Astrophysics

Compulsory Home Exercises. Problem Set 2.

Return by Tuesday, March 1, 2022.

Please, write down **every step in your line of thinking** and state assumptions etc.
A sole answer is not enough.

Problem 2.1

What is the energy released by the nuclear reactions of carbon burning (fusion of 2 carbon nuclei)? Give the answer in MeV and ergs per gram.

Solution:

Let's consider the simplest reaction ${}^{12}_{6}\mathrm{C} + {}^{12}_{6}\mathrm{C} \rightarrow {}^{24}_{12}\mathrm{Mg} + \gamma$

$m({}^{12}_{6}\mathrm{C}) = 12.000 \text{ au}$ $m({}^{24}_{12}\mathrm{Mg}) = 23.985 \text{ au}$

$$[2m({}^{12}_{6}\mathrm{C}) - m({}^{24}_{12}\mathrm{Mg})]c^2 = 0.015 \times 931.1 \text{ MeV} = 13.97 \text{ MeV}$$

$$= 13.97 \times 1.602 \times 10^{-6} \text{ erg} = 2.24 \times 10^{-5} \text{ erg}$$

Problem 2.2

Calculate the mean molecular weight μ for

1) the completely ionized stellar interior, where we have 45% hydrogen, 52% helium, and 3% heavy elements by mass,
2) completely ionized hydrogen,
3) completely ionized helium,
4) neutral gas at the solar interior abundance, 73% hydrogen, 25% helium, and 2% heavy elements by mass.

Solution:

From Lecture 9, slide 87:

$$\mu^{-1} = 2X + \frac{3}{4}Y + \frac{1}{2}Z$$

1) X=0.45, Y=0.52, Z=0.03 → μ=0.766
2) X=1.0 → μ=0.5
3) Y=1.0 → μ=1.33

4) X=0.73, Y=0.25, Z=0.02

But now it is neutral gas: $\mu^{-1} = X + \frac{1}{4}Y + \frac{1}{A}Z$

A is the atomic mass of heavy elements. Most abundant heavy elements in the Milky Way are Oxygen (1.04%) and Carbon (0.46%). Then A=(16*1.04+12*0.46)/(1.04+0.46)=14.8

Thus, μ=1.26

Problem 2.3

Prove that for the case when Z is negligible, the mean molecular weight per electron, $\mu_e = \frac{\rho}{n_e m_H}$, can be approximately expressed as

$$\mu_e \approx \frac{2}{1+X}$$

Solution:

Z=0, *Y*=1-*X*

From Lecture 9, slide 87:

$$\mu_e^{-1} = X + \frac{2}{4}Y = X + \frac{1}{2} - \frac{X}{2} = \frac{X+1}{2}$$

Problem 2.4

Does a lower Gamow energy E_G increase or decrease the probability of penetration?

Solution:

A lower E_G increases the probability of penetration: $P_{pen} \approx e^{-\sqrt{E_G/E}}$

Problem 2.5

a) Calculate the Gamow energy E_G (in electronvolts) for the collision of two α-particles (helium-4 nuclei, ${}^{2}_{2}\mathrm{He}$) and find the penetration probability P_{pen} for the typical kinetic energy of particles in the Sun's core, E ~ 1 keV. Compare the results with the case of two protons. Explain the result.

b) What temperature is required to have the probability of penetration of two α-particles similar to that of two protons in the Sun's core?

Solution:

From Lecture 10, slide 113:

a) $P_{pen} \approx e^{-\sqrt{\frac{E_G}{E}}}$; $E_G = 2m_r c^2(\pi\alpha Z_1 Z_2)^2$

$m_r = 4/2 \times m_p = 2m_p$

$E_G = 2 \times 2 \times 931.1\ \mathrm{MeV} \times \left(\frac{3.14}{137} * 4\right)^2 = 31.3\ \mathrm{MeV}$

$P_{pen} \approx 1.3 \times 10^{-77}$

b) For 2 protons: E_G=0.490 MeV; $P_{pen} \approx 2.4 \times 10^{-10}$

$e^{-\sqrt{E_G(2He)/E}} = P_{pp}$

$E = E_G(2He)/[\ln(P_{pp})]^2 = 31.3/490 = 63.8\ \mathrm{keV} = 740 \times 10^6\ \mathrm{K}$

Problem 2.6

We have seen that a polytropic model of the Sun shows quite good agreement with the results of a detailed solution of the equations of stellar structure. Using the n=3 polytrope and the solar mass and radius, find the central pressure P_c, central density ρ_c, and temperature T_c at the centre of the Sun.

Solution:

From Lecture 12, slide 154:

$P_c = (4\pi)^{1/3} B_n G M^{2/3} \rho_c^{4/3}$

$\bar{\rho} = \frac{M}{4/3\pi R^3}$ $\rho_c = \bar{\rho} D_N$ D_N=54.18 (from the Table)

Thus $\rho_c = 76.3$ g cm^{-3} $P_c = 1.25 \times 10^{18}$ dyn cm^{-2}

$P_c = \frac{\Re T \rho}{\mu} + \frac{a T^4}{3} \cong \frac{\Re T_c \rho_c}{\mu}$ $T_c = \frac{P_c \mu}{\Re \rho_c}$ assuming μ=0.6 T_c=11.8×10^6 K

Problem 2.7

Prove that according to the virial theorem, the mean temperature of a star can be expressed as

$$\bar{T} \propto M^{2/3}\rho^{1/3}$$

Solution:

From Lecture 3, slide 103:

$\bar{T} = \frac{e_G}{3}\frac{\mu m_p}{k}\frac{GM}{R}$

$\bar{\rho} = \frac{M}{4/3\pi R^3}$ → $R = \left(\frac{M}{4/3\pi\bar{\rho}}\right)^{1/3}$ → $\bar{T} \propto M^{2/3}\rho^{1/3}$