Equilibrium in stellar interiors

BASIC ASSUMPTIONS
MASS CONSERVATION
HYDROSTATIC EQUILIBRIUM
VIRIAL THEOREM
STELLAR TIME-SCALES

Introduction and recap (1)

Definition of **a star** as an object:

- Bound by self-gravity
- Radiates energy that is primarily released by nuclear fusion reactions in the stellar interior

Other energy sources are dominant during star formation and stellar death:

- **Star formation** - before the interior is hot enough for significant fusion, gravitational potential energy is radiated as the radius of the forming star contracts. *Protostellar or pre-main-sequence evolution*.
- **Stellar death** - remnants of stars (white dwarfs and neutron stars) radiate stored thermal energy and slowly cool down. Sometimes refer to these objects as stars but more frequently as *stellar remnants*.

Introduction and recap (2)

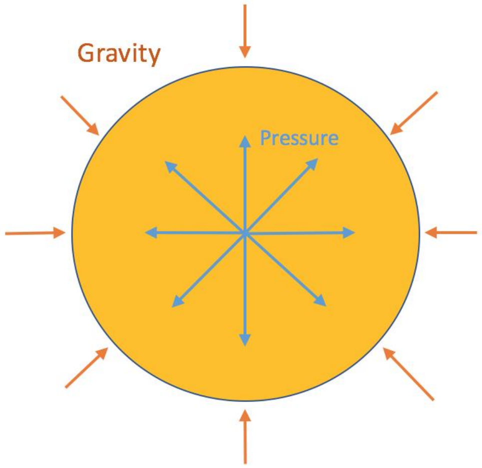
With this definition:

- **planets** are not stars - no nuclear fusion.
- objects in which release of gravitational potential energy is always greater than fusion are not stars either – these are called **brown dwarfs.**

Distinction between brown dwarfs and planets is less clear, most people reserve "planet" to mean very low mass bodies in orbit around a star.

Irrespective of what we call them, physics of stars, planets, stellar remnants is similar.
Balance between:

- **Gravity**
- **Pressure**



Basic assumptions (1)

What are the **main** physical processes which **determine** the structure of **stars**?

- Stars are **held together** by **gravitation** – attraction exerted on each part of the star by all other parts
- Collapse is **resisted** by internal thermal **pressure**.
- These two forces play the principal role in determining stellar structure – they must be (at least almost) in balance. If they are not, the star will explode or collapse on very short (dynamical) time-scale. Since stars do seem to be rather stable on time-scale of millennium, the balance is good.
- Stars **continually radiating** energy into space. As they do not seem to cool dramatically on the civilization lifetime-scale, an energy source must exist (we will see later that thermal energy is not enough).
- Theory must describe - **origin of energy** and **transport to surface**.

Basic assumptions (2)

We make two fundamental assumptions :

1. Neglect the rate of change of properties – assume constant with time.
2. All stars are spherical and symmetric about their centres. Thus, all quantities (e.g., density, temperature, pressure) depend only on the distance from the centre of the star - radius r.

Density as function of radius is $\rho\,(r)$.

If m is the mass interior to r, then: $m(r) = \int_0^r 4\pi r^2 \rho(r)\, dr$

Differential form of this equation is: $dm = 4\pi r^2 \rho dr$

Two equivalent ways of describing the star:

- Properties as $f(r)$: e.g. temperature $T(r)$
- Properties as $f(m)$: e.g. $T(m)$

Second way often more convenient, because (ignoring mass loss) total mass M of the star is fixed, while radius R evolves with time.

We will start with these assumptions and later reconsider their validity.

Stellar structure

For our stars – which are isolated, static, and spherically symmetric – there are **four** basic equations to describe structure. All physical quantities depend on the distance from the centre of the star alone

- **Conservation of mass**
- **Equation of hydrostatic equilibrium**: at each radius, forces due to pressure differences balance gravity
- **Conservation of energy**: at each radius, the change in the energy flux equals the local rate of energy release
- **Equation of energy transport**: relation between the energy flux and the local gradient of temperature

These basic equations supplemented with

- **Equation of state** (pressure of a gas as a function of its density and temperature)
- **Opacity** (how opaque the gas is to the radiation field)
- Nuclear energy generation rate as $f(\rho,T)$.

Equation of mass conservation

Mass $m(r)$ contained within a star of radius r is determined by the density of the gas $\rho(r)$.

Consider a thin shell inside the star with radius r and outer radius $r+dr$:

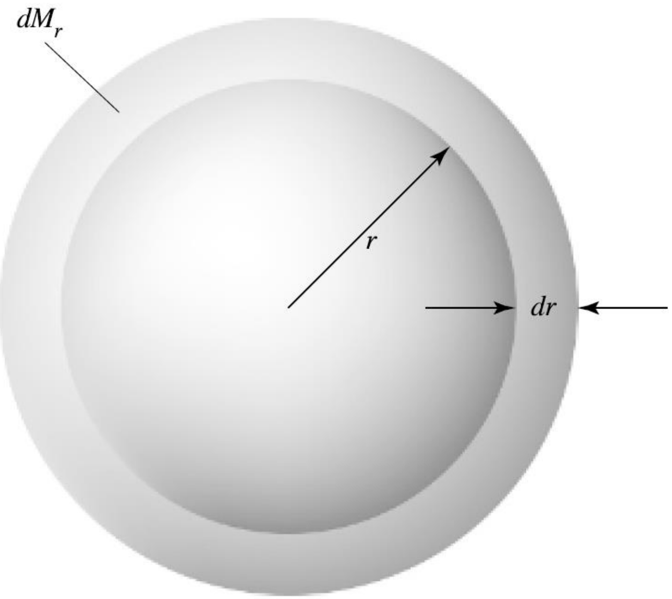
$dV = 4\pi r^2 dr$

$dM = dV\rho(r) = 4\pi r^2 \rho(r) dr$

In the limit where $dr \rightarrow 0$:

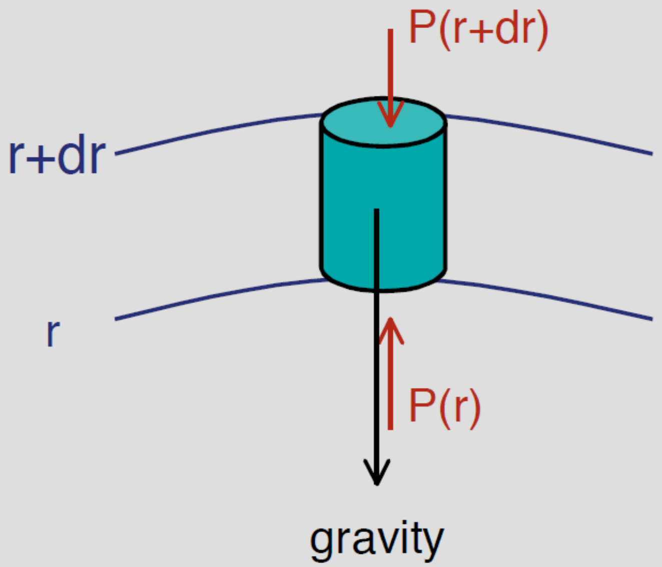
$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

This the **equation of mass conservation**.



Hydrostatic equilibrium (1)

- Balance between gravity and gradient of internal pressure is known as hydrostatic equilibrium.



- Consider a small cylindrical element between radius r and radius $r + dr$ in the star.
  - Its surface area = ds
  - Mass of the element: $dm = \rho(r)\, ds\, dr$
  - Mass of gas in the star at smaller radii: m = m(r)

Hydrostatic equilibrium (2)

Consider forces acting in radial direction:

- Outward force: pressure exerted by stellar material on the bottom face:

$$F_{P,b} = P(r)ds$$

- Inward forces:
  - Gravity (gravitational attraction of all stellar material lying within r):

$$F_g = -\frac{Gm}{r^2}dm = \frac{Gm}{r^2}\rho(r)\,ds\,dr$$

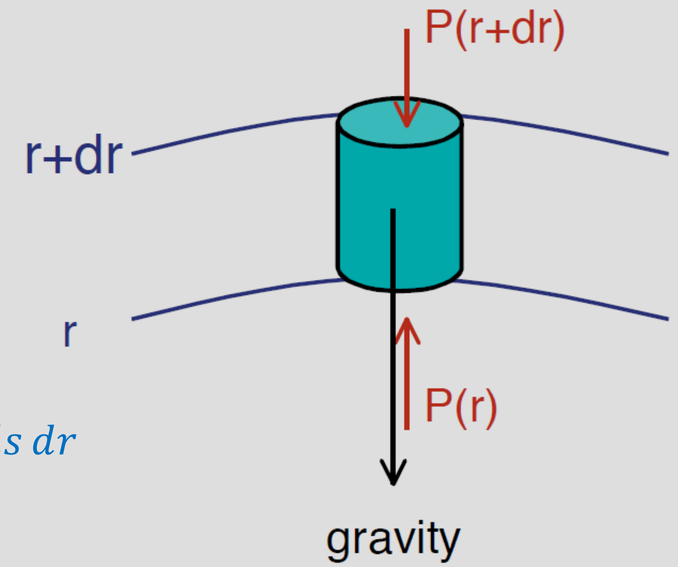
  - Pressure exerted by stellar material on the top face:

$$F_{P,t} = P(r+dr)ds$$

- In hydrostatic equilibrium:

$$F_{P,b} = F_{P,t} + F_g$$

$$P(r)ds = P(r+dr)ds + \frac{Gm}{r^2}\rho(r)\,ds\,dr$$



Hydrostatic equilibrium (3)

$$P(r)ds = P(r+dr)ds + \frac{Gm}{r^2}\rho(r)\,ds\,dr$$

$$\Rightarrow \quad P(r+dr) - P(r) = -\frac{Gm}{r^2}\rho(r)\,dr$$

If we consider an infinitesimal element, we write for $dr \to 0$

$$\frac{P(r+dr) - P(r)}{dr} = \frac{dP(r)}{dr}$$

Hence rearranging above, we get **the equation of hydrostatic equilibrium:**

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

Hydrostatic equilibrium (4)

The equation of hydrostatic equilibrium:

$$\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$$

Combining it with the equation of mass conservation, we obtain an **alternate** form of hydrostatic equilibrium equation, in which enclosed mass m is used as the dependent variable:

$$\frac{dP(r)}{dm} = \frac{dP(r)}{dr} \times \frac{dr}{dm} = -\frac{Gm}{r^2}\rho \times \frac{1}{4\pi r^2 \rho}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

$$\frac{dP(r)}{dm} = -\frac{Gm}{4\pi r^4}$$

Hydrostatic equilibrium (5)

Properties of the equation of hydrostatic equilibrium: $\frac{dP(r)}{dr} = -\frac{Gm}{r^2}\rho(r)$

1) Pressure always **decreases** outward
2) Pressure gradient vanishes at $r = 0$
3) Condition at surface of star: $P = 0$ (to a good first approximation)

(2) and (3) are **boundary conditions** for the hydrostatic equilibrium equation.

Accuracy of hydrostatic assumption (1)

We have assumed that the gravity and pressure forces are balanced – how valid is that ?

Consider the case where the outward and inward forces are not equal,
there will be a resultant force acting on the element which will give rise to an acceleration *a:*

$$P(r+dr)ds + \frac{Gm}{r^2}\rho(r)\,ds\,dr - P(r)ds = dm \times a = a\,\rho(r)\,ds\,dr$$

[Applying Newton's second law (*F*=*ma*) to the cylinder]

acceleration = 0 everywhere if star **static**

$$\frac{dP(r)}{dr} + \frac{Gm}{r^2}\rho(r) = a\,\rho(r)$$

Now acceleration due to gravity is $g = \frac{Gm}{r^2}$

$$\frac{dP(r)}{dr} + g\rho(r) = a\rho(r)$$

This is a generalized form of the equation of hydrostatic support.

Accuracy of hydrostatic assumption (2)

$$\frac{dP(r)}{dr} + g\rho(r) = a\rho(r)$$

Now suppose there is a resultant force on the element (LHS≠0).

Suppose their sum is small fraction of gravitational term (β): $\beta g\rho(r) = a\rho(r)$

Hence there is an inward acceleration of

$$a = \beta g$$

Assuming it begins at rest, the spatial displacement d after a time t is

$$d = \frac{1}{2}at^2 = \frac{1}{2}\beta g t^2$$

Calculate!

Accuracy of spherical symmetry assumption

Stars are rotating gaseous bodies – to what extent are they flattened at the poles?

If so, departures from spherical symmetry must be accounted for.

Consider mass m near the surface of a star of mass M and radius r.

Element will be acted on by centrifugal force $F_c = m\omega^2 r$, where ω= angular velocity of the star.

There will be **no** departure from spherical symmetry provided that

$$\frac{F_c}{F_g} = m\omega^2 r / \frac{GMm}{r^2} \ll 1 \quad \text{or} \quad \omega^2 \ll \frac{GM}{r^3}$$

Solar rotation period is about P ≈ 27 days.

Angular velocity $\omega = 2\pi/P \approx 2.7\times10^{-6}\ s^{-1}$ $\qquad F_c/F_g \sim 2\times10^{-5}$

...even rotation rates much faster than that of the Sun are negligibly small to influence star's structure.

Accuracy of spherical symmetry assumption

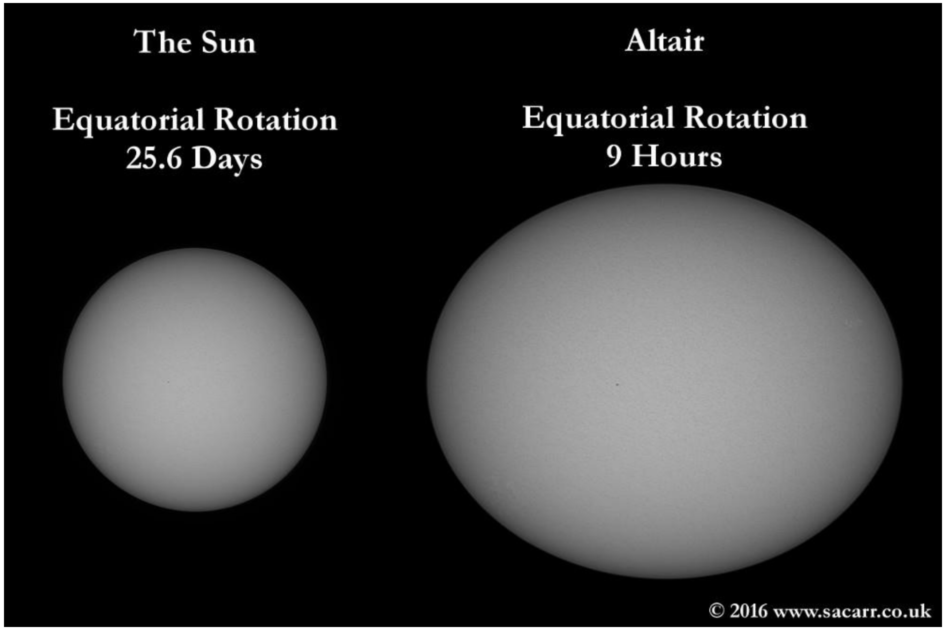
$V = (4/3)\,\pi\, R^3$

In terms of mean density, we get

$$\omega^2 \ll \frac{GM}{r^3} \approx \pi G \bar{\rho} \quad \Rightarrow \quad \bar{\rho} \gg \frac{4\pi}{GP^2} \approx \frac{1.9 \times 10^8}{P^2}$$

For the majority of stars, departures from spherical symmetry can be **ignored**.

However, some stars do rotate rapidly and rotational effects must be included in the structure equations –
can change the output of models.



Accuracy of spherical symmetry assumption

Isolation?

In the Solar neighborhood, distances between stars are enormous: e.g. Sun's nearest stellar companion is Proxima Centauri at $d = 1.3$ pc. Ratio of Solar radius to this distance is:

$$\frac{R_{sun}}{d} \approx 2 \times 10^{-8}$$

Two important implications:

- Can ignore the gravitational field and radiation of other stars when considering stellar structure.
- Stars (almost) never collide with each other.

Once star has formed, initial conditions rather than interactions with other stars determine evolution.

However, stars in double systems are elongated due to gravitational attraction.

The Virial theorem (1)

Let's again take the hydrostatic equilibrium equation, in which enclosed mass m is used as the dependent variable (or combine the equation of hydrostatic equilibrium with the equation of mass conservation):

$$\frac{dP(r)}{dm} = -\frac{Gm}{4\pi r^4}$$

$$\frac{dP(r)}{dr} \times \frac{dr}{dm} = -\frac{Gm}{r^2}\rho \times \frac{1}{4\pi r^2 \rho}$$

$$\frac{dM}{dr} = 4\pi r^2 \rho(r)$$

Now multiply both sides by volume V=(4/3)πr^3:

$$3V(r)dP = -\frac{Gm}{r}dm$$

And integrate over the whole star:

$$3\int_{P_c}^{P_s} V(r)dP = -\int_0^M \frac{Gm}{r}dm$$

integrating by parts

$$3[PV]_c^s - 3\int_{V_c}^{V_s} PdV = -\int_0^M \frac{Gm}{r}dm$$

At centre, V_c=0 and at surface P_s=0

The Virial theorem (2)

Hence, we have

$$3\int_0^V P\,dV = \int_0^M \frac{Gm}{r}\,dm = -E_G$$

Now the right-hand term = - total gravitational binding energy of the star, or it is the energy needed to spread the star to infinity, or to assemble the star by bringing gas from infinity.

$$3\int_0^V P\,dV = -E_G$$

← version of the virial theorem

The left-hand side contains pressure integral. With some assumptions about the pressure, we can progress further.

The Virial theorem (3)

$$3\int_0^V P dV = -E_G$$

For **ideal** gas, $P = NkT$,

where N is concentration, T is the temperature, k is Boltzmann's constant,

while the thermal (kinetic) energy per particle is $e_{kin} = \frac{3}{2}NkT$

Thus, the LHS is

$$3\int_0^V P dV = 2\int_0^V e_{kin} dV = 2E_T$$

where E_T is the thermal energy of the star.

Thus, we can write the Virial Theorem:

or for the total energy $E = E_T + E_G$:

$$2E_T + E_G = 0$$

$$E = -E_T$$

This is of great importance in astrophysics and has many applications.