

# The equations of stellar structure - II

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EQUATION OF STATE (EOS)  
STELLAR OPACITY

# Introduction

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- We have 4 differential equations of stellar structure.
- Accurate expressions for pressure, opacity and energy generation are extremely complicated, but we can find simple approximate forms.
- Equations of stellar structure too complicated to find exact analytical solution, hence must be solved with computer.
- Sometimes simplifications can be made to find analytical solutions that still have most of the physics.

# The equations of stellar structure

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- $\frac{dm}{dr} = 4\pi r^2 \rho(r)$

- $\frac{dP(r)}{dr} = -\frac{Gm}{r^2} \rho(r)$

- $\frac{dL(r)}{dr} = 4\pi r^2 \rho(r) \varepsilon(r)$

- $\frac{dT(r)}{dr} = -\frac{3}{64\pi\sigma r^2} \frac{\rho(r)\kappa_R(r)}{T^3(r)} L(r)$

- $\frac{P}{T} \frac{dT}{dP} = \frac{\gamma-1}{\gamma}$

- $r$  = radius
- $P$  = pressure at  $r$
- $m$  = mass of material within  $r$
- $\rho$  = density at  $r$
- $L$  = luminosity at  $r$  (rate of energy flow across sphere of radius  $r$ )
- $T$  = temperature at  $r$
- $\kappa_R$  = Rosseland mean opacity at  $r$
- $\varepsilon$  = energy release per unit mass per unit time

To these four differential equations we **need to add** three equations connecting the pressure, the opacity, and the energy production rate of the gas with its density, temperature, and composition:

$P = P(\rho, T, \text{chemical composition}) \longrightarrow$  **usually called the equation of state (EOS)**

$\kappa_R = \kappa_R(\rho, T, \text{chemical composition})$

$\varepsilon = \varepsilon(\rho, T, \text{chemical composition})$

# The equation of state (EOS)

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- The equation of state (EOS) describes the microscopic properties of stellar matter for given density  $\rho$ , temperature  $T$  and composition  $X_i$ .
- It is usually expressed as the function that relates the pressure  $P$  to  $\rho$ ,  $T$ , and mean molecular weight  $\mu$  at any place in the star.
- Since it is a solely an internal property of the gas, it can, in principle, be computed once externally, and used via a lookup table, i.e.,  $P_{\text{gas}} = P(\rho, \mu, T)$ .

# EOS in stars

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- We have seen that stellar gas is ionized plasma, and although density is so high that typical inter-particle spacing is of the order of an atomic radius, the effective particle size is more like a nuclear radius ( $10^5$ ) times smaller.
- Thus, interior of a star contains a mixture of ions, electrons, and radiation (photons). For most stars (except for very low mass stars and stellar remnants), the ions and electrons can be treated as **an ideal gas** and quantum effects can be neglected.
- The net pressure can be divided into three components, pressure from ions, pressure from electrons, and pressure from radiation.

Total pressure:  $P = P_i + P_e + P_{\text{rad}} = P_{\text{gas}} + P_{\text{rad}}$

$P_i$  is the pressure of the ions

$P_e$  is the electron pressure

$P_{\text{rad}}$  is the radiation pressure

However,  $P_{\text{gas}}$  may **not** obey the ideal gas law due to the effects of degeneracy.

# EOS of an ideal gas

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The equation of state for an ideal gas is:

$$P_{\text{gas}} = nkT$$

where  $n$  is concentration (number of particles per  $\text{cm}^3 = n_i + n_e$ , where  $n_i$  and  $n_e$  are the number densities of ions and electrons respectively),  $T$  is the temperature,  $k$  is Boltzmann's constant.

But we want this equation in the form:  $P = P(\rho, T, \text{chemical composition})$

This can be written as:

$$P_{\text{gas}} = \frac{\rho kT}{\mu m_p} = \frac{\mathfrak{R} \rho T}{\mu} \quad \text{where} \quad \mathfrak{R} = \frac{k}{m_p} \quad \text{is the gas constant, and}$$

$\mu$  = mean molecular weight,  
i.e. the average mass of particles in unit of proton mass  $m_p$ .

# Mean molecular weight (1)

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The mean molecular weight  $\mu$  (the average mass of particles in unit of proton mass  $m_p$ ) depends upon the composition of the gas and the state of ionization. For example:

- Neutral hydrogen:  $\mu = 1$
- Fully ionized hydrogen:  $\mu = 0.5$

An exact solution is **complex**, depending on fractional ionization of all the elements in all parts of the star.

For simplicity, let's now assume that all of the material in the star is fully ionized. This is justified as **hydrogen** and **helium** are most abundant and they are certainly fully ionized in stellar interiors (however, this **assumption will break down** near stellar surface).

# Mean molecular weight (2)

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Denote abundances of different elements per unit mass by:

$X$  = fraction of material by mass of H

$Y$  = fraction of material by mass of He

$Z$  = fraction of material by mass of all heavier elements (“metals”)

$$X + Y + Z = 1$$

Hence in  $1 \text{ cm}^3$  of stellar gas of density  $\rho$ , there is mass  $X \times (\rho \text{ of H})$ ,  $Y \times (\rho \text{ of He})$ ,  $Z \times (\rho \text{ of metals})$ . In a fully ionized gas,

H gives 2 particles per  $m_{\text{H}}$

He gives  $3/4$  particles per  $m_{\text{H}}$  ( $\alpha$  particle, plus two  $e^-$ )

Metals, average mass  $A m_{\text{H}}$ , give  $\sim 1/2$  particles per  $m_{\text{H}}$

( $^{12}\text{C}$  has nucleus plus  $6e^- = 7/12$ )

( $^{16}\text{O}$  has nucleus plus  $8e^- = 9/16$ )

where  $A$  is the atomic weight of the species.



# Mean molecular weight (3)

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If the density of the plasma is  $\rho$ , then add up number densities of hydrogen, helium, and metal nuclei, plus electrons from each species:

	H	He	metals
Number density of nuclei	$\frac{X\rho}{m_H}$	$\frac{Y\rho}{4m_H}$	$\frac{Z\rho}{Am_H}$
Number density of electrons	$\frac{X\rho}{m_H}$	$\frac{2Y\rho}{4m_H}$	$\frac{A}{2} \times \frac{Z\rho}{Am_H}$

The total number of particles per  $\text{cm}^3$  is then given by the sum:

$$n = 2 \frac{X\rho}{m_H} + \frac{3}{4} \frac{Y\rho}{m_H} + \frac{2Z\rho + AZ\rho}{2Am_H} \approx \frac{\rho}{m_H} \left[ 2X + \frac{3}{4}Y + \frac{1}{2}Z \right] = \frac{\rho}{\mu m_H}$$

...assuming that  $A \gg 1$

Thus,

$$\mu = \left[ 2X + \frac{3}{4}Y + \frac{1}{2}Z \right]^{-1}$$

# Mean molecular weight (4)

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$$\mu = \left[ 2X + \frac{3}{4}Y + \frac{1}{2}Z \right]^{-1}$$

This is a good approximation to  $\mu$  except in cool, outer stellar regions.

For solar abundances,  $X = 0.73$ ,  $Y = 0.25$ ,  $Z = 0.02$ , and therefore  $\mu = 0.60$ , i.e. the mean mass of particles in a star of solar composition is a little over half the mass of the proton.

In the **central** regions of the Sun, about half of the hydrogen has already been converted into helium by nuclear reactions, and as a result  $X = 0.34$ ,  $Y = 0.64$ , and  $Z = 0.02$ , giving  $\mu = 0.85$ .

When  $Z$  is negligible:  $Y = 1 - X$ ;  $\mu = 4/(3 + 5X)$

The electron number density  $n_e$  plays a considerable role for the properties of the gas. It is convenient to introduce the mean molecular weight per electron,  $\mu_e$ , such that

$$n_e = \frac{\rho}{\mu_e m_H} \quad \Rightarrow \quad \mu_e \approx \frac{2}{1 + X}$$

**Prove it!**

# The Ionization Fraction

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- The **accurate** calculation of mean molecular weight  $\mu$  requires knowledge of the chemical composition of the material and **the ionization fraction**. To calculate ionization fraction, one needs **the Saha equation**, which we will derive later, in the Stellar atmospheres part of this course :

$$\frac{N_1^+}{N_1} = \frac{2g_1^+}{N_e g_1} \frac{(2\pi m_e kT)^{3/2}}{h^3} e^{-\chi_{ion}/kT}$$

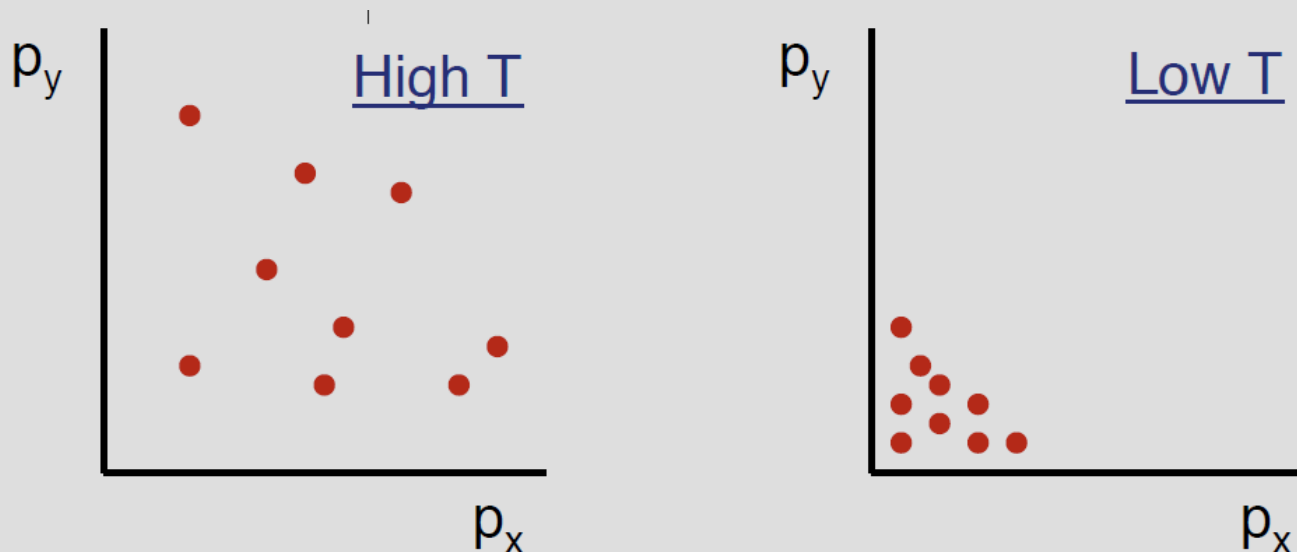
where  $m_e$  is the mass of the electron,  $\chi_{ion}$  is the ionization energy,  $N_1^+$  and  $N_1$  are the number density of ions and neutral atoms in their ground state,  $N_e$  is the electron number density,  $g_1^+$  and  $g_1$  are the statistical weight of the ground state of the ion and neutral atom.

- In general, the Saha equation can be used to compute ionization fractions over most of the star. It does, however, require that the gas be in the **thermodynamic equilibrium**. This is true throughout almost the whole star, as at high densities, collisions will control the level populations. This approximation only breaks down in the solar corona, where the densities become very low.
- **However**, the Saha equation also breaks down in the centers of stars, where high densities cause the ionization energies of atoms to be reduced. Indeed, if the mean distance between atoms is  $d$ , then there can be no bound states with radii greater than  $\sim d/2$ ). In practice, the Saha equation begins to break down at nuclear distances of  $\sim 10a_0$  ( $\sim 10$  Bohr radii).
- To correct for this effect, the Saha equation is normally used until it begins to show decreasing ionization fractions toward the center of the star. When this happens, complete ionization is assumed.

# Degeneracy Pressure (1)

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- An ordinary classical gas:  $P_{\text{gas}} \propto T \rightarrow 0$  as  $T \rightarrow 0$
- Simultaneously, the mean speed of particles in the gas also goes to zero:  
$$v = \sqrt{2kT/m}$$
- The momenta are given by:  $p_x = mv_x$ ;  $p_y = mv_y$ ;  $p_z = mv_z$
- ... if we plot the momenta of particles in a 3D space of  $p_x$ ,  $p_y$ , and  $p_z$  then as  $T$  decreases the particles become concentrated near the origin:



# Degeneracy Pressure (2)

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At low enough temperatures / high enough densities, the concentration of particles with similar (low) momenta would violate the Pauli exclusion principle:

**No two electrons can occupy the same quantum state**

i.e. have the same momentum, spin, and location.

To avoid violating the exclusion principle, electrons in a dense, cold gas **must have larger momenta than we would predict classically.**

Since the pressure  $P$  is mean rate of transport of momentum across unit area

$$P = \frac{1}{3} \int_0^{\infty} v p n(p) dp$$

...where  $n(p)dp$  is the number of particles with momentum between  $p$  and  $p+dp$

... larger momentum means higher pressure.

This quantum mechanical source of pressure is **degeneracy pressure.**

# Degeneracy Pressure (3)

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This quantum mechanical source of pressure is **degeneracy pressure**. We will discuss it later.

- **Non-relativistic degeneracy pressure** (speeds  $v \ll c$ ) :

$$P_{\text{deg}} = K_1 \rho^{5/3} = K_1 \rho^{1 + \frac{1}{n}}$$

← A polytrope of index  $n=1.5$

- $K_1$  is constant
- Does not depend upon temperature for low enough  $T$
- Depends upon composition via the relation between  $N_e$  and  $\rho$

- **Relativistic degeneracy pressure** :

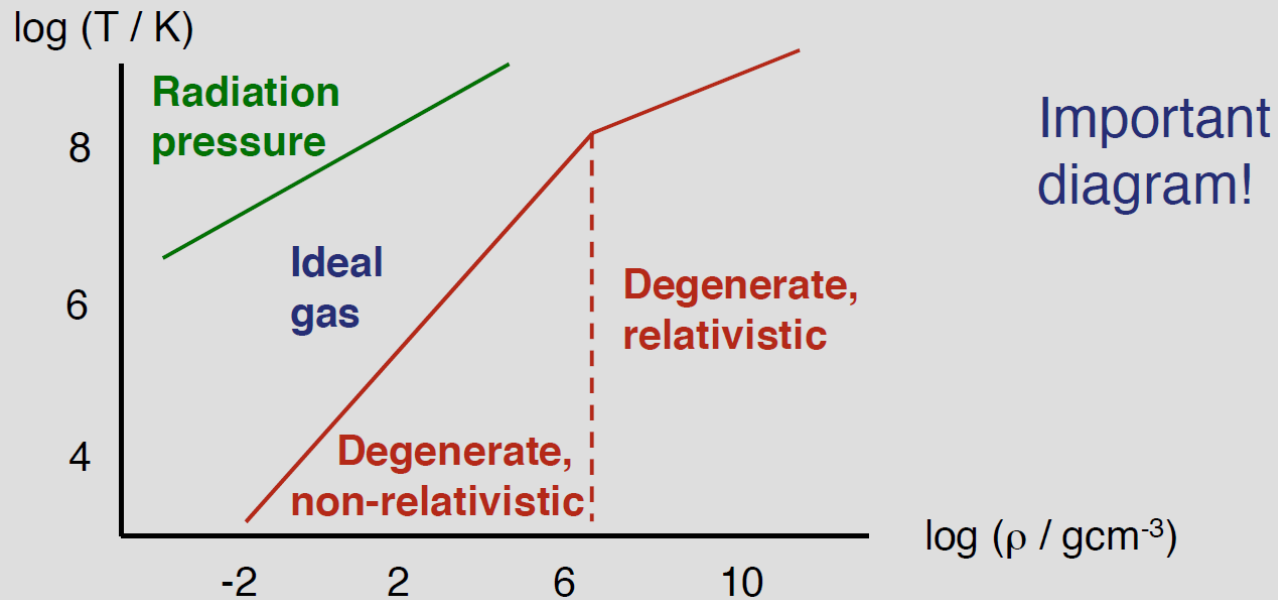
$$P_{\text{deg}} = K_2 \rho^{4/3}$$

← A polytrope of index  $n=3$

- $K_2$  is another constant
- Equation of state for relativistic degenerate matter, which applies at high density. This is a “softer” equation of state, since  $P$  rises more slowly with increasing density than for the non-relativistic case.

# When do the different pressures matter?

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Different types of star occupy different portions of the plane:

- Solar-type stars - ideal gas throughout
- Massive stars - radiation pressure
- White dwarfs - non-relativistic degeneracy pressure

Relativistic degeneracy implies an **unstable** equation of state, so **no stable stars** in that part of the plane.

# Radiation Pressure

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We have already showed before that radiation pressure can be neglected for Solar-type stars:

$$\frac{P_{rad}}{P_g} = \frac{aT^4}{3} / \frac{kT\rho}{\mu m_p} = \frac{\mu a}{3\mathcal{R}} \frac{T^3}{\rho} \approx 10^{-4} \text{ (for the Sun)}$$

But becomes **very important** for early-type stars due to the  $T^4$  sensitivity.

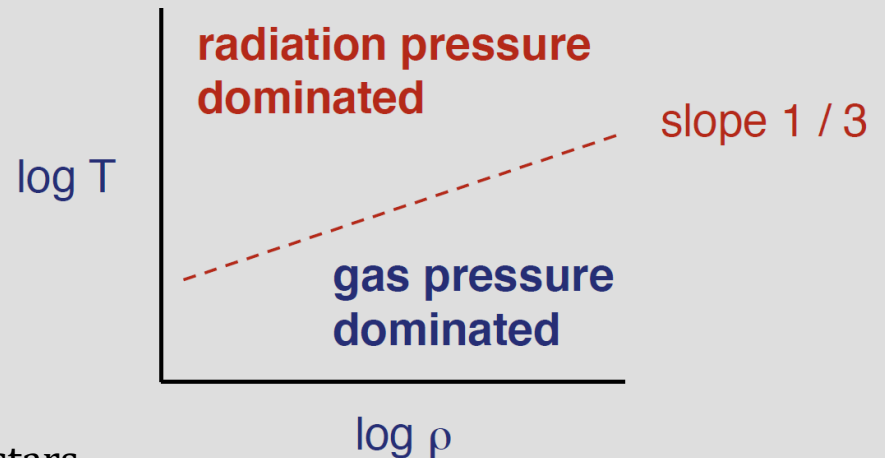
**In which stars are gas and radiation pressure important?**

$$\left. \begin{aligned} P_{rad} &= \frac{aT^4}{3} \\ P_g &= \frac{\mathcal{R}T\rho}{\mu} \end{aligned} \right\} \text{ equal when } T^3 = \frac{3\mathcal{R}}{a\mu} \rho$$

From the virial theorem (see Lecture 4):

$$\bar{T} \propto \frac{M}{R} \Rightarrow \frac{P_{rad}}{P_g} \propto M^2$$

i.e.  $P_{rad}$  becomes more significant in higher mass stars.





# Effect of radiation pressure

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For stars in which radiation pressure plays a non-negligible role we can write the generalized form of the equation of hydrostatic support (Lecture 3):

$$\frac{dP(r)}{dr} + g\rho(r) = a\rho(r)$$

Then

$$\frac{dP(r)}{dr} = -g\rho(r) - \frac{dP_{rad}}{dr} = -g_{\text{eff}}(r)\rho(r)$$

From Lecture 8:

$$\frac{dP_{rad}}{dr} = -\frac{\rho\kappa_R}{c}F \Rightarrow g_{\text{eff}}(r) = g - \frac{\kappa_R}{c}F$$

Consider relative contributions of radiation and (ideal) gas pressures:

$$P_g = \beta P = \frac{\mathfrak{R}T\rho}{\mu}, \quad P_{rad} = (1 - \beta)P = \frac{aT^4}{3}$$

Exclude temperature:  $P = \left[ \left( \frac{\mathfrak{R}}{\mu} \right)^4 \frac{3}{a} \frac{1-\beta}{\beta^4} \right]^{1/3} \rho^{4/3} \Rightarrow P = K\rho^{1+\frac{1}{n}}$  ← A polytrope of index  $n=3$

# Opacity (1)

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Concept of opacity was introduced when deriving the equation of radiation transport. It will be discussed extensively in the [Stellar atmospheres](#) part of this course.

Opacity is the resistance of material to the flow of radiation through it. In most stellar interiors it is determined by all the processes which scatter and absorb photons.

## Four main processes:

- **Bound-bound absorption:**
  - is related to photon-induced transitions of a (bound) electron in atoms or ions to a higher energy state by the absorption of a photon. The atom is then de-excited either spontaneously or by collision with another particle, whereby a photon is emitted. Although this is limited to certain transition frequencies, the process can be efficient because the absorption lines are strongly broadened by collisions.
- **Bound-free absorption:**
  - which is another name for photoionization - the removal of an electron from an atom (ion) caused by the absorption of a photon. The inverse process is radiative recombination.
- **Free-free absorption:**
  - the absorption of a photon by a free electron, which makes a transition to a higher energy state by briefly interacting with a nucleus or an ion. The inverse process, leading to the emission of a photon, is known as bremsstrahlung.
- **[Electron] Scattering:**
  - the scattering of a photon by a free electron, the photon's energy remaining unchanged (known as Thomson scattering).

# Opacity (2)

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Which process is most important in the deep stellar interiors?

- Temperatures are very high there, thus the last two processes are dominant, simply because the material is almost completely ionized, there are very few bound electrons.
- Furthermore, the energy of most photons in the Planck distribution is of the order of keV, whereas the separation energy of atomic levels is only a few tens eV. Hence most photons interacting with bound electrons would set them free. Thus bound-bound (and even bound-free) transitions have extremely low probabilities, interactions occurring predominantly between photons and free electrons.
- Thus, the most dominant process in the deep stellar interiors is electron scattering.

The opacity per unit mass of material in this case is  $\kappa_e = \frac{n_e \sigma_T}{\rho}$ ,

where  $\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$  is the Thomson cross section. The opacity is therefore:

$$\kappa_e \cong 0.2(1 + X) \text{ cm}^2 \text{ g}^{-1}$$

$$n_e = \frac{\rho}{\mu_e m_H}$$

The opacity resulting from electron scattering is temperature and density **independent!**

# Opacity (3)

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We need an expression for opacity to solve the equations of stellar structure. For stars in thermodynamic equilibrium with only a slow outward flow of energy, the opacity should have the form  $\kappa = \kappa(\rho, T, \text{chemical composition})$

Opacity coefficients may be calculated, taking into account all possible interactions between the elements and photons of different frequencies. This requires an enormous amount of calculation and is beyond the scope of this course. Such calculations are done e.g. by the OPAL opacity project at Lawrence Livermore National Laboratory.

When it is done, the results are usually approximated by the relatively simple formula:

$$\kappa = \kappa_0 \rho^\alpha T^\beta$$

where  $\alpha, \beta$  are slowly varying functions of density and temperature and  $\kappa_0$  is a constant for a given chemical composition.

# Opacity (4)

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Figure shows opacity as a function of temperature for a star of given  $\rho$  ( $10^{-4} \text{ g cm}^{-3}$ ). Solid curve is from detailed opacity calculations. Dotted lines are approximate power-law forms.

- At high  $T$ :  $\kappa$  is low and remains constant. Most atoms are fully ionized, high photon energy, hence free-free absorption unlikely. Dominant mechanism is electron scattering, independent of  $T$ ,  $\alpha=\beta=0$ :

$$\kappa = \kappa_1 = \sigma_T / m_H \mu_e \quad (\text{curve c})$$

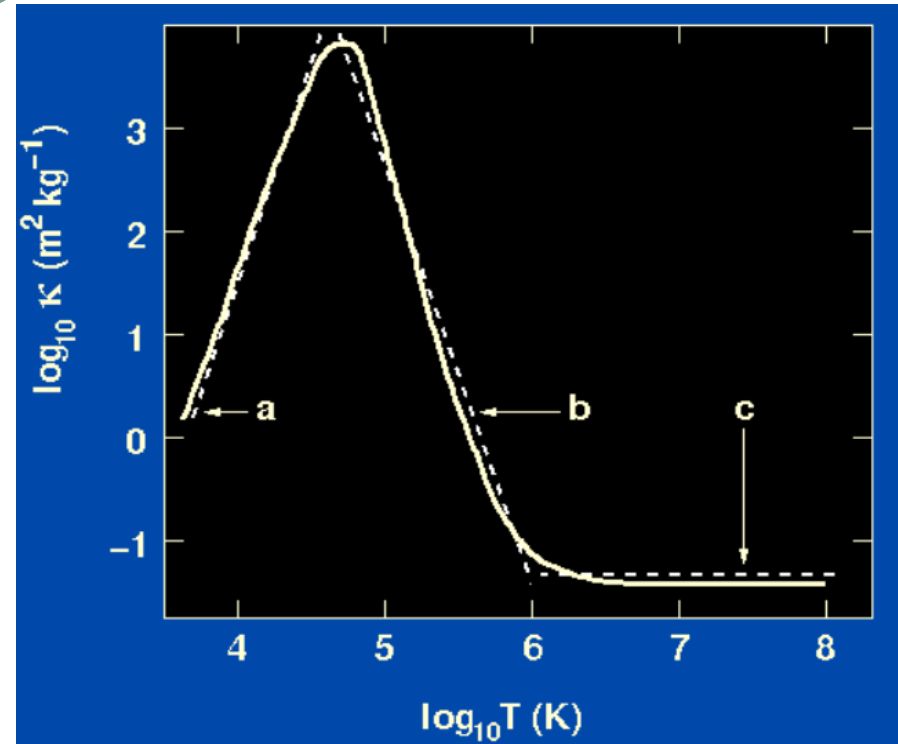
- Opacity is low a low  $T$ , and increases towards higher  $T$ . Most atoms are not ionized, few electrons available to scatter photons or for free-free absorption. Approximate analytical form is  $\alpha=1/2$ ,  $\beta=4$ :

$$\kappa = \kappa_2 \rho^{0.5} T^4 \quad (\text{curve a})$$

- At intermediate  $T$ ,  $\kappa$  peaks, when bound-free and free-free absorption are very important, then decreases with  $T$  (Kramers opacity law):

$$\kappa = \kappa_3 T^{-7/2} \quad (\text{curve b})$$

- $\kappa_1, \kappa_2, \kappa_3$  are constants for stars of given chemical composition but all depend on composition.



# Summary and conclusions



- We have learned the approximate forms of the equation of state and the opacity.
- Next lecture: A method of simplifying the solution of the stellar structure equations.
- After that we will discuss nuclear reactions and move on to discussing the output of full numerical solutions of the equations and realistic predictions of modern theory