

# Astrophysics

## Compulsory Home Exercises. Problem Set 1.

Return by Wednesday, February 9, 2022 (before the lecture).

Please, write down every step in your line of thinking and state assumptions etc.  
A sole answer is not enough.

### Problem 1.1

In a star of mass  $M$ , the density decreases from the centre to the surface as a function of radial distance  $r$ , according to

$$\rho = \rho_0 \left[ 1 - \left( \frac{r}{R} \right)^2 \right]$$

where  $\rho_0$  is a given constant and  $R$  is the star's radius,

- Find  $m(r)$ .
- Derive the relation between  $M$  and  $R$ .
- Show that the average density of the star (total mass divided by total volume) is  $0.4\rho_0$ .

*Solution:*

From Lecture 3, slide 73:

$$\text{a) } m(r) = \int_0^r 4\pi r^2 \rho(r) dr = 4\pi\rho_0 \int_0^r r^2 \left[ 1 - \left( \frac{r}{R} \right)^2 \right] dr = 4\pi\rho_0 \left[ \frac{r^3}{3} - \frac{r^5}{5R^2} \right]$$

$$\text{b) } M = m(R) = 4\pi\rho_0 \left[ \frac{R^3}{3} - \frac{R^3}{5} \right] = 8\pi\rho_0 R^3 / 15$$

$$\text{c) } \bar{\rho} = \frac{M}{V} = \frac{8\pi\rho_0 R^3}{15 \cdot 4/3\pi R^3} = 0.4\rho_0$$

## Problem 1.2

For a star of mass  $M$  and radius  $R$ , find the central pressure and check the validity of inequality  $P_c \geq \frac{3}{8\pi} \frac{GM^2}{R^4}$  for the following cases:

- (a) a uniform density
- (b) a density profile as in Exercise 1.1.

*Solution:*

From Lecture 3, slide 78:

The equation of hydrostatic equilibrium:  $\frac{dP(r)}{dr} = -\frac{Gm(r)}{r^2} \rho(r)$

We need to integrate it from the centre where  $P=P_c$  to the surface where  $P_s=0$ :

$$P = P_c - P_s = \int_0^R \frac{Gm(r)}{r^2} \rho(r) dr$$

a)

$$\rho = \bar{\rho} = \frac{M}{V} = \frac{3M}{4\pi R^3}$$

$$m(r) = \frac{4\pi r^3}{3} \bar{\rho} = \frac{Mr^3}{R^3}$$

$$P = G \int_0^R \frac{Mr^3}{R^3 r^2} \frac{3M}{4\pi R^3} dr = G \frac{3M^2}{4\pi R^6} \int_0^R r dr = \frac{3GM^2}{8\pi R^4}$$

b) From problem 1.1:

$$m(r) = 4\pi\rho_0 \left[ \frac{r^3}{3} - \frac{r^5}{5R^2} \right]$$

$$M = \frac{8\pi\rho_0 R^3}{15} \rightarrow \rho_0 = \frac{15M}{8\pi R^3}$$

$$\begin{aligned} P &= 4\pi\rho_0^2 G \int_0^R \left[ \frac{r}{3} - \frac{r^3}{5R^2} \right] \left[ 1 - \frac{r^2}{R^2} \right] dr = 4\pi\rho_0^2 G \int_0^R \left[ \frac{r}{3} - \frac{r^3}{3R^2} - \frac{r^3}{5R^2} + \frac{r^5}{5R^4} \right] dr = \\ &= 4\pi\rho_0^2 G \int_0^R \left[ \frac{r}{3} - \frac{8r^3}{15R^2} + \frac{r^5}{5R^4} \right] dr = 4\pi\rho_0^2 G \left[ \frac{r^2}{6} - \frac{8r^4}{60R^2} + \frac{r^6}{30R^4} \right]_0^R \\ &= \frac{4\pi G 15^2 M^2 R^2}{8^2 \pi^2 R^6} = \frac{15GM^2}{16\pi R^4} > \frac{3}{8\pi} \frac{GM^2}{R^4} \end{aligned}$$

### Problem 1.3

In Lecture 4 we obtained the mean temperature of the star to be

$$\bar{T} = \frac{e_G \mu m_p GM}{3 k R}$$

For a star of mass  $M$  and radius  $R$ , find the value of  $e_G$  for two cases:

- (a) a uniform density
- (b) a density profile as in Exercise 1.1.

*Solution:*

$$e_G = \int_0^1 \frac{q}{x} dq \quad q=m/M, x=r/R$$

$$e_G = \int_0^M \frac{mR}{M^2 r} dm = \frac{R}{M^2} \int_0^M \frac{m}{r} dm$$

a) From Problem 1.2a:

$$m(r) = \frac{Mr^3}{R^3} \rightarrow r = R \left( \frac{m}{M} \right)^{1/3}$$

$$e_G = \frac{R}{M^2} \int_0^M \frac{m}{R \left( \frac{m}{M} \right)^{1/3}} dm = M^{-5/3} \int_0^M m^{2/3} dm = M^{-5/3} \frac{3}{5} M^{5/3} = \frac{3}{5}$$

b) From Problem 1.1:

$$\rho_0 = \frac{15M}{8\pi R^3}$$

$$m(r) = 4\pi\rho_0 \left[ \frac{r^3}{3} - \frac{r^5}{5R^2} \right] = \frac{15M}{2R^3} \left[ \frac{r^3}{3} - \frac{r^5}{5R^2} \right]$$

$$e_G = \frac{R}{M^2} \int_0^M \frac{m}{r} dm$$

$$= \frac{R}{M^2} \frac{15M}{2R^3} \int_0^R \left[ \frac{r^2}{3} - \frac{r^4}{5R^2} \right] d \left[ \frac{15M}{2R^3} \left( \frac{r^3}{3} - \frac{r^5}{5R^2} \right) \right] =$$

$$= \frac{15}{2MR^2} \frac{15M}{2R^3} \int_0^R \left[ \frac{r^2}{3} - \frac{r^4}{5R^2} \right] \left[ r^2 - \frac{r^4}{R^2} \right] dr = \frac{15^2}{4R^5} \int_0^R \left[ \frac{r^4}{3} - \frac{8r^6}{15R^2} + \frac{r^8}{5R^4} \right] dr$$

$$= \frac{15^2}{4R^5} \left[ \frac{r^5}{15} - \frac{8r^7}{105R^2} + \frac{r^9}{45R^4} \right]_0^R = \frac{15^2}{4R^5} \left( \frac{7R^5}{105} - \frac{8R^5}{105} + \frac{R^5}{45} \right) = \frac{15}{4} \left( -\frac{1}{7} + \frac{1}{3} \right) = \frac{15}{21}$$

**Problem 1.4** (from lecture 5):

Assume that the whole mass of a star is concentrated in the centre. Please show that the time for collapse from radius  $R$  to  $0$  is

$$t_{dyn} = \frac{\pi}{2\sqrt{2}} \left( \frac{R^3}{GM} \right)^{1/2}$$

*Solution:*

We can simply use Kepler's Third law:

$$\frac{a^3}{T^2} = \frac{G(M+m)}{4\pi^2} \Rightarrow T^2 = \frac{4\pi^2}{G(M+m)} a^3$$

Assuming eccentricity  $e \rightarrow 1$  and  $m \rightarrow 0$ , the free-fall time will be  $\tau_{ff} = T/2$  and  $a = R/2$ :

$$\tau_{ff} = t_{dyn} = \frac{\pi}{2\sqrt{2}} \left( \frac{R^3}{GM} \right)^{1/2}$$

**Problem 1.5** (from lecture 5):

How much mass per time must the Sun accrete in order for its luminosity to equal that of observed? Show the answer in Solar masses. Discuss how this process of accretion will affect Earth?

*Solution:*

$$L_{\odot} = 3.85 \cdot 10^{33} \text{ erg/s}$$

The accreted gas must lose the gravitational potential energy liberated as it falls toward the mass gaining star. If this energy is radiated, luminosity is:

$$L = \frac{GM dm/dt}{R} = \frac{GM\dot{M}}{R} \rightarrow \dot{M} = \frac{LR}{GM} = \frac{3.85 \cdot 10^{33} \times 6.96 \cdot 10^{10}}{6.67 \cdot 10^{-8} \times 1.99 \cdot 10^{33}} = 2.0 \times 10^{18} \text{ g/s}$$

$$\dot{M} = 3.2 \times 10^{-8} M_{\odot} / \text{yr}$$

On a relatively short time scale ( $\sim 30$  million years) the solar mass would be doubled which will result in changing of the Earth year duration and the distance from Earth to the Sun.

### Problem 1.6 (Expanded problem from lecture 7):

A B5V star in the LMC – distance 50 kpc – has  $V=13.5$  mag,  $B-V=-0.07$  mag.

- (a) What is its bolometric luminosity, relative to the Sun?
- (b) What is its stellar radius?
- (c) At what wavelength does its radiation peak (assuming Wien's law)?

*Solution:*

Slides 51 and 45 from Lecture 7:

$E(B-V)=(B-V)-(B-V)_0=-0.07+0.17=0.1$  mag for a B5V star, so  $A_V \approx 3.1 \times E(B-V)=0.31$  mag.

$M_V - m_V = 5 - 5 \log(d/\text{pc}) - A_V = -18.80 \rightarrow M_V = -5.3$

From the same table  $BC = -1.46$ . Then from  $BC = M_{\text{bol}} - M_V \rightarrow M_{\text{bol}} = -1.46 - 5.3 = -6.76$  mag

From  $M_{\text{bol}} = 4.74 - 2.5 \log(L/L_\odot) \rightarrow L/L_\odot = 39800$

From the same table  $T_{\text{eff}} = 15200\text{K}$ .  $L = 4\pi R^2 F_s = 4\pi R^2 \sigma T^4 \rightarrow R = 28.8 R_\odot$

From Wien's displacement law,  $\lambda_{\text{max}} = 1970\text{\AA} \rightarrow \text{UV}$