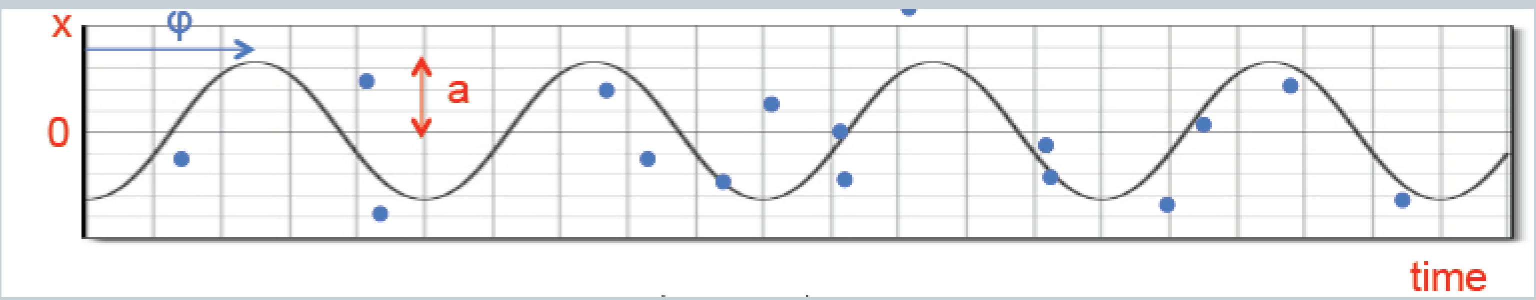
Curve-Fitting Approach

- The simplest periodic data are those consisting of a single cosine (sine) wave:

$$x(t) = a \cos(\omega t - \varphi) = A \cos \omega t + B \sin \omega t$$



Frequency

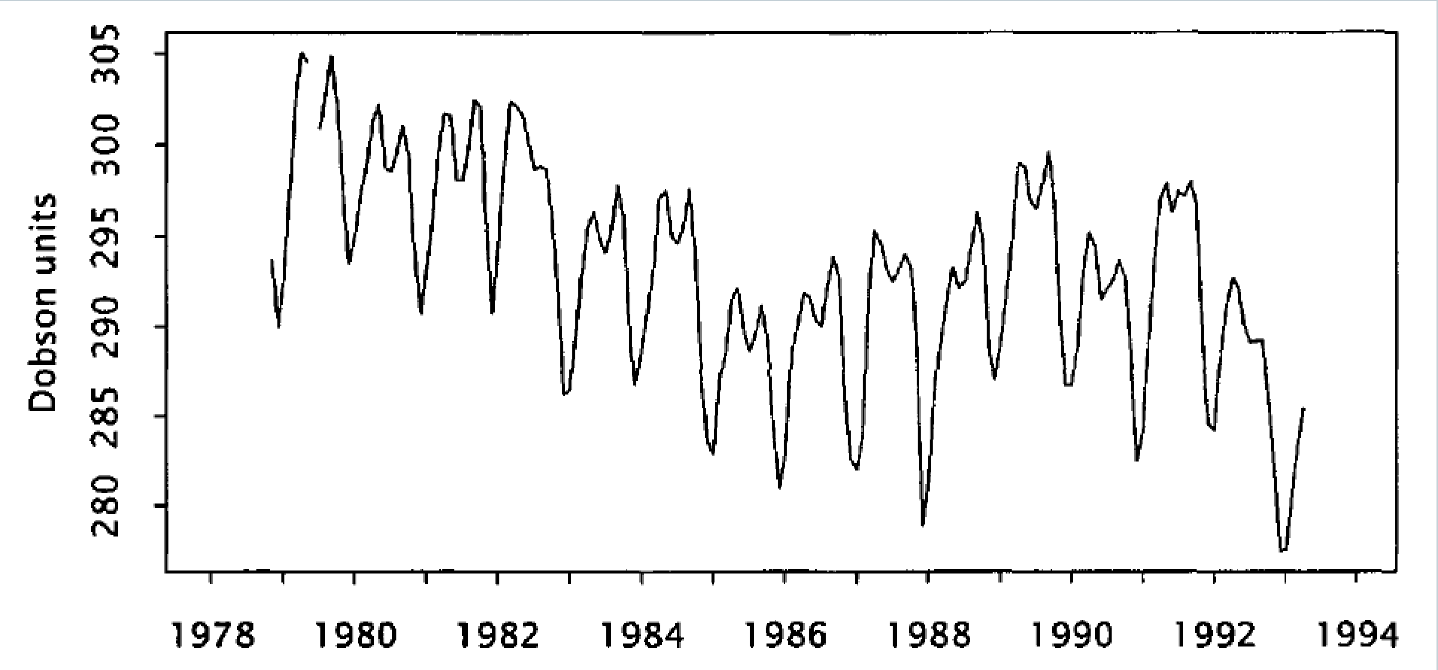
- The basic relation:

$$Frequency = \frac{1}{Period} \quad \text{or} \quad \nu = \frac{1}{P}$$

- If the Period is in seconds, then Frequency will be in Herz [Hz]
- If the Period is in days, then Frequency will be in 1/day [Cycles per day]

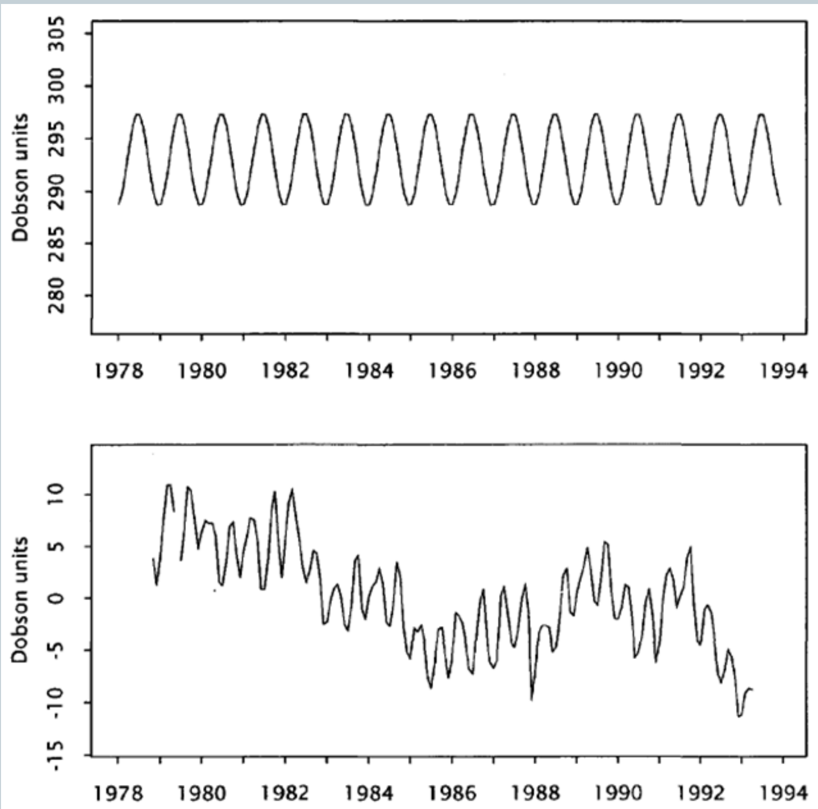
- Angular frequency $\omega = 2\pi\ \nu$ [radiands per second]

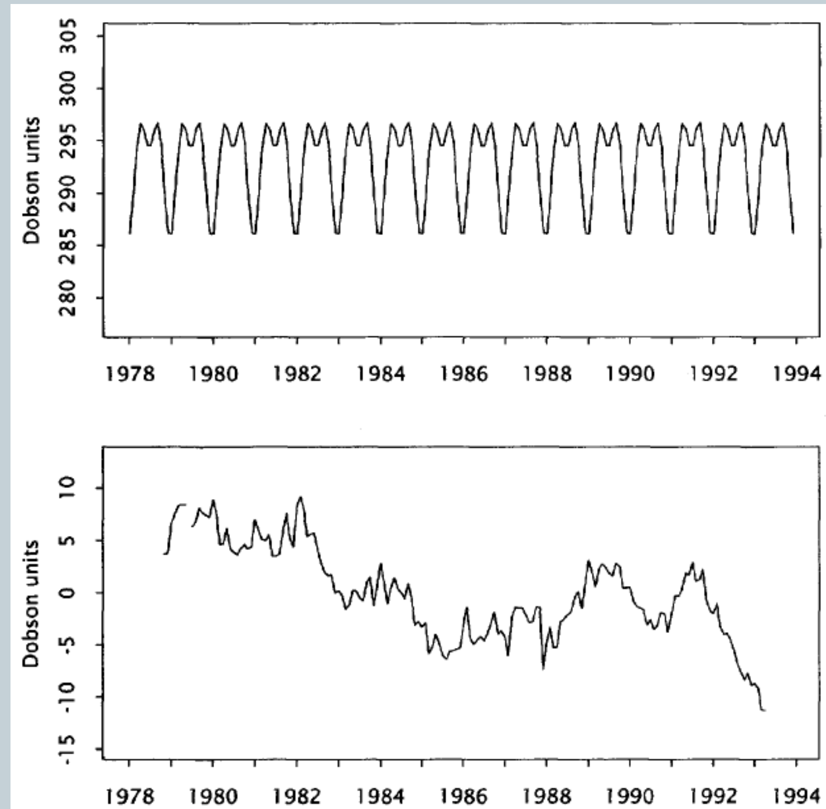
Curve-Fitting Approach



Monthly average total ozone levels, 65° S to 65° N

Curve-Fitting Approach





Fitting one (left) and two (right) sinusoids with **known** periods.
If the period is unknown then the fitting is not simple.

Harmonic Analysis

CONTINUOUS AND DISCRETE FOURIER TRANSFORM

POWER SPECTRUM

Time and Frequency domains

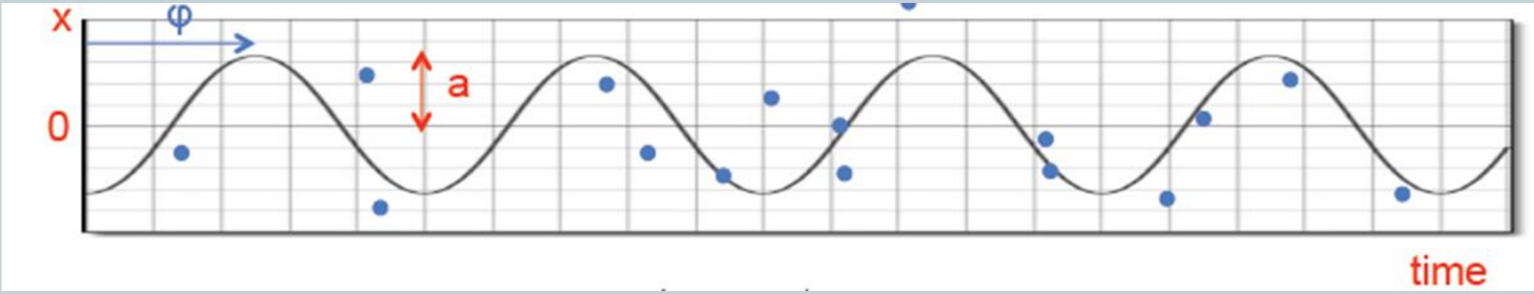
- A physical process can be described either in the time domain, by the values of some quantity x as a function of time t, e.g., x(t), or else in the frequency domain, where the process is specified by giving its amplitude X (generally a complex number indicating phase also) as a function of frequency ν, that is X(ν), with $-\infty < \nu < +\infty$. For many purposes it is useful to think of x(t) and X(ν) as being two different representations of the same function.
- One goes back and forth between these two representations by means of the Fourier transform equations.

FOURIER TRANSFORM

- A Workhorse of the Timing World (or a part of this).
- The Additive Model for a Time Series:
  data are realizations of random variables Y_t that are themselves sums of different components (for example, signal and noise).

FOURIER TRANSFORM

- A Fourier transform is a decomposition of the signal into sine waves



- At ω, best-fit sinusoid is:

$$x(t) = a\cos(\omega t - \varphi) = A\cos\omega t + B\sin\omega t$$

$$a = \sqrt{A^2 + B^2} \text{ and } \tan\varphi = -B/A$$

FOURIER TRANSFORM

- Do this at many frequencies ω_j, then

$$x(t) = \frac{1}{N}\sum_j a_j \cos\left(\omega_j t - \varphi_j\right) = \frac{1}{N}\sum_j \left(A_j \cos \omega_j t + B_j \sin \omega_j t\right)$$

- The Fourier coefficients A_j and B_j can be straightforwardly computed as:

$$A_j = \sum_k x_k \cos \omega_j t_k$$

$$B_j = \sum_k x_k \sin \omega_j t_k$$

where $x_k = x(t_k)$

FOURIER TRANSFORM

- A_j and B_j are simply the correlation of the signal x_k with a sine or cosine wave of frequency ω_j;
- If there is a good correlation then the corresponding Fourier coefficient is large and gives a large contribution to the sum;
- So, good correlation:
  large A, B — bad correlation: small A, B.

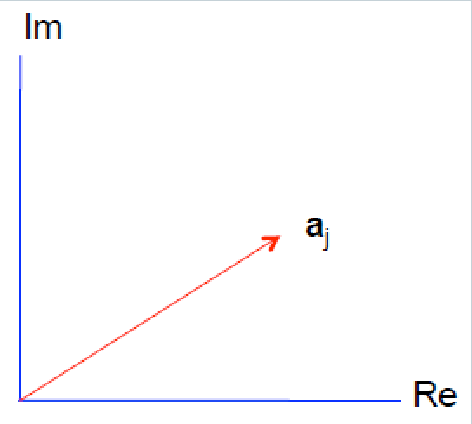
Complex representation

- At each ω we get two numbers: (A, B) or (a, φ). For easier handling, it is possible to represent the Fourier transform in terms of complex numbers:

$$a_j = \sum_k x_k \, e^{i\omega_j t_k}$$

$$x_k = \sum_j a_j \, e^{-i\omega_j t_k}$$

$$i^2 = -1$$



The complex numbers a_j – complex Fourier amplitudes:

$$a_j = |a_j| \; e^{i\varphi_j} = |a_j| \left(\cos\varphi_j + i \sin\varphi_j\right)$$

Complex representation

The Euler relation:

$$e^{ix} = \cos x + i \sin x$$

and its inverse:

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

$$\sin x = \frac{1}{2i}\left(e^{ix} + e^{-ix}\right)$$

Do not worry! Observed data are strictly real-valued.
We consider both positive and negative frequencies, $\omega_{-j} = -\omega_j$
Imaginary terms at $+j$ and $-j$ cancel out in Σ_j to produce strictly **real** terms $2|a_j| \cos(\omega_j t_k - \varphi_j)$

Complex representation

Good explanation:

http://betterexplained.com/articles/an-interactive-guide-to-the-fourier-transform/

Continuous Fourier transform

- Decomposes a function into an infinite number of sinusoidal waves.
- Signal $x(t)$: $-\infty < t < +\infty$
- Transform $a(\nu)$: $-\infty < \nu < +\infty$

$$a(\nu) = \int_{-\infty}^{\infty} x(t) e^{2\pi\nu i t}\, dt \qquad -\infty < \nu < \infty$$

$$x(t) = \int_{-\infty}^{\infty} a(\nu) e^{-2\pi\nu i t}\, d\nu \qquad -\infty < t < \infty$$

Basic properties of Fourier transform

Let's consider transform pairs: $x(t) \Leftrightarrow a(v)$

- **Linearity:** The transform of the sum of two functions is equal to the sum of the transforms:

  $$x(t) + y(t) \Leftrightarrow a(v) + b(v)$$

  We can analyse complex optical systems by looking at different frequencies separately.
- **Time scaling:** The transform of a constant times a function is that same constant times the transform of the function: $x(ct) \Leftrightarrow \frac{1}{|c|} a\left(\frac{v}{c}\right)$

Basic properties of Fourier transform

- **Convolution:**
  The convolution of the two functions *x(t)* and *y(t)* is

$$x(t) * y(t) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} x(\tau) y(t-\tau)\, d\tau$$

  Basic properties are
  commutativity: $x * y = y * x$
  distributivity over addition: $x * (y+z) = x * y + x * z$

- **Convolution theorem:** The Fourier transform of the convolution is the product of the individual Fourier transforms: $x(t) * y(t) \Leftrightarrow a(\nu)\, b(\nu)$