\tau_{\lambda,\mathrm{v}} - t)/\mu} \frac{\mathrm{d}t}{\mu},$$

Solution:
The source function S as a function of τ and F (from Lecture 18):

$$S(\tau) = \frac{3}{4\pi}\left(\tau + \frac{2}{3}\right) F$$

Substituting in the above expression (pay attention that t is a dummy integration variable) for S, we have:

$$I(\tau, \mu > 0) = \int_\tau^\infty \frac{3}{4\pi}(t + \frac{2}{3}) F e^{(\tau - t)/\mu} \frac{dt}{\mu} = \frac{3}{4\pi} F \int_\tau^\infty \left(t + \frac{2}{3}\right) e^{\frac{\tau - t}{\mu}} \frac{dt}{\mu} \Rightarrow$$

Substitute: $u = \frac{\tau - t}{\mu}$, t=τ-μu :

$$\Rightarrow -\frac{3}{4\pi} F \int_\tau^\infty \left(\tau - \mu u + \frac{2}{3}\right) e^u du = \frac{3}{4\pi} F \int_\tau^\infty \left(\mu u e^u du - \left(\tau + \frac{2}{3}\right) e^u du\right) \Rightarrow$$

$$=> \frac{3}{4\pi} F \left[-\left(\tau + \frac{2}{3}\right) e^{\frac{\tau - t}{\mu}} \right]_{\tau}^{\infty} + \frac{3}{4\pi} F \, \mu \int_{\tau}^{\infty} u \, e^{u} du \;=>$$

A standard integral $\int x e^{x} dx = e^{x}(x-1)$:

$$=> \frac{3}{4\pi} F \left(\tau + \frac{2}{3}\right) + \frac{3}{4\pi} F \left[\left(\frac{\tau - t}{\mu} - 1 \right) e^{\frac{\tau - t}{\mu}} \right]_{\tau}^{\infty}$$

$$I(\tau, \mu > 0) = \frac{3}{4\pi} F \left(\tau + \frac{2}{3} + \mu \right)$$

b. Using your expression for the upward specific intensity $I(\tau, \mu)$, evaluate the upward component of the radiative flux,

$$F^{+} = 2\pi \int_{0}^{1} I(\mu) \, \mu \, d\mu$$

Solution:

$$F^{+} = 2\pi \frac{3}{4\pi} F \int_{0}^{1} \left(\tau + \frac{2}{3} + \mu \right) \mu \, d\mu = \frac{3}{2} F \left[\frac{\tau \mu^{2}}{2} + \frac{2}{3} \frac{\mu^{2}}{2} + \frac{\mu^{3}}{3} \right]_{0}^{1}$$

$$F^{+} = F \left(1 + \frac{3}{4} \tau \right)$$

c. Find the corresponding downward component of the flux, F^{-}.

Solution:

$$F = F^{+} - F^{-}$$

$$F^{-} = F^{+} - F = F \left(1 + \frac{3}{4} \tau \right) - F = \frac{3}{4} F \tau$$