Stellar Atmospheres
Lecture 9

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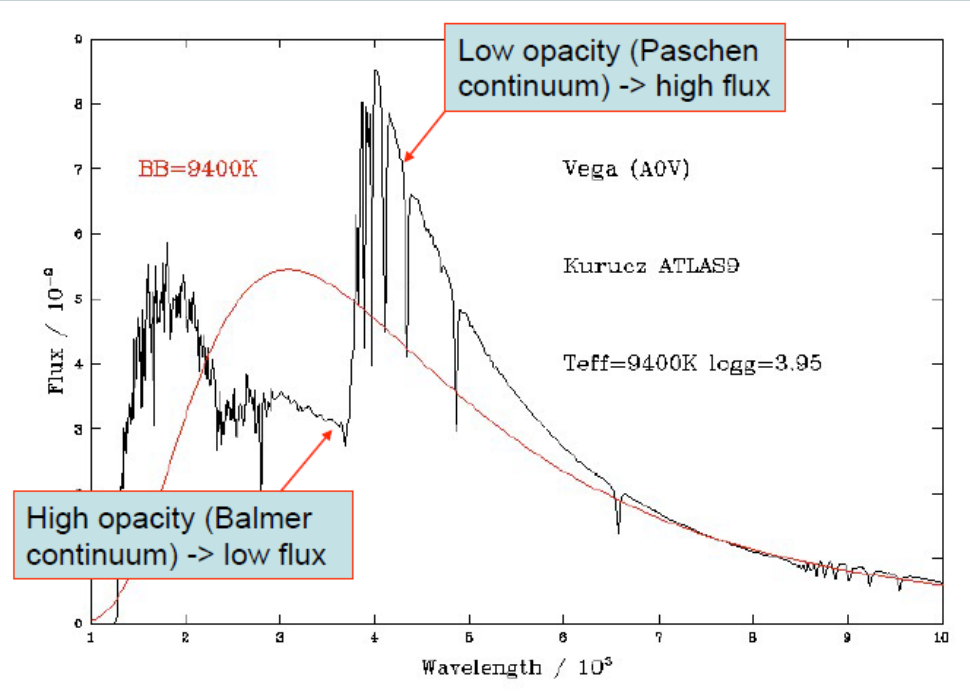
Summary from the last lecture

- Opacity depends strongly on wavelengths → the atmosphere is **NOT** grey.
- Nongreyness changes the temperature structure.



More about mean opacity

ROSSELAND MEAN OPACITY
ROSSELAND DEPTH

The mean absorption coefficient

- The grey approximation (α=const) is very coarse, but can still be useful. Is there a sensible mean value $\bar{\alpha}$ to use? What choice to make for a mean value?
- We demand flux conservation and hope to keeps the temperature structure.

- **From the third radiative equilibrium condition:**

$$K_\lambda(\tau_\lambda) = \frac{1}{3} J_\lambda(\tau_\lambda) = \frac{1}{3} B_\lambda$$

the Eddington approximation

$$F = \int F_\lambda d\lambda = 4\pi \int \frac{dK_\lambda}{d\tau_\lambda} d\lambda = 4\pi \int \frac{dK_\lambda}{\alpha_\lambda ds} d\lambda = \frac{4\pi}{3} \int \frac{dB_\lambda}{\alpha_\lambda ds} d\lambda$$

$$F = \frac{4\pi}{3} \frac{1}{\alpha_R} \int \frac{dB_\lambda}{ds} d\lambda = \frac{4\pi}{3} \frac{1}{\alpha_R} \frac{dB}{ds}$$

$$\frac{1}{\alpha_R} = \frac{\int \frac{dB_\lambda}{\alpha_\iota ds} d\lambda}{\frac{dB}{ds}}$$

Rosseland mean opacity

$$\frac{1}{\alpha_R} = \frac{\int \frac{1}{\alpha_\lambda}\frac{dB_\lambda}{ds}d\lambda}{\frac{dB}{ds}}$$

$$\frac{dB}{ds} = \frac{dB}{dT}\frac{dT}{ds} \quad \text{and} \quad \frac{dB}{dT} = \frac{d}{dT}\left(\frac{\sigma}{\pi}T^4\right) = \frac{4\sigma}{\pi}T^3$$

$$\frac{1}{\alpha_R} = \frac{\int_0^\infty \frac{1}{\alpha_\lambda}\frac{dB_\lambda}{dT}d\lambda}{\frac{4\sigma}{\pi}T^3}$$

Definition of Rosseland mean opacity

The Rosseland mean $1/\alpha_R$ is a weighted (harmonic) mean of opacity, for which there is a corresponding optical depth (Rosseland depth):

$$\tau_{Ross}(s) = \int_0^s \alpha_R(z)dz$$

We hoped for the temperature structure:

$$T^4(\tau) = \frac{3}{4}(\tau + \frac{2}{3})T_{eff}^4 = \frac{3}{4}(\tau_{Ross} + \frac{2}{3})T_{eff}^4$$

Eddington approximation

The grey approximation is very good for $\tau_{Ross} \gg 1$.

Towards the Model Photosphere

HYDROSTATIC EQUILIBRIUM
GAS PRESSURE
ELECTRON PRESSURE
RADIATION PRESSURE
EDDINGTON LIMIT

Model atmospheres (example)

$\log \tau_0$	T (K)	$\log P_g$ (dyne/cm^2)	$\log P_e$ (dyne/cm^2)	$\log \kappa_0/P_e$ (cm^2/g per dyne/cm^2)	x (km)
Solar model, $S_0 = 1.0$, $\log g = 4.438$ cm/s^2					
−4.0	4310	2.87	−1.16	−1.22	−509
−3.8	4325	3.03	−1.02	−1.23	−476
−3.6	4345	3.17	−0.89	−1.24	−448
−3.4	4370	3.29	−0.78	−1.25	−422
−3.2	4405	3.41	−0.66	−1.26	−397
−3.0	4445	3.52	−0.55	−1.28	−373
−2.8	4488	3.64	−0.44	−1.30	−349
−2.6	4524	3.75	−0.33	−1.32	−325
−2.4	4561	3.86	−0.23	−1.33	−301
−2.2	4608	3.97	−0.12	−1.35	−277
−2.0	4660	4.08	−0.01	−1.37	−252
−1.8	4720	4.19	0.10	−1.40	−228
−1.6	4800	4.30	0.22	−1.43	−203
−1.4	4878	4.41	0.34	−1.46	−177
−1.2	4995	4.52	0.47	−1.50	−151
−1.0	5132	4.63	0.61	−1.55	−124
−0.8	5294	4.74	0.76	−1.60	−97
−0.6	5490	4.85	0.93	−1.66	−70
−0.4	5733	4.95	1.15	−1.73	−43
−0.2	6043	5.03	1.43	−1.81	−19
0.0	6429	5.10	1.78	−1.91	0
0.2	6904	5.15	2.18	−2.01	15
0.4	7467	5.18	2.59	−2.11	27
0.6	7962	5.21	2.92	−2.18	37
0.8	8358	5.23	3.16	−2.23	46
1.0	8630	5.26	3.32	−2.25	56
1.2	8811	5.29	3.42	−2.27	68
$S_0 = 0.7$, $\log g = 4.6$, normal abundances					
−4.0	3017	3.22	−2.12	−0.46	−246
−3.0	3111	3.89	−1.51	−0.53	−179

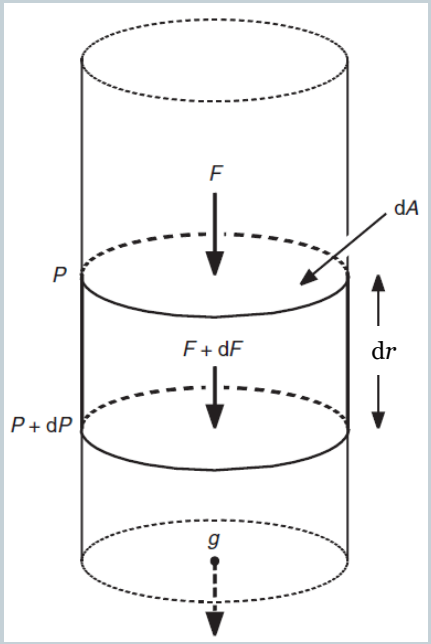
Usual assumptions to start with:

1. Plane parallel geometry, making all physical variables a function of only one space coordinate.
2. Hydrostatic equilibrium, meaning that the photosphere is not undergoing largescale accelerations comparable to the surface gravity; there is no dynamically significant mass loss.
3. Structures such as granulation or starspots are negligible, or at least can be adequately represented by mean values of the physical parameters.
4. Magnetic fields are excluded.

Ideal gas

We require a knowledge of the **electron pressure** in order to use the Saha equation, which is related to the **gas pressure**. How do we calculate this in stellar atmospheres?

We start with **hydrostatic equilibrium**.



Forces acting upon the volume element of density ρ(r) are *gravity*:

$$dF_g = -\frac{GM_r dm}{r^2} = -\frac{GM_r \rho(r)}{r^2} dAdr$$

plus *buoyancy* (pressure difference × area):

$$dF_P = -dPdA$$

Since the mass of the atmosphere is negligible compared to the stellar mass and the radius of the photosphere is negligible vs the stellar radius R,

$$dF_g = -\frac{GM_r \rho(r)}{R^2} dAdr = -g\rho(r)dAdr$$

since

$$g = \frac{GM}{R^2}$$

Hydrostatic equilibrium

The Earth's atmosphere is neither lost to space nor pulled to the surface of the Earth via a balance between gravity ($\mathrm{d}F_g = -g\rho\,\mathrm{d}A\,\mathrm{d}r$) and gas pressure ($\mathrm{d}F_p = -\mathrm{d}P\,\mathrm{d}A$) i.e. **hydrostatic equilibrium** which is the balance between gravitational and pressure forces ($\mathrm{d}F_g + \mathrm{d}F_p = 0$). If the gas pressure is P_g the differential form of the hydrostatic equilibrium equation for an ideal gas is then

$$\frac{dP_g}{dr} = -g\rho(r)$$

We can eliminate $\rho\,(r)$ with the ideal gas equation, $P_g = \rho\, kT/\mu m_{\mathrm{H}}$

$$\frac{dP_g}{dr} = -g\,\frac{\mu(r) m_H}{kT(r)}\,P_g(r) = -g\,\frac{\mu(r)}{R_g T(r)}\,P_g(r)$$

having introduced the gas constant $R_g = k/m_{\mathrm{H}} = 8.3\mathrm{x}10^7$ erg/mol/K

μ - mean molecular weight

Pressure Scale Height

We obtain $$\frac{1}{P_g}\frac{dP_g}{dr} = \frac{d\ln P_g}{dr} = -\frac{g\mu(r)}{R_g T(r)}$$

For an idealized isothermal (T(r)=constant) atmosphere with $\mu(r)$=const, we can integrate this expression

$$P_g(r) = P_g(r_0)e^{-(r-r_0)g\mu/R_g T} = P_g(r_0)e^{-(r-r_0)/H}$$

where we have introduced the scale height *H*, $$H = \frac{kT}{g\mu m_H} = \frac{R_g T}{g\mu}$$

i.e. gas *pressure changes by a factor of e over a scale height.*

For a (ficticious) atmosphere of constant density, corresponding to the gas pressure at the base of the real atmosphere, we can put the total mass of the real atmosphere into a layer of height *H*.

Examples

Betelgeuse	μ=1 (H)	T=3600K	Log g=0	H=3$R_\odot$
Sun	μ=1 (H)	T=6000K	Log g=4.4	H=180 km
Earth	μ=28 (N_2)	T=300K	Log g=3	H=9 km
White Dwarf	μ=0.5 (H^++N_e)	T=1.5x10^4 K	Log g=8	H=0.25 km
Neutron Star	μ=0.5 (H^++N_e)	T=10^6 – 10^7 K	Log g=15	H=1.7mm

Gas Pressure $P_g(\rho)$

When using the Saha equation, we need T and P_g in a particular layer of the atmosphere, which can be described by geometric depth t or optical depth τ. Temperature dependence on average optical depth is known

$$T^4(\bar{\tau}) \approx \frac{3}{4}(\bar{\tau} + \frac{2}{3})T_{eff}^4$$

(from Lecture 5).

The average optical depth $d\bar{\tau} = -\kappa_R \rho \, dr$ may be expressed via the Rosseland mean opacity per unit mass (cm^2/g), κ_R.

Thus we generally express the gas pressure as a function of optical depth. From hydrostatic equilibrium we obtain

$$\frac{dP_g}{dr} = -g\rho(r) \quad \Rightarrow \quad \frac{dP_g}{d\bar{\tau}} = \frac{g}{\kappa_R}$$

The gas pressure can now be obtained by integrating this differential equation, although in general κ_R, is a complicated function of temperature and pressure.

Integration of hydrostatic equation

- In the simplest case, assuming a constant mean opacity (which is not a very sensible approximation, but ok for electron scattering), with τ=0 and P_g=0 at the surface:

$$P_g = \frac{g}{\kappa_R}\bar{\tau}$$

- Knowing $T(\tau)$ for a given $T_{\rm eff}$, we can assume a value for κ_R insert this into the above equation and compute a value for the gas pressure.
- More realistically, we can approximate the mean opacity with a power-law expression,

$$\kappa_R = \kappa_{R,0}\,(P_g / P_{g,0})^{\rm n}$$

with $\kappa_{R,0}$ the opacity at a reference wavelength, and n=1 usually acceptable (especially if H^- is responsible for the continuous opacity).

Gravity dependence of P_g

We can readily integrate this expression (P=0 at surface):

$$P_g^n dP_g = \frac{g}{\kappa_{g,o}} P_{g,0}^n d\bar{\tau}$$

to give

$$\frac{1}{n+1} P_g^{n+1} = \frac{g}{\kappa_{g,o}} P_{g,0}^n \bar{\tau}$$

For n=1

$$P_g = \sqrt{\frac{2g\bar{\tau}P_{g,0}}{\kappa_{g,0}}}$$

i.e. the gas pressure for a given optical depth increases with $g^{1/2}$

For different stars we see down to τ=2/3, whose pressure varies approximately as $g^{1/2}$.

The larger the pressure, the greater the Rosseland mean opacity, so we see geometrically higher layers in stars with higher gravity.

Giants have deep atmospheres, dwarfs thin ones.

Electron pressure

So far we have dealt with the gas pressure, but it is the electron pressure that is needed in the Saha equation.

We can generally say,

$$P_g = NkT$$

where N is the sum of all particles/cm^3 , and

$$P_e = n_e kT$$

with n_e=number of electrons/cm^3. Of course,

$$n_e = n^+ + 2n^{2+} + 3n^{3+} \text{ etc.}$$

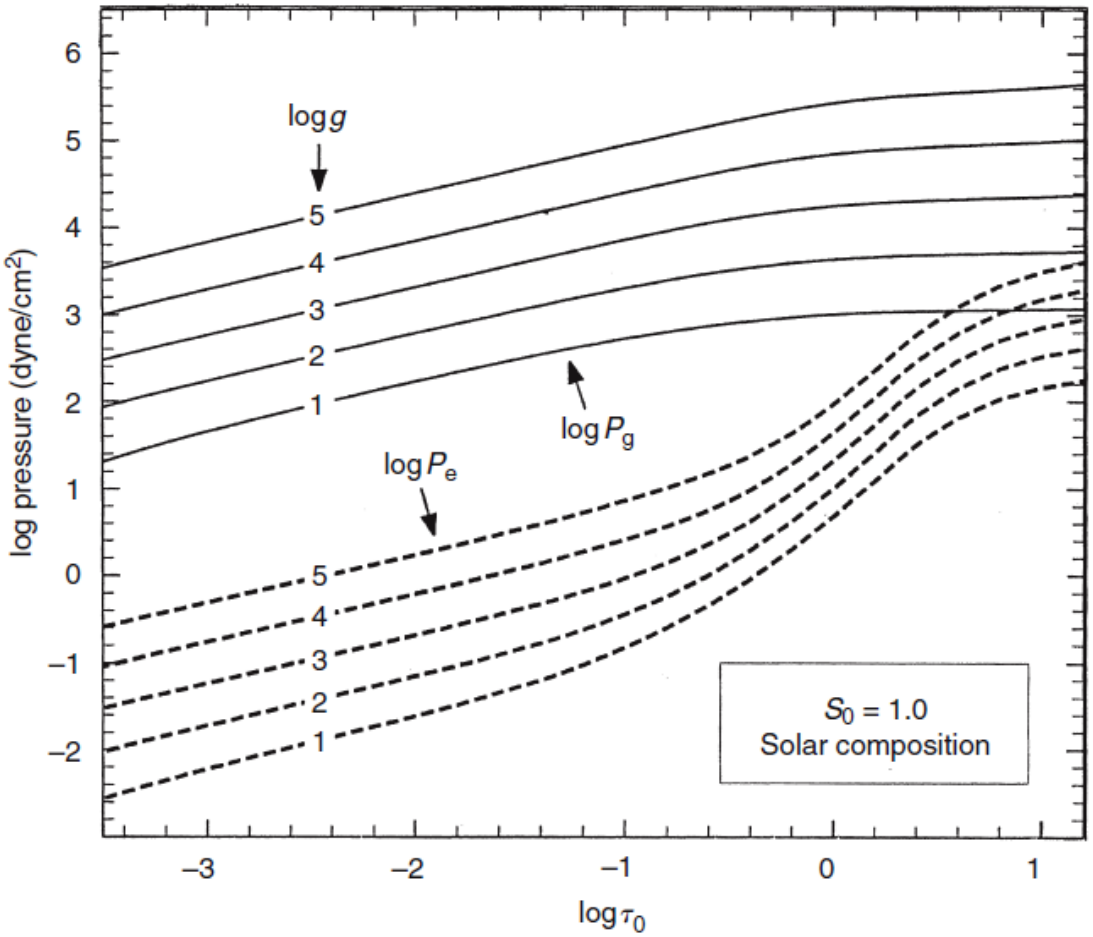
In the simplest case of pure hydrogen,

$$N = N(\mathrm{H}) + N(\mathrm{H}^+) + n_e = N(\mathrm{H}) + 2N(\mathrm{H}^+) = P_g / kT$$

since from charge conservation $n_e = N(\mathrm{H}^+)$.

Given $N(\mathrm{H}^+) n_e / N(\mathrm{H}) = f(T)$ from Saha equation, we may solve for $N(\mathrm{H}^+) = n_e$ and $N(\mathrm{H})$, if T and P_g are known.

Numerical examples



In fact, the gas pressure exponent is not 1/2, but numerical results show it ranges from 0.57 to 0.64 from shallow to deep layers.

The electron pressure dependence on gravity is not so linear, with an exponent of 1/3 predicted ($P_e^2 \propto P_g$ for solar-type stars). Numerical calculations show 0.45 to 0.3 from shallow to deep layers (Gray Fig. 9.13)..

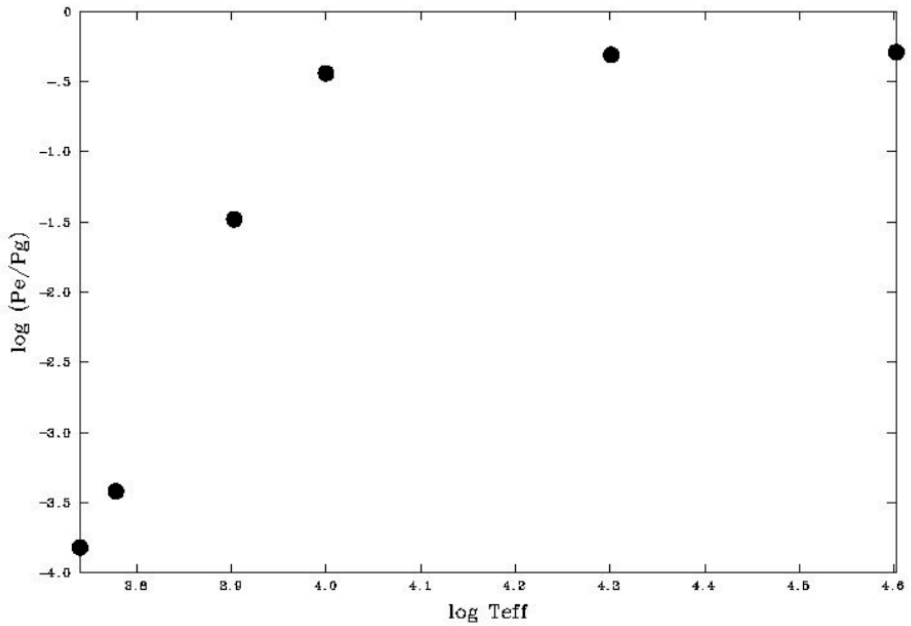
Role of Metals?

For a pure H atmosphere in the case of the Solar photosphere, the gas pressure greatly exceeds the electron pressure. Although metals are few in number, some are very easily ionized e.g. $Na/H=2x10^{-6}$, $Mg/H=3x10^{-5}$, $Al/H=2.7x10^{-6}$, $Ca/H=2x10^{-6}$, $Si/H=3x10^{-5}$. These will **contribute** electrons to the atmosphere, increasing P_e and suppress ionization.

		Stage of ionization												
Atom		I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	XIII	XIV
1	H	13.598 44													
2	He	24.587 41	54.417 78												
3	Li	5.391 72	75.640 18	122.454											
4	Be	9.322 63	18.211 16	153.897	217.713										
5	B	8.298 03	25.154 84	37.931	259.366	340.22									
6	C	11.260 30	24.383 32	47.888	64.492	392.08	489.98								
7	N	14.534 14	29.601 3	47.449	77.472	97.89	552.06	667.03							
8	O	13.618 06	35.117 30	54.936	77.413	113.90	138.12	739.29	871.41						
9	F	17.422 82	34.970 82	62.708	87.140	114.24	157.17	185.19	953.91	1 103.1					
10	Ne	21.564 54	40.963 28	63.45	97.12	126.21	157.93	207.28	239.10	1 195.8	1 362.2				
11	Na	5.139 08	47.286 4	71.620	98.91	138.40	172.18	208.50	264.25	299.9	1 465.1	1 648.7			
12	Mg	7.646 24	15.035 28	80.144	109.265	141.27	186.76	225.02	265.96	328.1	367.5	1 761.8	1 963		
13	Al	5.985 77	18.828 56	28.448	119.99	153.83	190.49	241.76	284.66	330.1	398.8	442.0	2 086	2 304	
14	Si	8.151 69	16.345 85	33.493	45.142	166.77	205.27	246.49	303.54	351.1	401.4	476.4	523	2 438	2 673
15	P	10.486 69	19.769 4	30.203	51.444	65.03	220.42	263.57	309.60	372.1	424.4	479.5	561	612	2 817
16	S	10.360 01	23.337 9	34.79	47.222	72.59	88.05	280.95	328.75	379.6	447.5	504.8	564	652	707
17	Cl	12.967 64	23.814	39.61	53.465	67.8	97.03	114.20	348.28	400.1	455.6	529.3	592	657	750
18	Ar	15.759 62	27.629 67	40.74	59.81	75.02	91.01	124.32	143.46	422.5	478.7	539.0	618	686	756
19	K	4.340 66	31.63	45.806	60.91	82.66	99.4	117.56	154.88	175.8	503.8	564.7	629	715	787
20	Ca	6.113 16	11.871 72	50.913	67.27	84.50	108.78	127.2	147.24	188.5	211.3	591.9	657	727	818
21	Sc	6.561 44	12.799 67	24.757	73.489	91.65	111.68	138.0	158.1	180.0	225.2	249.8	688	757	831
22	Ti	6.828 2	13.575 5	27.492	43.267	99.30	119.53	140.8	170.4	192.1	215.9	265.1	292	788	863
23	V	6.746 3	14.66	29.311	46.71	65.28	128.1	150.6	173.4	205.8	230.5	255.1	308	336	896
24	Cr	6.766 64	16.485 7	30.96	49.16	69.46	90.64	161.18	184.7	209.3	244.4	270.7	298	355	384
25	Mn	7.434 02	15.639 99	33.668	51.2	72.4	95.6	119.20	194.5	221.8	248.3	286.0	314	344	404
26	Fe	7.902 4	16.187 8	30.652	54.8	75.0	99.1	124.98	151.06	233.6	262.1	290.2	331	361	392
27	Co	7.881 0	17.083	33.50	51.3	79.5	103	131	160	186.2	276.2	305	336	379	411
28	Ni	7.639 8	18.168 84	35.19	54.9	75.5	108	134	164	193	224.6	321	352	384	430
29	Cu	7.726 38	20.292 40	36.841	55.2	79.9	103	139	167	199	232	266	369	401	435
30	Zn	9.394 05	17.964 40	39.723	59.4	82.6	108	136	175	203	238	274	311	412	454

Gas and electron pressures

To calculate the electron density properly, all low ionization energy species and their corresponding abundances should be included.



For ionized hydrogen, we find $P_e = 0.5 P_g$,
For doubly ionized helium, $P_e = 2/3 P_g$.

Gas and electron pressures

Teff	Log g	Log $P_g(\tau=2/3)$	Log $P_e(\tau=2/3)$
5500	4	4.83	1.01
6000	4	4.76	1.34
8000	4	3.94	2.46
10000	4	3.03	2.59
20000	4	3.40	3.09
40000	4	3.58	3.29

Summary

- Hydrostatic equilibrium – P_g changes by a factor of e=2.71 over the scale height.
- $P_g(\rho)$ scales with $g^{1/2}$ in Solar-type stars (n=1 approx if H^- dominant opacity source). Dwarfs have high P_g & high mean opacities (thin atmospheres) whilst (super)giants have low P_g and low mean opacities (deep atmospheres).
- Increased P_e in Solar-type stars from readily ionized metals versus pure H case. Ratio of electron to gas pressure is strong function of T.

Next time we will talk about Radiation pressure and Eddington limit