Astrophysics

Compulsory Home Exercises.
Solution of Problem Set 3.

1. Assume a typical photon mean free path in the Sun of l=0.3 cm.
   Calculate the average time it would take for the photon to escape from the centre of the Sun if this mean free path remained constant for the photon's journey to the surface.

   Solution:
   Distance d moved by a photon that random walks through N steps of the mean free path l is $d = l\sqrt{N}$. From the center of Sun to the surface, with l = constant, $N = R_\odot^2/l^2$. Then the total path length traveled by the photon is $D = Nl = R_\odot^2/l$.
   It will take time $t = D/c$ (the speed of light) = $R_\odot^2/lc$

   Answer: t=$(6.957\times10^{10})^2$ / $(0.3\times2.998\times10^{10})$ = 5.38×10^{11} sec ≈ 17000 yr.
   Without interaction with matter, the photon would leave the Sun in $R_\odot/c = 2.3$ sec.

2. For neutral hydrogen gas, at what temperature is the number of atoms in the first excited state only 1% of the number of atoms in the ground state?

   Solution:
   The Boltzmann equation gives ratios of level populations as a function of temperature:
   $$\frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-(E_u - E_l)/kT}$$
   Solving for T yields:
   $$T = \frac{E_l - E_u}{k \ln\left(\frac{N_u}{N_l}\frac{g_l}{g_u}\right)}$$

   l=1, u=2, $g=2n^2$ → g_2=8, g_1=2, E_1-E_2= -10.2 eV, k=8.6174x10^{-5} eV/K
   Answer: For N_2/N_1 = 0.01, we get T=19756 K

3. At what temperature is the number of atoms in the first excited state equal to 5% of the number in the ground state?

   Answer: For N_2/N_1 = 0.05, we get T=27012 K

4. For neutral hydrogen, at what temperature will equal numbers of atoms have electrons in the ground state (n=1) and the first excited state (n=2)? What is the energy required to excite the electron from the ground state to n=2?

   Answer: For N_2/N_1 = 1.0, we get T=85382 K.
   The energy required to excite an electron from n=1 to n=2 is
   E_2-E_1= -3.4 eV+13.6 eV = 10.2 eV.

5. As temperature approaches ∞, what is the predicted distribution of electrons in each orbital according to the Boltzmann equation? Will this be the distribution that actually occurs? Why or why not?

   Solution:

   If T→∞, the Boltzmann equation approaches the ratio of the statistical weights:

$$\frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-(E_u - E_l)/kT} \rightarrow \frac{g_u}{g_l} = \frac{n_u^2}{n_l^2}$$

   However, this will not be the distribution that actually occurs at T→∞ because at such high temperatures all of the hydrogen atoms will have ionized!

6. What are the ionization energy of hydrogen, and the first and the second ionization energies of Helium? Give the numbers and explain in words.

   The ionization energy of hydrogen is the energy required to remove the electron from the ground state – a transition from n=1 to n=inf, which simply corresponds to the energy of the ground state: 13.6 eV.
   For Helium, the first and the second ionization energies are 24.6 eV and 54.4 eV, respectively, much higher than for hydrogen. This is because the nucleus of Helium has 2 protons attracting the electrons instead of 1, and the electrons are being removed from the same orbital as in hydrogen's case.
   The second ionization energy of helium is 2^2 times larger than the ionization energy of hydrogen: 13.6×4=54.4 eV. However, for the first ionization energy of 24.6 eV of helium it seems no one was able to formulate any successful formula.

7. Consider your results from above as well as your answer to Problem 5. Would you expect a significant number of Hydrogen atoms to be ionized at T=10000 K?
   At 40000 K? Why or why not?

   Answer: in Problem 4 you could find, using the Boltzmann equation, that at T=85000 K only half of the atoms have been excited to n=2. This transition requires energy of 10.2 eV, while 13.6 eV is needed to ionize hydrogen. Consequently, based solely on the answers to above problems 2–4, NO significant number of hydrogen atoms would be expected to be ionized at just 10000 K and even 40000 K.

8. Calculate the fraction of atoms that would be ionized in a stellar atmosphere of pure hydrogen at T=8000K. What about an atmosphere at T=12000K? Assume the electron pressure is constant, P_e=200 dyn/cm^2.

   Solution:

   The Saha equation gives the ratio of the total number of ions (N^+) to the total number of neutral atoms (N^0) as a function of temperature T and electron pressure P_e:

$$\log\frac{N^+}{N^0} = \log\frac{u^+}{u^0} + \log 2 + \frac{5}{2}\log T - \chi_{ion}\Theta - \log P_e - 0.48$$

$$\log\frac{N^+}{N^0} = \log u^+ - \log u^0 + \log 2 + \frac{5}{2}\log T - \chi_{ion}\Theta - \log P_e - 0.48$$

P_e=200 dyn/cm^2, χ_{ion}=13.6 eV, the partition function for the ion u^+=1, i.e. log u^+=0
For T=8000 K: Θ=5040/T = 0.63, log u^0=0.301
Answer: $N^+/N^0 \approx$ 2.5%
For T=12000 K: Θ=5040/T = 0.42, log u^0=0.301
Answer: $N^+/N^0 \approx$ 50.6%
Then, the fraction of ionized atoms is

$$\frac{N^+}{N_{total}} = \frac{N^+}{N^+ + N^0} = \frac{N^+/N^0}{N^+/N^0 + 1}$$

For T=8000 K, $N^+/N_{total} \approx$ 2.5%
For T=12000 K, $N^+/N_{total} \approx$98%
Thus, ionization occurs at a much lower temperature than predicted by the Boltzmann equation and our answer to Problem 7 is incorrect. The reason is that we should have taken into account the likelihood of ionization from energy states n>1.