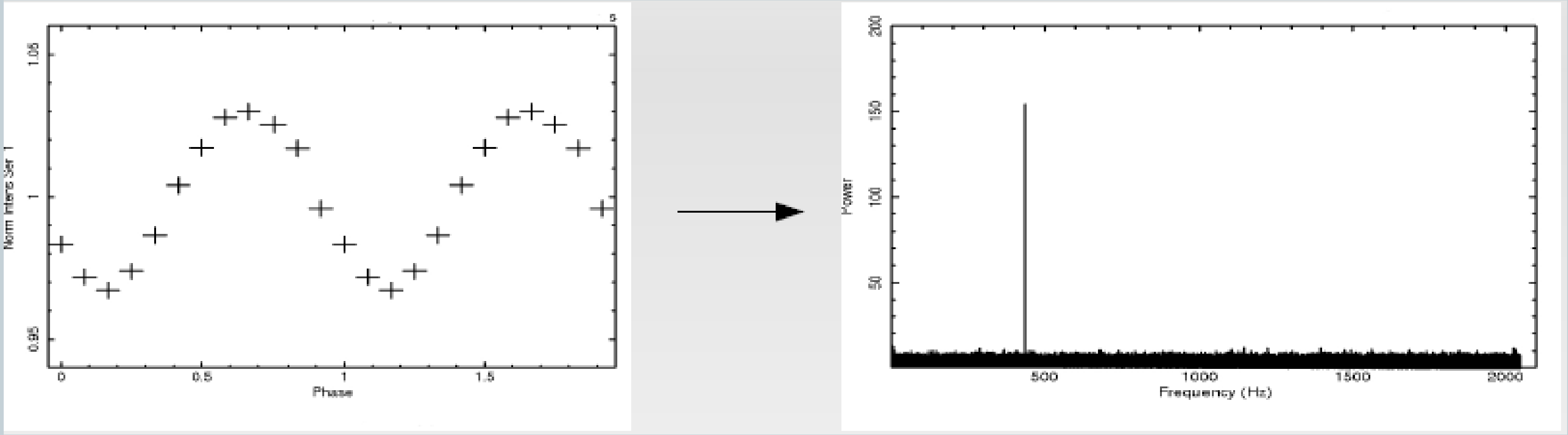
Continuous Fourier transform

- The continuous Fourier transform of an infinitely extended sine (or cosine) wave is a delta function (this is not in general true for the discrete Fourier transform).



Basic properties of Fourier transform

- Dirac delta function (δ-function):

$$\delta(x) = 0, \quad x \neq 0$$
$$\delta(x) = \infty, \quad x = 0$$

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1$$

Continuous Fourier transform

- The continuous Fourier transform has a number of pleasing properties.
- Therefore, theoretical predictions of the shape of the Fourier transform of a signal are usually in terms of the continuous Fourier transform.
- … but we don't have either continuous or infinite signals.

- **Fourier theorem:** the **discrete** Fourier transform gives a complete description of the discrete signal.

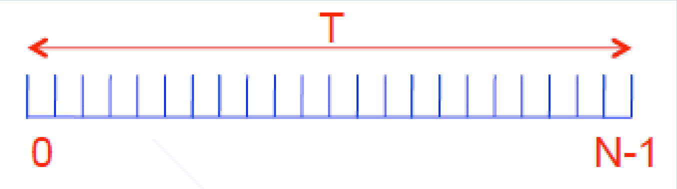
Discrete Fourier transform of real time series

- Time series, x_k, $k=0, \ldots, N-1$
- The discrete Fourier transform decomposes this signal into N sine waves, a_j, $j= -N/2+1, \ldots, N/2$

$$a_j = \sum_{k=0}^{N-1} x_k \, e^{2\pi i j k/N} \qquad j = -\frac{N}{2}+1, \ldots, \frac{N}{2}$$

$$x_k = \frac{1}{N} \sum_{j=-\frac{N}{2}+1}^{N/2} a_j \, e^{-2\pi i j k/N} \qquad k = 0, \ldots, N-1$$

Discrete Fourier transform of real time series



- Time step, $\delta t = T/N$
- Frequency step, $\delta\nu = 1/T$
- x_k refers to time $t_k = kT/N$
- a_j refers to frequency $\omega_j = 2\pi\nu_j = 2\pi j/T$
- *So, for* $e^{i\omega_j t_k}$ we have $e^{2\pi ijk/N}$

- **Note that the number (N) of input values x_k equals the number of output values a_j.**
- **If x_k are uncorrelated, then a_j are uncorrelated as well.**

Discrete Fourier transform of real time series

- The highest frequency you need for a complete description of the discrete signal is the **Nyquist frequency**

  $\nu_{Ny}=\nu_{N/2}=\frac{N}{2T}$ – half the "sampling“ frequency
- Lowest frequency (>0) = frequency of the first frequency step = $1/T$ = frequency of sinusoid that fits exactly once on T
- At zero frequency you get

  $a_0=\sum_k x_k=N_{ph}$ – the total number of counts (photons detected), always real for real x_k

Fast Fourier Transform (FFT)

- FFT is an efficient algorithm to compute the discrete Fourier transform and its inverse;
- The FFT has been called the most important numerical algorithm of our lifetime;
- The computation time can be reduced by several orders of magnitude (especially for long data);
- **Number of points $N_{bin} = 2^n$; n – integer.**

Power spectrum

- **Parseval's theorem:**
  The total power in a signal is the same in the time domain and in the frequency domain:

$$Total\ Power \equiv \int_{-\infty}^{+\infty} |x(t)|^2\, dt = \int_{-\infty}^{+\infty} |a(\nu)|^2\, d\nu$$

- For a given signal, the **power spectrum** gives a plot of the portion of a signal's power (energy per unit time) falling within given frequency bins.

Power Spectral Density (PSD)

- How much ***power*** is contained in the frequency interval bentween ν and $\nu+d\nu$?

  Power spectral density (PSD):

  $$P(\nu) \equiv |a(\nu)|^2, \ -\infty < \nu < \infty$$

  One-sided PSD:

  $$P(\nu) \equiv |a(\nu)|^2 + |a(-\nu)|^2, 0 \leq \nu < \infty$$

  We have real *x(t)*, then:

  $$P(\nu) \equiv 2|a(\nu)|^2$$

Power Density Spectrum

- **Parseval's theorem**:

$$\sum_{k=0}^{N-1} {x_k}^2 = \frac{1}{N} \sum_{j=-\frac{N}{2}+1}^{N/2} |a_j|^2$$

- A Power Density Spectrum is computed as the squared Fourier amplitudes with **some normalization**.
- A light curve of N bins (x_k) is then translated into a PDS of N/2+1 independent amplitudes.

Power Density Spectrum

- **Non-linear transformation:**
  Suppose that the data x_k are the sum of y_k and z_k.

$$\sum_k x_k\, e^{i\omega_j t_k} = \sum_k (y_k + z_k)\, e^{i\omega_j t_k} = \sum_k y_k\, e^{i\omega_j t_k} + \sum_k z_k\, e^{i\omega_j t_k}$$

- **But!**

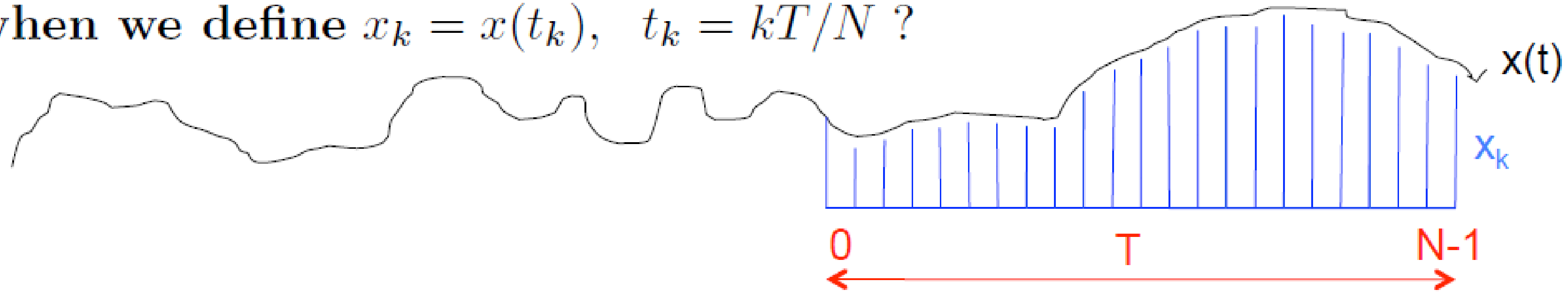
$$x_k = y_k + z_k \;\Rightarrow\; \begin{aligned} a_j &= b_j + c_j \\ |a_j|^2 &= |b_j|^2 + |c_j|^2 + crossterms \end{aligned}$$

- If independent (random noise added), cross terms average out to zero

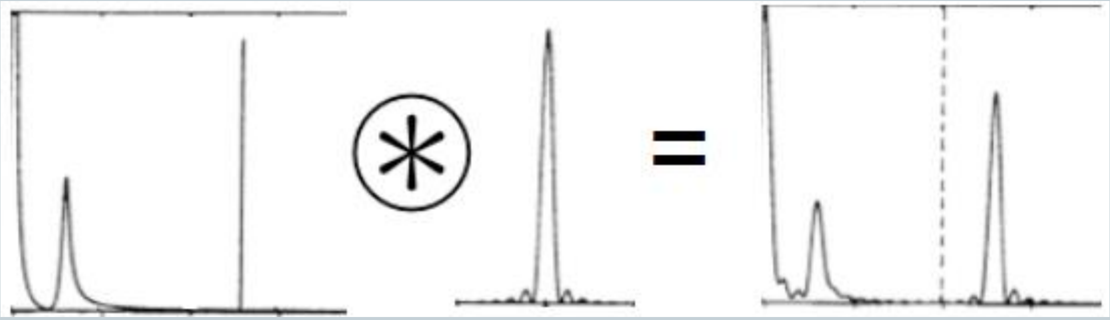
Continuous FT ⇔ Discrete FT

- How can we connect continuous and discrete FT?

What is the relation of this 'ideal case' with the discrete Fourier transform **when we define** $x_k = x(t_k),\quad t_k = kT/N$?



- We use the convolution theorem: " the transform of the product is the convolution of the transforms".



Continuous FT ⇔ Discrete FT

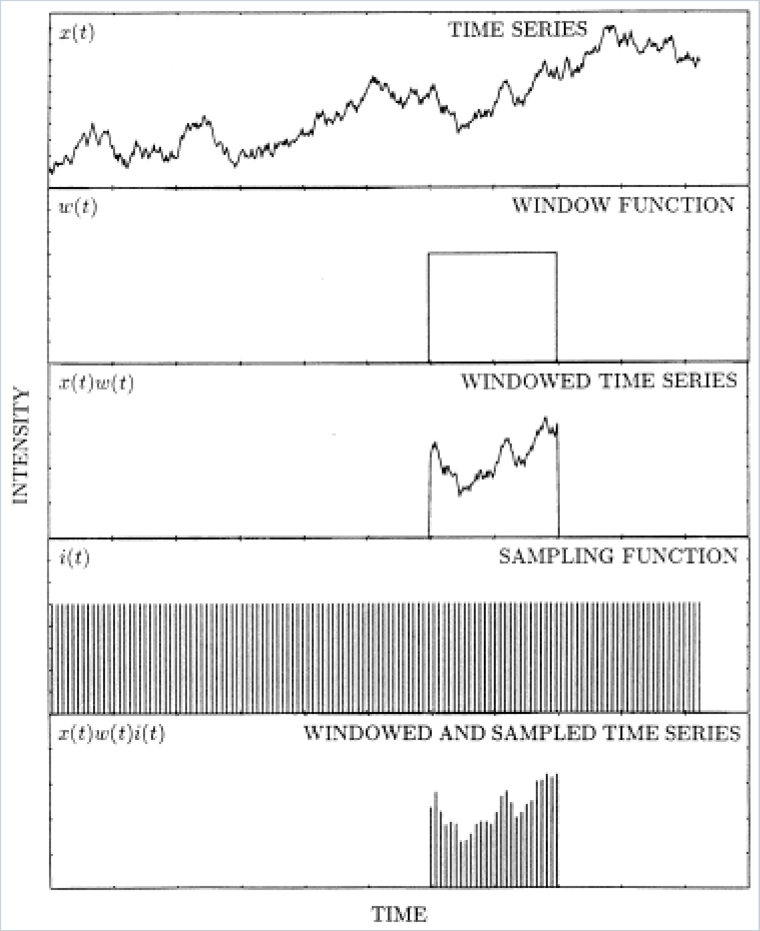
- We have:
  - Time-series $x(t)$
  - Window function $w(t)$:

$$w(t) = \begin{cases} 1, & 0 \leq t < T \\ 0, & \text{otherwise} \end{cases}$$

  - Sampling function $i(t)$:

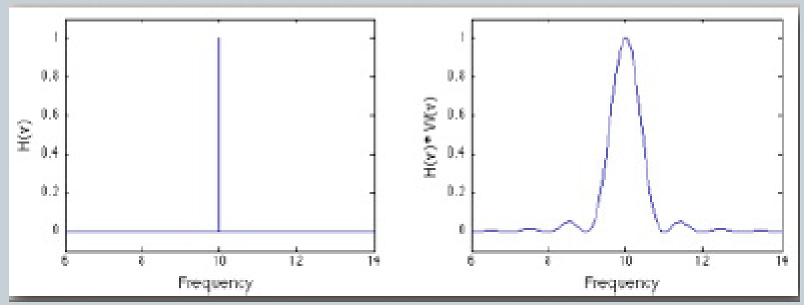
$$i(t) = \sum_{-\infty}^{+\infty} \delta\left(t - \frac{kT}{N}\right)$$

- Now we multiply:
$x_k = x(t)\, w(t)\, i(t)$

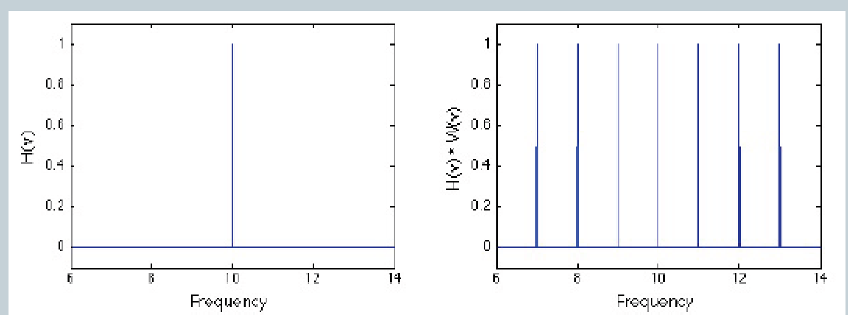


Continuous FT ⇔ Discrete FT

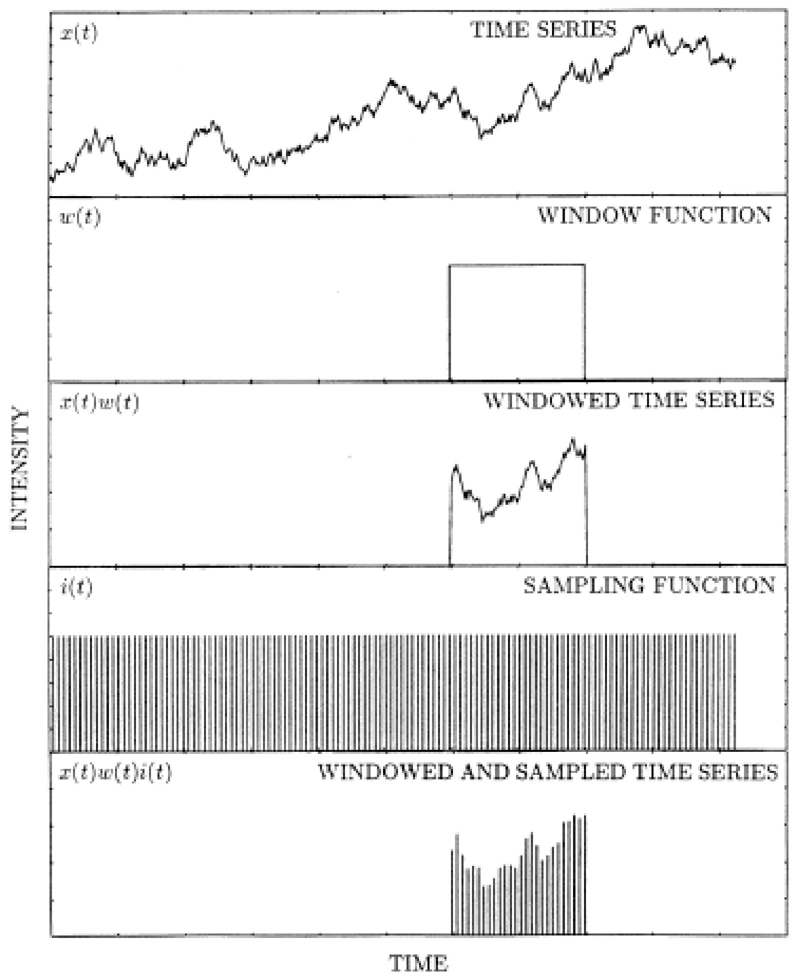
- The power spectrum of *W(ν)*: $$|W(\nu)|^2 \equiv \left| \int_{-\infty}^{\infty} w(t) e^{2\pi\nu i t}\, dt \right|^2 = \left| \frac{\sin \pi\nu T}{\pi\nu} \right|^2$$
  - Broadening of peaks

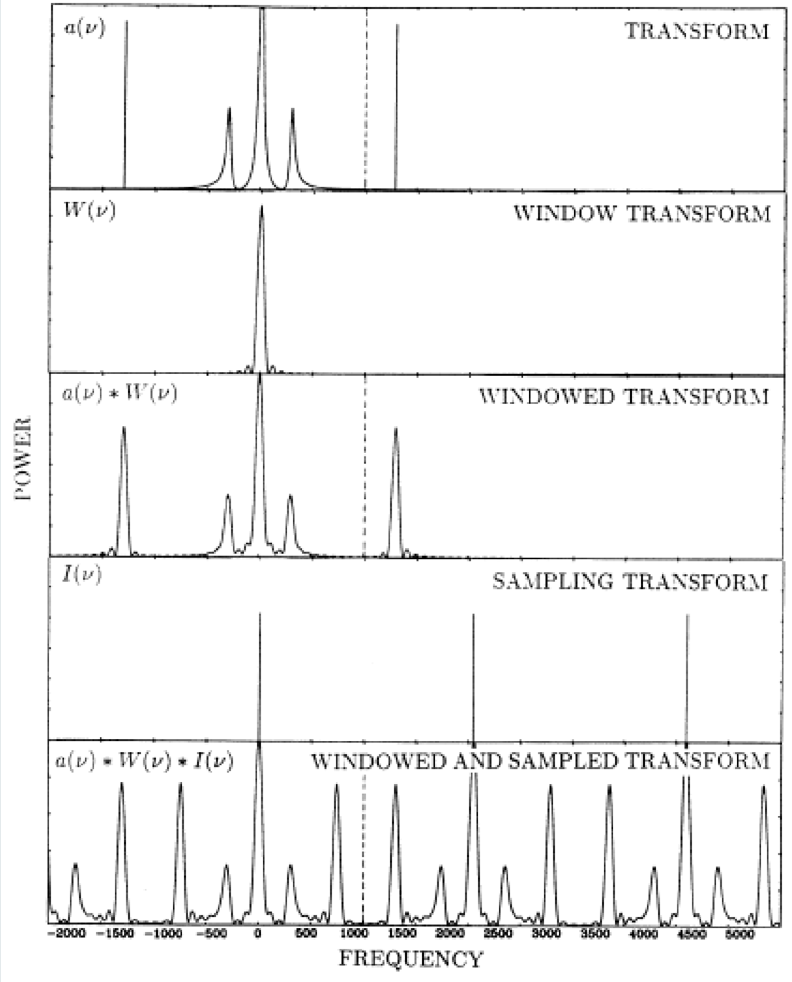


- The Fourier transform of *i(t)*: $$I(\nu) \equiv \int_{-\infty}^{\infty} i(t) e^{2\pi\nu i t}\, dt = \frac{N}{T} \sum_{\ell=-\infty}^{\infty} \delta\left(\nu - \ell \frac{N}{T}\right)$$
  - Infinite series of δ functions, with spacing N/T = $\mathbf{2\,\nu_{Ny}}$



Continuous FT ⇔ Discrete FT





Summary of discrete FT effects

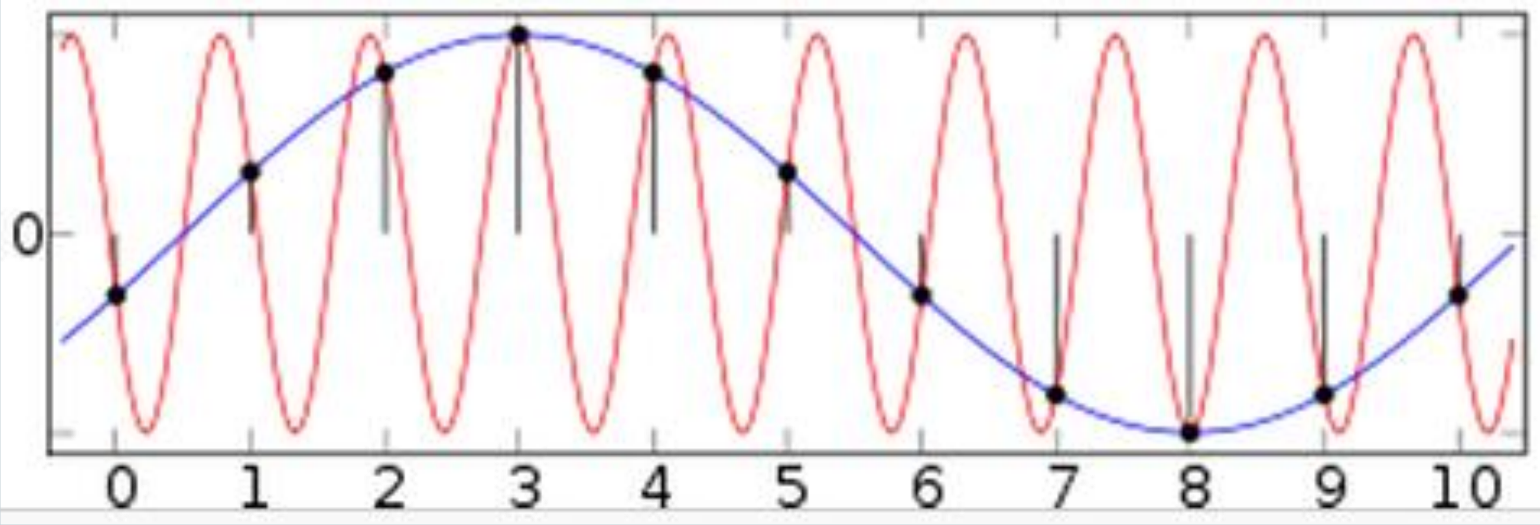
- WINDOW: broadening & sidebands
- SAMPLING: aliasing

Aliasing

- FT is symmetric in frequency for a real signal;
- Alias repeats it every $2\nu_{Ny}$:
- The power spectrum of the convolved function $|a(\nu) * I(\nu)|^2$ is reflected around the Nyquist frequency ν_{Ny}.
- This causes features with a frequency exceeding ν_{Ny} by ν_x (so, located at $\nu = \nu_{Ny} + \nu_x$) to also appear at a frequency (ν_{Ny} - ν_x) — a phenomenon known as aliasing; the reflected feature is called the alias of the original one.

Aliasing

- Two different sinusoids fit the same set of samples:



Is aliasing a problem?

- Not such a big problem for high-energy astronomy;
- In practice, we do not really discretely sample the data, but rather bin the data up;
- That means that before discrete sampling we convolve the *x(t)* with the bin width.
- So, in the frequency domain, we multiply $a(\nu)$ with

$$B(\nu) = \frac{\sin(\frac{\pi\nu N}{T})}{\pi\nu N/T}$$

- Suppression of high frequencies.

Is windowing a problem?

- Yes:
  - It broadens delta peaks;
  - It flattens the slopes of noise components (sidelobes);
  - For steep spectra the ”leakage” can be severe.

- Solution: The longer the observation, the better.