Astrophysics of interacting binary stars

Problem Set 2. Solutions.

Problem 3

How much mass per time must a neutron star of 1.4 $M_\odot$ accrete in order for its luminosity to equal that of Betelgeuse? The luminosity of Betelgeuse is about 140000 times that of the Sun.

Solution 3

$L=L_{\text{Betelgeuse}}=140000\ L_\odot=140000*3.85\cdot10^{33}$ erg/s$=5.39\cdot10^{38}$ erg/s

$$L=\frac{GM\dot{M}}{R} \quad \rightarrow \quad \dot{M}=\frac{LR}{GM}=\frac{5.39\cdot10^{38}\times\ 1.2\cdot10^{6}}{6.67\cdot10^{-8}\ \times1.4\cdot1.99\cdot10^{33}}=3.48\times10^{18}\ \mathrm{g/s}$$

$$\dot{M}=5.52\times10^{-8}\ \ M_\odot\ /\ \mathrm{yr}$$

Problems 4 and 5

Write down the accretion disk’s relative thickness as a function of the accretor’s mass, the mass flow rate, and the disk’s inner and outer radii (assuming that an average effective temperature applies everywhere in the disk), and discuss the influences of the different quantities onto the relative thickness.

Assume an accretion disk around a white dwarf of 0.85 $M_\odot$ (R_{wd}=0.01 $R_\odot$) with an accretion rate of 10^{-9} $M_\odot$/year and outer radius of the disk R_{out}=0.5 $R_\odot$. Estimate, how thick is such a disk at R_{in} and R_{out}, assuming a uniform effective temperature everywhere?

Solutions 4 and 5

(4) $\frac{h}{R} \sim \frac{C_s}{V_{rot}}$ $\quad C_s \sim T^{1/2} \cdot 10^4$ cm/s

$$V_{rot}^2 = \frac{GM}{R}$$

$$\frac{h^2}{R^2} \triangleq \frac{10^8\, T \cdot R}{GM} \qquad L_d = \frac{GM\dot{M}}{2R_*} = \underset{\substack{\downarrow \\ \text{2 sides}}}{2}\,\pi\underbrace{\left(R_d^2 - R_*^2\right)}_{\text{area of the disk}} \sigma T^4$$

$$T = \left[\frac{GM\cdot\dot{M}}{2R_*\, 2\pi\sigma\left(R_d^2 - \cancel{R_*^2}\right)}\right]^{1/4} = \frac{G^{1/4}}{\sqrt{2}\,\pi^{1/4}\sigma^{1/4}}\frac{M^{1/4}\dot{M}^{1/4}}{R_d^{1/2}R_*^{1/4}}$$

$$\frac{h^2}{R^2} = \frac{10^8}{G}\frac{G^{1/4}}{\sqrt{2}\,\pi^{1/4}\sigma^{1/4}}\frac{M^{1/4}\dot{M}^{1/4}\,R}{M\,R_d^{1/2}R_*^{1/4}} = [\;]\, M^{-3/4}\dot{M}^{1/4}R_d^{-1/2}R_*^{-1/4}R$$

$$h = \underbrace{\sqrt{[\;]}}_{\sim 1{,}2\cdot 10^7} M^{-3/8}\dot{M}^{1/8}R_d^{-1/4}R_*^{-1/8}R^{3/2} \quad \text{cm}$$

(5)

$M = 0{,}85\, M_\odot = 1{,}69 \cdot 10^{33}$ g

$\dot{M} = 10^{-9}\, M_\odot/yr = 6{,}3 \cdot 10^{16}$ g/s

$R_* = 6{,}96 \cdot 10^8$ cm

$R_{out} = 3{,}48 \cdot 10^{10}$ cm

$$h_{in} = 1{,}2\cdot 10^7 M^{-3/8}\dot{M}^{1/8}R_{out}^{-1/4}\cdot R_*^{11/8}$$

$$h_{out} = 1{,}2\cdot 10^7 M^{-3/8}\dot{M}^{1/8}R_*^{-1/8}R_{out}^{5/4}$$

$h_{in} = 1{,}7 \cdot 10^6$ cm $= 0{,}003\, R_{in}$

$h_{out} = 6{,}17 \cdot 10^8$ cm $= 0{,}02\, R_{out}$

Problem 6

In lecture we derived the following expression for the effective temperature T as a function of radial distance from the central compact star:

$$T(R) = \left\{ \frac{3GM\dot{M}}{8\pi R^3 \sigma} \left[1 - \left(\frac{R_*}{R} \right)^{1/2} \right] \right\}^{1/4}$$

where σ is the Stefan-Boltzmann constant.

1) Use this expression to find the location (i.e., the radial distance from the central star) where the temperature is a maximum. Express your answer in terms of R_*, the radius of the inner edge of the disk.

$$T(R) = \left[\frac{3GM\dot{M}}{8\pi R^3 \sigma} \left(1 - \left(\frac{R_*}{R} \right)^{1/2} \right) \right]^{1/4}$$

$$T_* = \left(\frac{3GM\dot{M}}{8\pi R_*^3 \sigma} \right)^{1/4}$$

$$* \quad T(R) = T_* \left(\frac{R_*}{R} \right)^{3/4} \left[1 - \sqrt{\frac{R_*}{R}} \right]^{1/4}$$

$$T' = T_* \left[-\frac{3}{4} \left(\frac{R_*}{R} \right)^{-1/4} \cdot \frac{R_*}{R^2} \left(1 - \sqrt{\frac{R_*}{R}} \right)^{1/4} + \left(\frac{R_*}{R} \right)^{3/4} \cdot \frac{1}{4} \left(1 - \sqrt{\frac{R_*}{R}} \right)^{-3/4} \times \frac{1}{2} \left(\frac{R_*}{R} \right)^{-1/2} \frac{R_*}{R^2} \right] =$$

$$= T_* \frac{R_*}{R^2} \left[\frac{1}{8} \left(\frac{R_*}{R} \right)^{1/4} \left(1 - \sqrt{\frac{R_*}{R}} \right)^{-3/4} - \frac{3}{4} \left(\frac{R_*}{R} \right)^{-1/4} \left(1 - \sqrt{\frac{R_*}{R}} \right)^{1/4} \right] = 0$$

$$\frac{1}{8} \left(\frac{R_*}{R} \right)^{1/4} \left(1 - \sqrt{\frac{R_*}{R}} \right)^{-3/4} = \frac{3}{4} \left(\frac{R_*}{R} \right)^{-1/4} \left(1 - \sqrt{\frac{R_*}{R}} \right)^{1/4}$$

$$\frac{4^4}{8^4 \cdot 3^4} \frac{R_*}{R} \left(1 - \sqrt{\frac{R_*}{R}} \right)^{-3} = \frac{R}{R_*} \left(1 - \sqrt{\frac{R_*}{R}} \right)$$

$$\frac{1}{2^4 \cdot 3^4} \frac{R_*^2}{R^2} = \left(1 - \sqrt{\frac{R_*}{R}} \right)^4$$

$$\sqrt{\frac{R_*}{R}} = 6 - 6\sqrt{\frac{R_*}{R}} \rightarrow 7\sqrt{\frac{R_*}{R}} = 6 \rightarrow 49 \frac{R_*}{R} = 36$$

$$\frac{R}{R_*} = \frac{49}{36} = 1{,}36$$

$$T = T_* \left(\frac{36}{49} \right)^{3/4} \left[1 - \frac{6}{7} \right]^{1/4} = T_* \cdot 0{,}615 \cdot 0{,}794 = 0{,}488\, T_*$$

2) Compute T_{max} for the following types of accreting sources:
   a) White dwarf: M=0.85 $M_\odot$; $\dot{M} = 10^{-9}$ $M_\odot$ / yr ; $R_* = 7 \times 10^8$ cm
   b) Neutron star: M=1.4 $M_\odot$; $\dot{M} = 10^{-9}$ $M_\odot$ / yr ; $R_* = 1.2 \times 10^6$ cm
   c) a non-rotating Black hole: M=5.0 $M_\odot$; $\dot{M} = 10^{-9}$ $M_\odot$ / yr

$$T_* = \left(\frac{3GM\dot{M}}{8\pi R_*^3 \sigma} \right)^{1/4}$$

$$T_{max} = 0.488\, T_*$$

a) $T_{wd,\,max}$= 39800 K
b) $T_{ns,\,max}$= $5.3 \cdot 10^6$ K
c) For a non-rotating black hole the innermost stable orbit is

$$R = 3R_{Sch} = \frac{6GM}{c^2}$$

$R_{bh,in}$ =$4.42 \cdot 10^6$ cm
$T_{bh,\,max}$= $2.76 \cdot 10^6$ K