Stellar Atmospheres
Lecture 5

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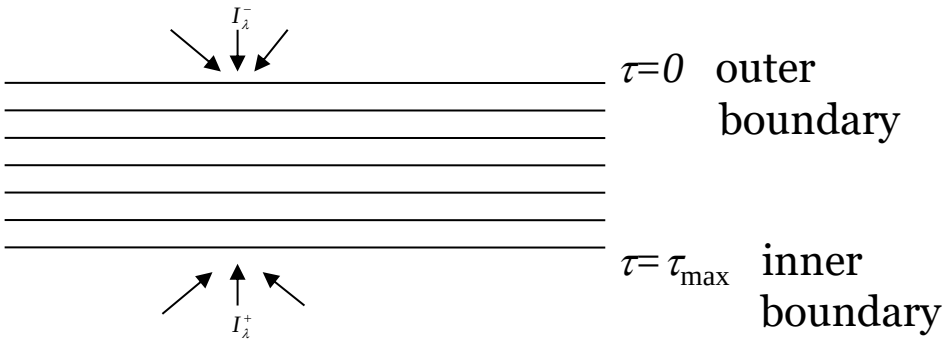
Summary of the previous lecture

- Solution to plane-parallel transfer equation at surface explains limb darkening in Sun
- Eddington-Barbier relation
- Grey atmosphere

Formal Solution to RTE

- The full intensity at the position τ_λ on the line of sight through the photosphere is

$$I_\nu(\tau_\nu) = I_\nu^{\text{out}}(\tau_\nu) + I_\nu^{\text{in}}(\tau_\nu)$$
$$= \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu$$
$$- \int_0^{\tau_\nu} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu$$

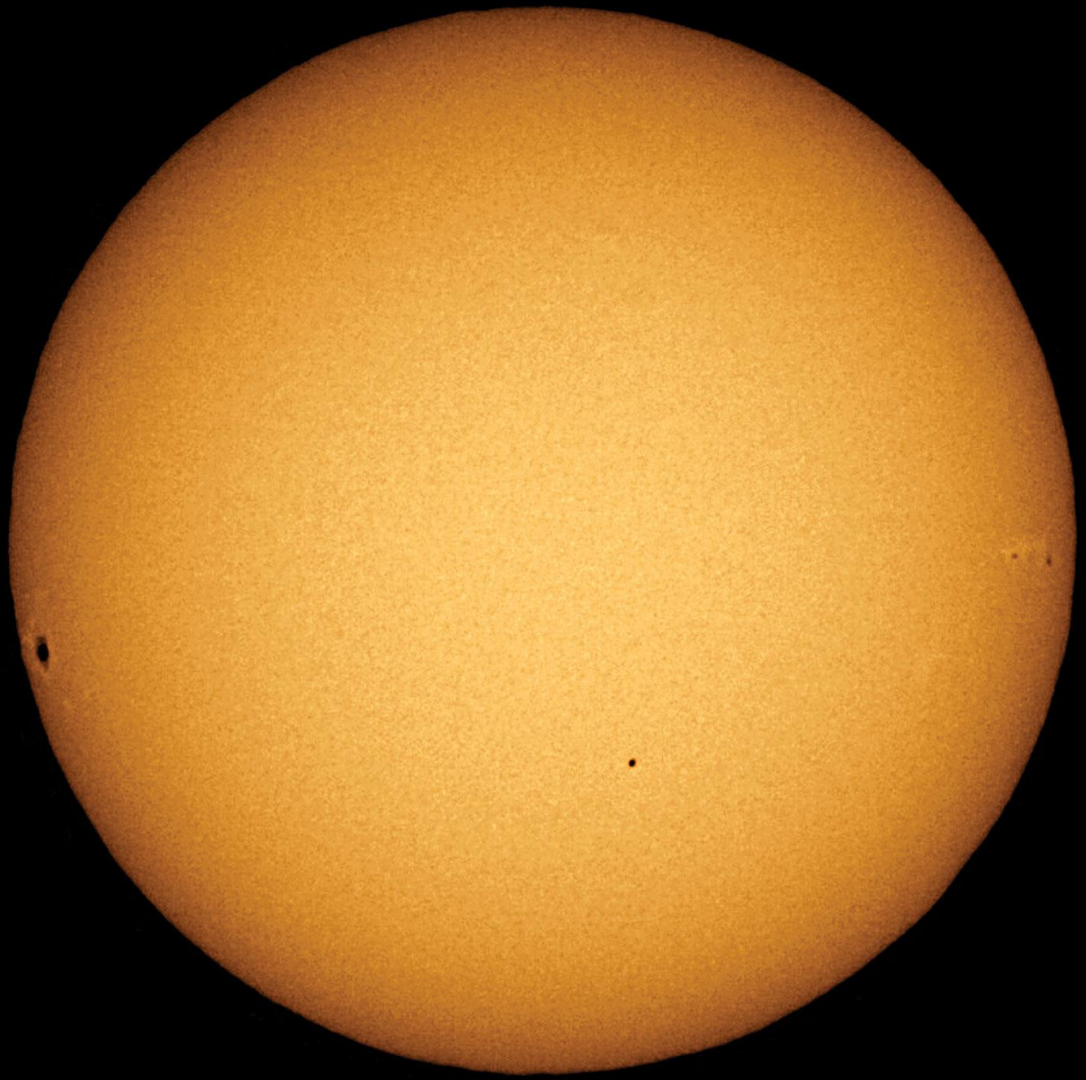


- An important special case occurs at the stellar surface. In this case

$$I_\nu^{\text{in}}(0) = 0$$
$$I_\nu^{\text{out}}(0) = \int_0^{\infty} S_\nu e^{-t_\nu \sec\theta} \sec\theta \, dt_\nu$$

where we assumed that the external radiation is completely negligible compared to the star's own radiation.

Solar limb darkening



Eddington-Barbier relation

- The emergent flux from the stellar surface is π times the Source function at an optical depth of 2/3

$$F_\lambda(0) = \pi S_\lambda(\tau_\lambda = 2/3)$$

- Assuming a grey atmosphere , we found that the “surface” of a star, which has temperature T_{eff} (by definition) is not at the very top of the atmosphere (where $\tau = 0$), but lies deeper down, at $\tau = 2/3$.

Radiative Equilibrium

GREY ATMOSPHERE
THERMAL (RADIATIVE) EQUILIBRIUM
THE DEPTH DEPENDENCE OF THE SOURCE FUNCTION
EDDINGTON APPROXIMATION
TEMPERATURE STRUCTURE OF THE GREY ATMOSPHERE

Grey atmosphere

- In the previous lecture, we assumed that the opacity can be independent of λ, i.e. $\kappa_\lambda = \kappa$. We call such a (hypothetical) atmosphere a **grey atmosphere**.
- In the theory of stellar atmospheres, much of the technical effort goes into iteration schemes using equations of radiative equilibrium (which we will discuss today) to find the source function S_λ.
- Often, a starting point for such iterations is the **grey** case.

Thermal (radiative) equilibrium

- In stellar atmospheres, radiation dominates transfer of energy, so we can discuss (three) conditions of radiative equilibrium, which can be used to derive the temperature structure in the photosphere.

- The radiation we see from the Sun comes from a layer of geometrical height of a few hundred km.

- In a column of 100 km height and 1 cm^2 cross-section there are 10^{24} particles (since $\rho \sim 10^{17}/cm^3$ in Sun), each of which has a thermal energy of $3kT/2$ (10^{-12} erg). The total thermal energy of this column is therefore 10^{12} erg/cm^2. The observed radiative energy loss (per cm^2) of the solar surface is $F_\odot = 6.3 \times 10^{10}$ erg/cm^2/s.

- If the Sun shines at a constant rate, the energy content of the solar photosphere can only last for 15 seconds without being replenished from below. Exactly the same amount of energy has to be supplied or else the photosphere would quickly change temperature.

First equation of radiative equilibrium

- Since this does **not** happen, $\mathrm{d}F/\mathrm{d}t=0$ or $\mathrm{d}F/\mathrm{d}x=0$ or $\mathrm{d}F/\mathrm{d}\tau=0$, i.e. the total flux must be constant at all depths of the photosphere (**conservation of energy**) – the **1st equation of radiative equilibrium**

$$\boldsymbol{F(x) = F(0) = const = \sigma T_{eff}^4}$$

- When all the energy is carried by radiation, we have

$$F(x) = \int_0^\infty F_\lambda(\tau_\lambda)d\lambda = F(0)$$

Although the shape of F_λ can be expected to change very significantly with depth, its integral remains invariant.

- If other sources of energy transport are significant, then a more general expression of flux constancy must be applied:

$$\Phi(x) + \int_0^\infty F_\lambda(\tau_\lambda)d\lambda = F(0)$$

$\Phi(x)$ is, for example, the convective flux

Radiative equilibrium

- We may integrate the plane-parallel transfer equation over solid angle ω.

$$\int \cos\theta \frac{dI_\lambda(\tau_\lambda,\theta)}{d\tau_\lambda} d\omega = \int I_\lambda(\tau_\lambda,\theta) d\omega - \int S_\lambda(\tau_\lambda) d\omega$$

$$\frac{d}{d\tau_\lambda}(F_\lambda(\tau_\lambda)) = 4\pi(J_\lambda(\tau_\lambda)) - \int S_\lambda(\tau_\lambda) d\omega$$

Based on the definition of mean intensity $J_\lambda = \frac{1}{4\pi}\oint I_\lambda d\omega$

- Finally, assuming S_λ to be isotropic we obtain,

$$\frac{1}{4\pi}\frac{d}{d\tau_\lambda}(F_\lambda(\tau_\lambda)) = J_\lambda(\tau_\lambda) - S_\lambda(\tau_\lambda)$$

Second equation of radiative equilibrium

- In the **grey** case, for which the opacity κ is independent of wavelength

$$\frac{1}{4\pi}\frac{d}{d\tau}F(\tau) = -S(\tau) + J(\tau) = 0$$

Since $\mathbf{dF/d\tau = 0}$, the Source function must be equal the mean intensity J.

- If the atmosphere is **not** grey, which is the situation for most stars, let's incorporate the opacity κ into the RHS, and integrating over wavelength

$$\frac{1}{4\pi}\frac{d}{dt}\left(\int_0^\infty F(\tau_\lambda)d\lambda\right) = \int_0^\infty (\kappa_\lambda S_\lambda - \kappa_\lambda J_\lambda)d\lambda = 0$$

Since dF/dt =0, we get the **radiative balance equation (energy conservation)**

$$\boxed{\int_0^\infty \kappa_\lambda S_\lambda d\lambda = \int_0^\infty \kappa_\lambda J_\lambda d\lambda}$$

- This is **the second equation of radiative equilibrium** and can be understood as the total energy absorbed (RHS) must equal the total energy re-emitted (LHS) if no heating or cooling is taking place.

Third equation of radiative equilibrium

- The third radiative equilibrium condition is obtained by multiplying the transfer equation by $\cos\theta$ and integrating over solid angle and then wavelength

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

$$\oint \cos^2\theta \frac{dI_\lambda(\tau_\lambda,\theta)}{d\tau_\lambda} d\omega = \oint \cos\theta \, dI_\lambda(\tau_\lambda,\theta)\omega - \oint \cos\theta \, S_\lambda(\tau_\lambda,\theta) d\omega$$

$$K_\lambda(\tau_\lambda) = \frac{1}{4\pi}\oint I_\lambda \cos^2\theta d\omega \qquad F_\lambda = \oint I_\lambda \cos\theta \, d\omega \qquad 0 \ (S_\lambda \text{ is isotropic})$$

$$4\pi \int \frac{dK_\lambda}{d\tau_\lambda} d\lambda = \int F_\lambda d\lambda = F(\tau)$$

The third radiative equilibrium condition:

$$\int_0^\infty \frac{dK_\lambda}{d\tau_\lambda} d\lambda = \frac{F(\tau)}{4\pi}$$

Equations of radiative equilibrium

- All the three radiative equilibrium conditions are not independent. S_λ that is a solution of one will be the solution of all three.
- The flux constant $F(0)$ is often expressed in terms of an effective temperature $F(0) = \sigma T_{eff}^4$.
- When model photospheres are constructed using flux constancy as a condition to be fulfilled by the model, the effective temperature becomes one of the fundamental parameters characterizing the model.
- In real stars, energy is created or lost from the radiation field through e.g. convection, magnetic fields, plus in supernovae atmospheres energy conservation is not valid (radioactive decay of Ni to Fe), so the energy constraints are more complicated in reality.

The depth dependence of the source function

- In a ***grey*** atmosphere, with $K(\tau) = \int_0^\infty K_\lambda d\lambda$, the 3rd equation implies:

$$\frac{dK(\tau)}{d\tau} = \frac{F(\tau)}{4\pi}$$

a new unknown function $K(\tau)$

- We can differentiate this, and insert our earlier result:

$$\frac{d^2K(\tau)}{d\tau^2} = \frac{1}{4\pi}\frac{dF(\tau)}{d\tau} = J(\tau) - S(\tau) = 0 \qquad \textbf{[1]}$$

- Integration of the equation with respect to τ gives $K(\tau)=c_1\tau+c_2$ **[2]**
  where $dK/d\tau = c_1 = F/4\pi$

- For a given ***F***, we now have two equations, [1] and [2], to determine the three unknowns: ***J***, ***S*** and ***K*** (or c_2). We need an additional relation between two of these variables in order to determine all three.

Eddington approximation (1)

- Previously we have seen that for the determination of the flux **the anisotropy in the radiation field is very important** because in the flux integral the inward-going intensities are subtracted from the outward-going ones, due to the factor $\cos\theta$.
- But for ***K***, a small anisotropy is unimportant because the intensities are multiplied by the factor $\cos^2\theta$, which does not change sign for inward and outward radiation.

- In order to evaluate K or c_2 we can approximate the radiation field by an isotropic radiation field of the mean intensity J: ***I = J*** (by definition). From the definition of K_λ we obtain

$$4\pi\, K_\lambda = \oint I_\lambda(\tau_\lambda,\theta)\cos^2\theta\, d\omega = J_\lambda(\tau_\lambda)\oint \cos^2\theta\, d\omega = \frac{4\pi}{3} J_\lambda(\tau_\lambda)$$

or after division by 4π,

$$K_\lambda(\tau_\lambda) = \frac{1}{3} J_\lambda(\tau_\lambda)$$

This approximation for the K function is known as the ***Eddington approximation.***

Eddington approximation (2)

- Inserting the Eddington approximation into the above equation $\frac{dK(\tau)}{d\tau} = \frac{F(\tau)}{4\pi}$ we find

$$\frac{dK(\tau)}{d\tau} = \frac{1}{3}\frac{dJ(\tau)}{d\tau} = \frac{F(\tau)}{4\pi} = c_1 \qquad \frac{dJ(\tau)}{d\tau} = \frac{3}{4\pi}F(\tau)$$

- Since the mean intensity J equals the source function S in a grey atmosphere, integrating the latter result we obtain

$$S(\tau) = \frac{3}{4\pi}\tau F(0) + C = J(\tau)$$

- From the conditions of radiative equilibrium, we finally obtained **the law for the depth dependence of the source function** (**for a grey atmosphere assuming the Eddington approximation**). We can evaluate **C** using boundary condition for the known emerging flux (there is no flux going into the star), i.e. $I(0,\theta)=I^{-}=0$ for $\pi/2<\theta<\pi$ plus we assume the outward intensity does not depend upon θ, ie. $I(0,\theta)=I^{+}$ for $0<\theta<\pi/2$,

Eddington approximation (3)

- Boundary condition: there is no flux going into the star, i.e. $I(0,\theta)=I^{-}=0$ for $\pi/2<\theta<\pi$
- We also assume that the outward intensity does not depend upon θ, ie. $I(0,\theta)=I^{+}=$ const for $0<\theta<\pi/2$

- It gives $$J(0)=\frac{1}{2\pi}I^{+}=\frac{1}{2\pi}F(0)$$

$$S(\tau)=\frac{3}{4\pi}\tau F(0)+C=J(\tau)$$

- Hence $C=J(0)=F(0)/2\pi$ so:

$$S(\tau)=\frac{1}{\pi}\left(\frac{3}{4}\tau+\frac{1}{2}\right)F(0)$$

$$S(\tau)=\frac{3}{4\pi}\left(\tau+\frac{2}{3}\right)F(0)$$

- To find the depth dependence of T, we also need to assume LTE.

Temperature structure of the grey atmosphere

In LTE, the source function is the Planck function, $S(\tau)=B(\tau)=\sigma T^4/\pi$

$$B(\tau)=\frac{\sigma}{\pi}T^4(\tau)=\frac{3}{4\pi}(\tau+\frac{2}{3})F(0)$$

Recall that $F(0)=\sigma T^4_{\text{eff,}}$ by definition, so

$$\frac{1}{\pi}\sigma T^4(\tau)=\frac{3}{4\pi}(\tau+\frac{2}{3})\sigma T_{eff}^4 \quad \text{or} \quad T^4(\tau)=\frac{3}{4}(\tau+\frac{2}{3})T_{eff}^4$$

We derived the **temperature dependence on optical depth**.

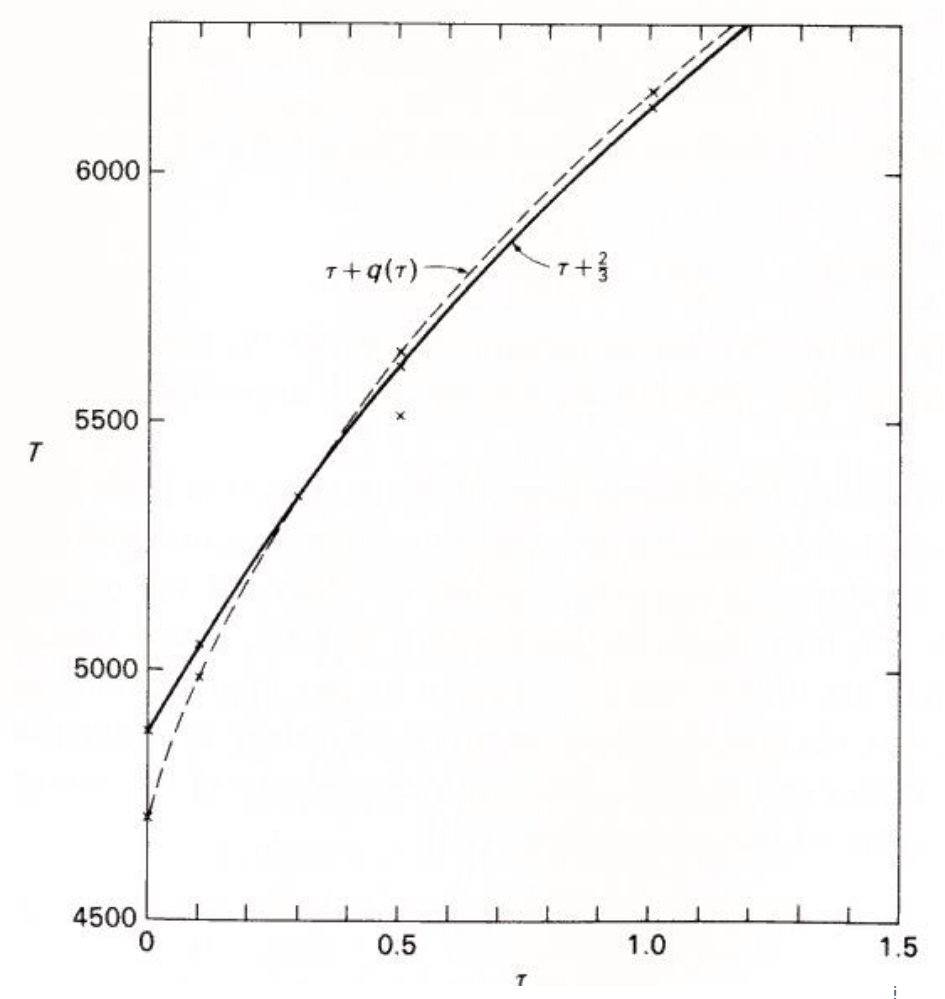
Note $T(\tau=2/3)=T_{\text{eff}}$ as we obtained earlier, and $T^4(\tau=0)=T_{\text{eff}}{}^4/2$

A complete solution of the grey case, using accurate boundary conditions, without Eddington approximation, leads to a solution only slightly different from this, usually expressed as

$$T^4(\tau)=\frac{3}{4}(\tau+q(\tau))T_{eff}^4$$

Here $q(\tau)$ is a slowly varying function (Hopf function), with $q=1/\sqrt{3}=0.577$ at $\tau=0$ to $q=0.710$ at $\tau=\infty$.

Grey Temperature Structure

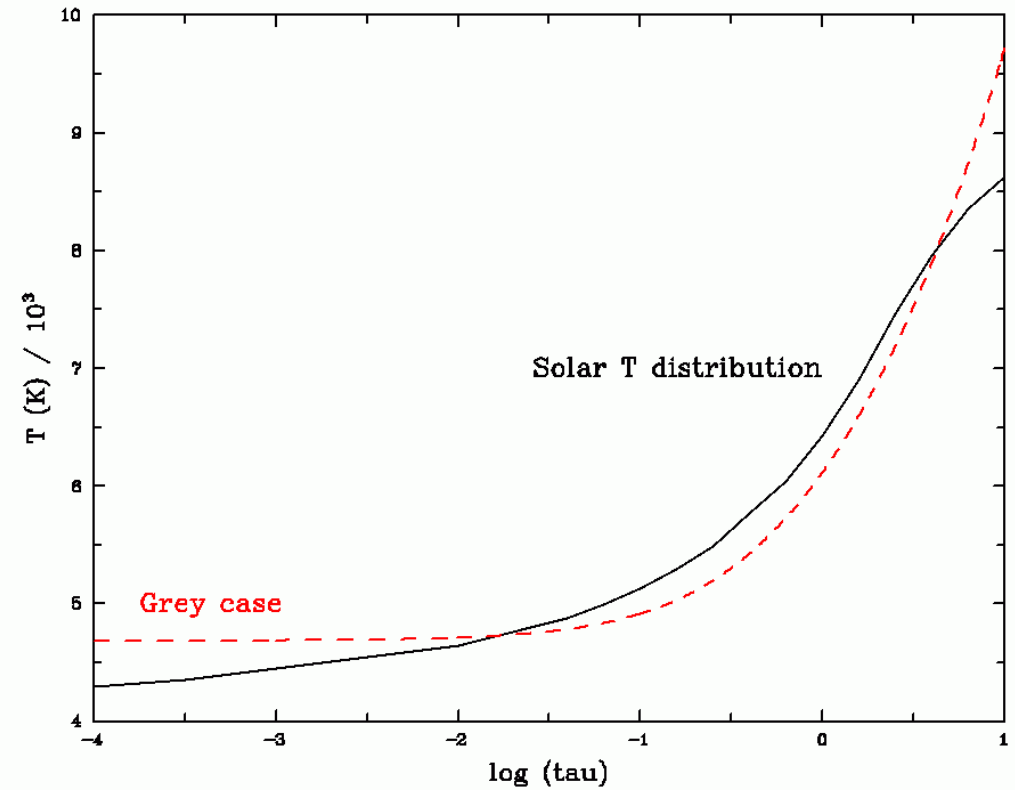


Comparison between $T(\tau)$ in the Solar atmosphere using the simplifying Eddington assumption (solid) versus the exact grey case (dashed) using the Hopf function, q(τ)

$$q(\tau) \approx 0.710 - 0.133e^{-2\tau}$$

How realistic is this?

- How good an approximate is the grey atmosphere? Next we must look at the frequency dependence of the sources of opacity.
- The grey temperature distribution is shown here versus the observed Solar temperature distribution as a function of optical depth τ at 5000A (D. Gray, Table 9.2)
- The poor match is because the opacity is **wavelength dependent**, as we shall see next lecture.



Summary

- Three equations of radiative equilibrium can be derived:
  (a) constant flux with depth;
  (b) energy absorbed equals energy emitted;
  (c) the K-integral is linear in τ.

- From these, the **grey** temperature distribution $T(\tau)$ may be derived, assuming:
  (a) the Eddington approximation and
  (b) LTE, in reasonable agreement with the exact case.

- On Friday, we will discuss LTE in more detail.