Stellar Atmospheres
Lecture 4

VITALY NEUSTROEV

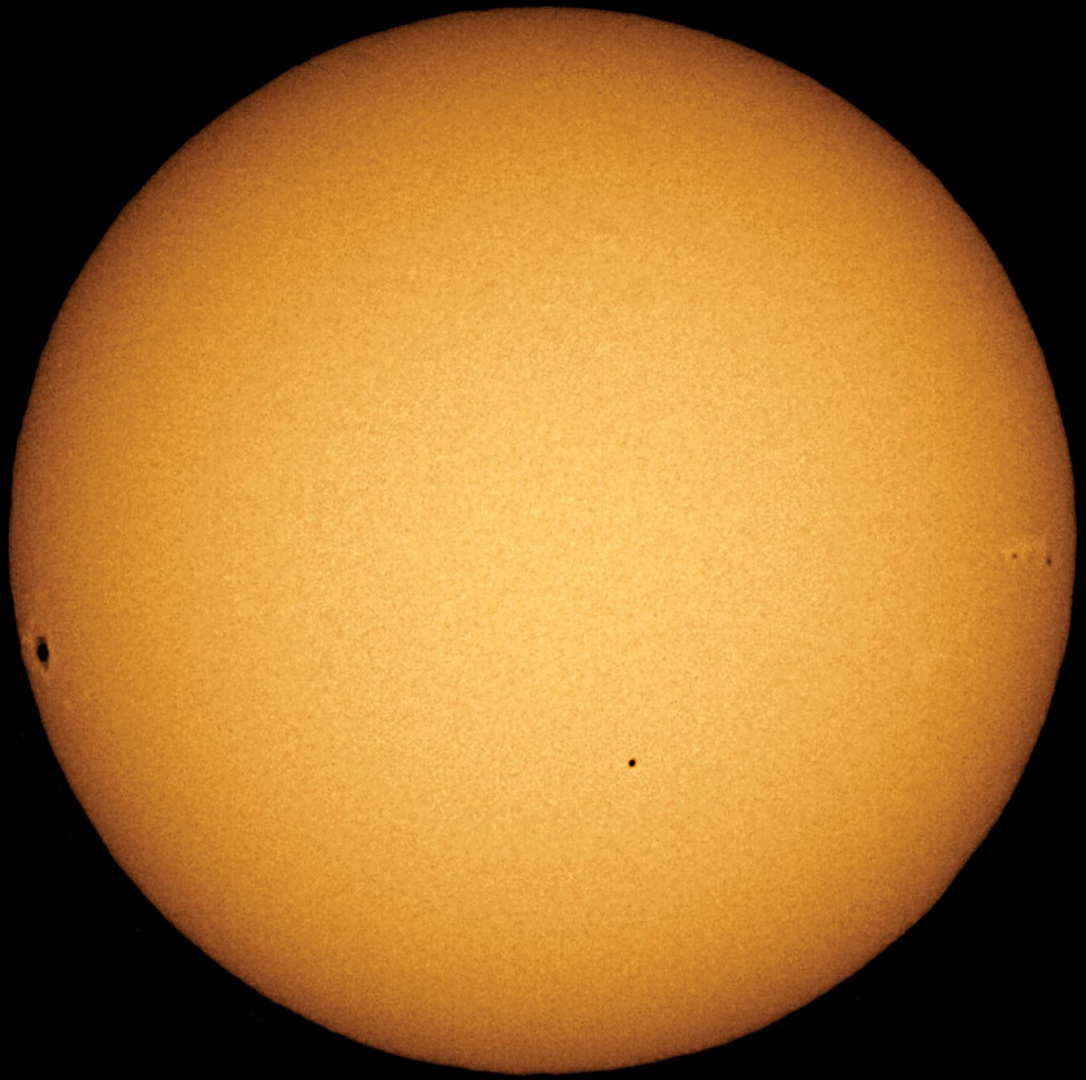
ASTRONOMY RESEARCH UNIT
UNIVERSITY OF OULU

2019

Radiative transfer II

RADIATIVE TRANSFER EQUATION IN PLANE-PARALLEL ATMOSPHERE. LIMB DARKENING.

Solar limb darkening



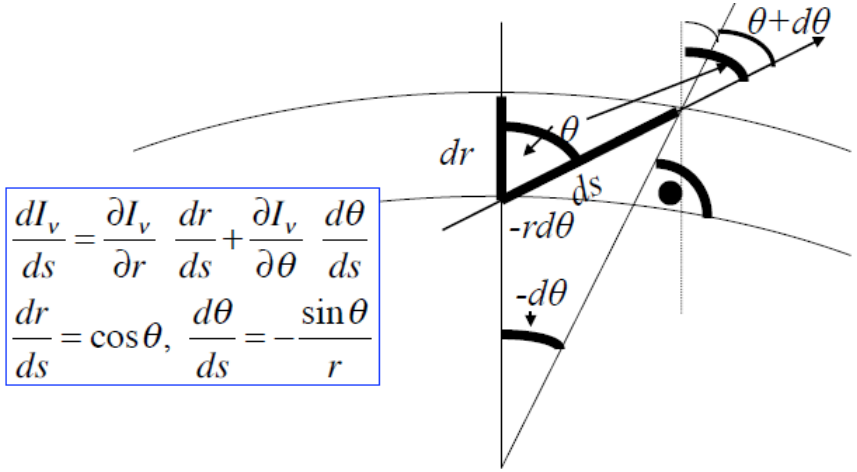
Plane-parallel vs spherical geometry

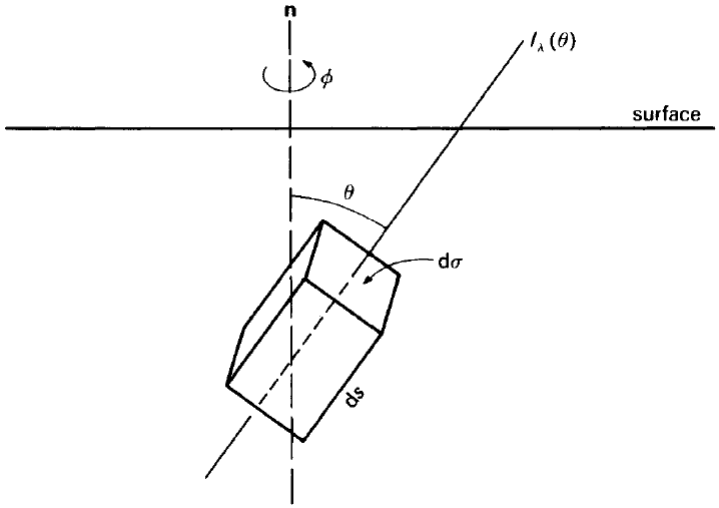
- In principal, we need to consider *spherical geometry* when calculating the transfer equation in stars
- Fortunately, the geometrical thickness of most photospheres is small compared to their radii, permitting the *plane-parallel* approximation, $r \rightarrow \infty$

$$\frac{dI_\nu}{ds} = -\cos\vartheta \frac{\partial I_\nu}{\partial t}$$

$$\frac{dI_\nu}{ds} = \frac{\partial I_\nu}{\partial r}\frac{dr}{ds} + \frac{\partial I_\nu}{\partial \theta}\frac{d\theta}{ds}$$

$$\frac{dr}{ds} = \cos\theta, \quad \frac{d\theta}{ds} = -\frac{\sin\theta}{r}$$





Transfer Equation for Stars

The plane-parallel transfer equation
(for stars with thin photospheres)

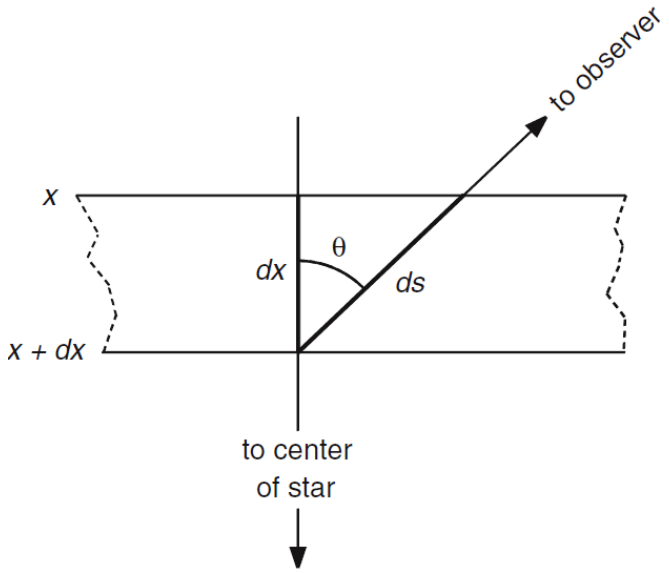
$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

is identical to the parallel-ray transfer equation (for ISM studies),

$$\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda + S_\lambda$$

except for

1. the cos(θ) term, because the optical depth is measured along the radial direction *x* and not along the line of sight, i.e $d\tau_\lambda = -\kappa_\lambda \rho dx$
2. sign change, since we are now looking from the outside in, along direction x.



The full spherical geometry transfer equation is necessary for supergiants.

Surface Intensity

- To derive the intensity at the **surface**, we multiply the plane-parallel transfer equation by an integrating factor $e^{-\tau/\cos\theta}=e^{-u}$,

$$\frac{dI_\lambda(\theta)}{du}e^{-u} - I_\lambda(\theta)\ e^{-u} = -S_\lambda e^{-u}$$

- This can be written as

$$\frac{d(I_\lambda(\theta)e^{-u})}{du} = -S_\lambda e^{-u}$$

- Integrating du from 0 to infinity

$$[I_\lambda(\theta)e^{-u}]_0^\infty = -\int_0^\infty S_\lambda(\tau_\lambda)e^{-u}du$$

$$I_\lambda(0,\theta) = \int_0^\infty S_\lambda(\tau_\lambda)e^{-u}du$$

Limb darkening

Let us assume a linear source function: $S_\lambda(\tau_\lambda) = a_\lambda + b_\lambda \tau_\lambda$

We derive, $$I_\lambda(0,\theta) = \int_0^\infty a_\lambda\, e^{-u} du \quad + \int_0^\infty b_\lambda \tau_\lambda\, e^{-u} du$$

$$I_\lambda(0,\theta) = a_\lambda \int_0^\infty e^{-u} du \quad + b_\lambda \cos\theta \int_0^\infty u\, e^{-u} du$$

Recall $u=\tau/\cos(\theta)$, so $\tau=u\cos(\theta)$ and we can use the standard integral

$$\int_0^\infty u^n\, e^{-u} du = n!$$

We obtain

$$\boldsymbol{I_\lambda(0,\theta) = a_\lambda + b_\lambda \cos\theta = S_\lambda(\tau_\lambda = \cos\theta)}$$

In the linear approximation for the Source function, the optical depth lies between 0 and 1. From the centre of the star we see radiation leaving the star perpendicular to the surface: $I_\lambda(0,0°)=a_\lambda+b_\lambda$, whilst at the limb the starlight leaves the surface at an angle $I_\lambda(0,90°)=a_\lambda$.

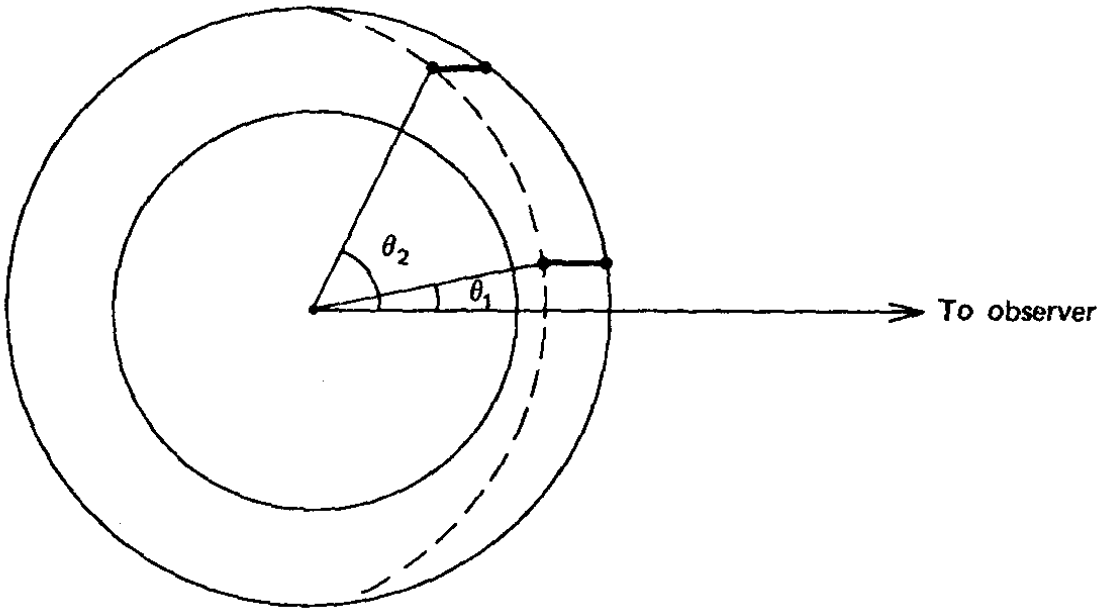
Limb darkening (less light from the limb versus the centre, if $b_\lambda>0$).

Solar limb darkening

- This optical image of the Sun clearly shows limb darkening. We see into the atmosphere down to a depth of $\tau=1$.
- Limb darkening exists because the continuum source function decreases outward.
- As we look towards the limb, we see higher photospheric layers, which are less bright.

Schematic of limb darkening

Schematic illustration of limb darkening – penetration of different lines of sight (heavy lines) to "unit optical depth" (dashed lines) corresponds to different depths in the photosphere, depending on θ. Radiation seen at θ_2 is characteristic of higher (cooler) layers than the radiation seen at position θ_1



Linear vs Quadratic source function

Up to now we assumed a linear source function. More generally, if:

$$S_\lambda(\tau_\lambda) = \sum_n a_{\lambda n}\,\tau_\lambda^n$$

Then

$$I_\lambda(0,\theta) = \sum_n A_n \cos^n\theta \qquad A_n = a_{\lambda n}\int_0^\infty u^n\, e^{-u} du = a_{\lambda n} n$$

We still get $S_\lambda(0)$ at the limb, but a more complicated result at the centre.
For example, a quadratic term requires the solution of

$$S(\tau_\lambda) = a_{0\lambda} + a_{1\lambda}\tau_\lambda + a_{2\lambda}\tau_\lambda^2$$
$$I_\lambda(0,\theta) = a_{0\lambda} + a_{1\lambda}\cos\theta + 2a_{2\lambda}\cos^2\theta$$

At $\theta = 90°$, $\tau_\lambda = 0$, whilst at $\theta = 0°$, $\tau_\lambda \sim 1 + 2a_{1\lambda}/a_{2\lambda}$ providing $a_{2\lambda} << a_{1\lambda}$.
The ratio of the limb-to-centre intensity is

$$\boldsymbol{I_\lambda(0,90°)/I_\lambda(0,0°) = a_{0\lambda}/(a_{0\lambda} + a_{1\lambda} + 2a_{2\lambda})}$$

Example for Solar Case:

The measured centre to limb variation of the solar intensity (Table 4.17, AQ 4th edition) is

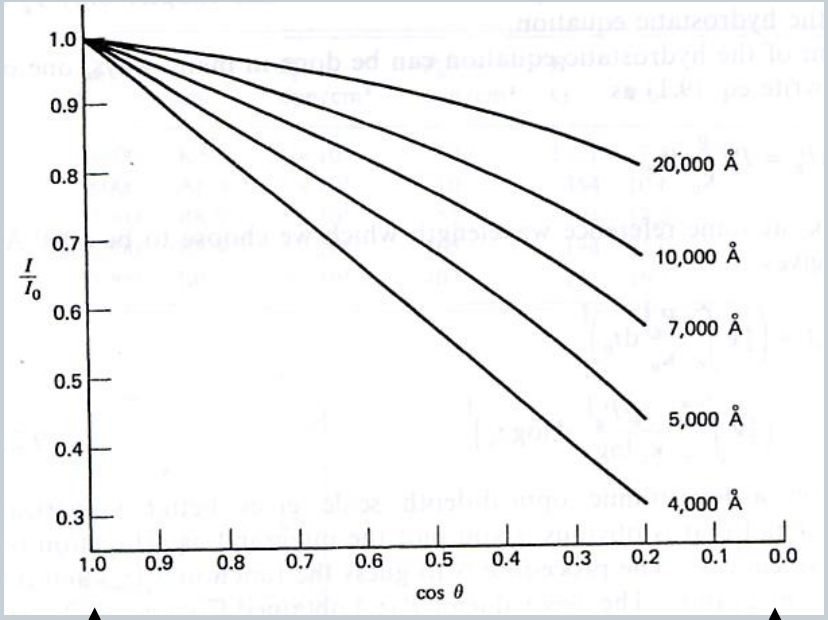
$$I_\lambda(0,\theta)/I_\lambda(0,0) = a_{0\lambda} + a_{1\lambda}\cos\theta + 2a_{2\lambda}\cos^2\theta$$

$\lambda(\mu m)$	a_0	a_1	$2a_2$
0.3	0.06	0.74	0.20
0.4	0.14	0.91	-0.05
0.6	0.35	0.88	-0.23
0.8	0.49	0.73	-0.22
1.5	0.56	0.64	-0.20
2.0	0.70	0.48	-0.18

Wavelength dependence

Limb darkening is observed to be greatest at shorter wavelengths in the Sun. The temperature distribution of the upper atmosphere of the Sun can be obtained from limb darkening measurements, carried out via e.g. multi-filter images of the Solar continuum (between the lines).

Until recently, the Sun was the only star for which limb darkening was observed, since one needs to spatially resolve the disk (most other stars appear as point sources!) to measure limb darkening. Other methods are now possible.



(Pierce & Waddell 1961).

Centre

Limb

Limb darkening for other stars

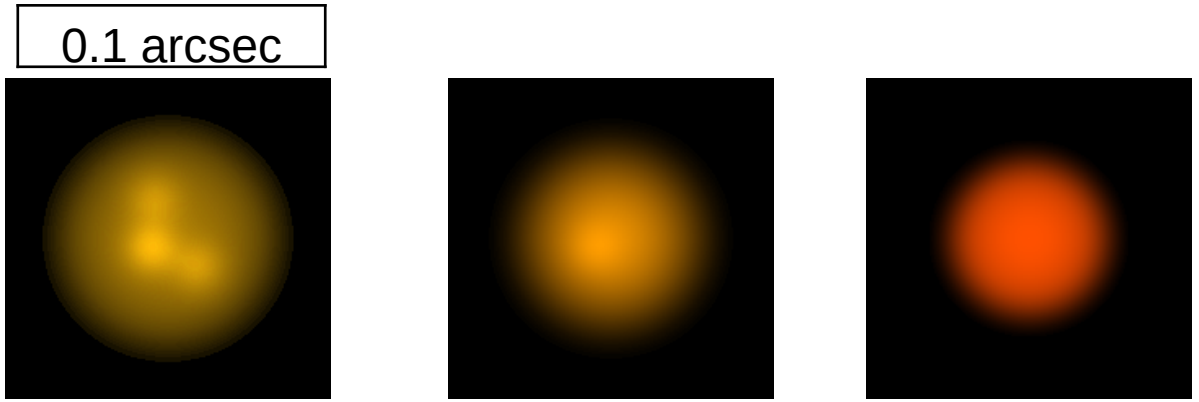
1. Direct interferometry, via high spatial resolution "imaging" – eg. COAST array, providing star is very large and nearby (a cool supergiant),

2. The light curve from an eclipsing binary system during secondary eclipse allows us to study limb darkening of the primary, although non-trivial! Similar approach followed by extra solar planets occulting parent star (e.g. HD209458)

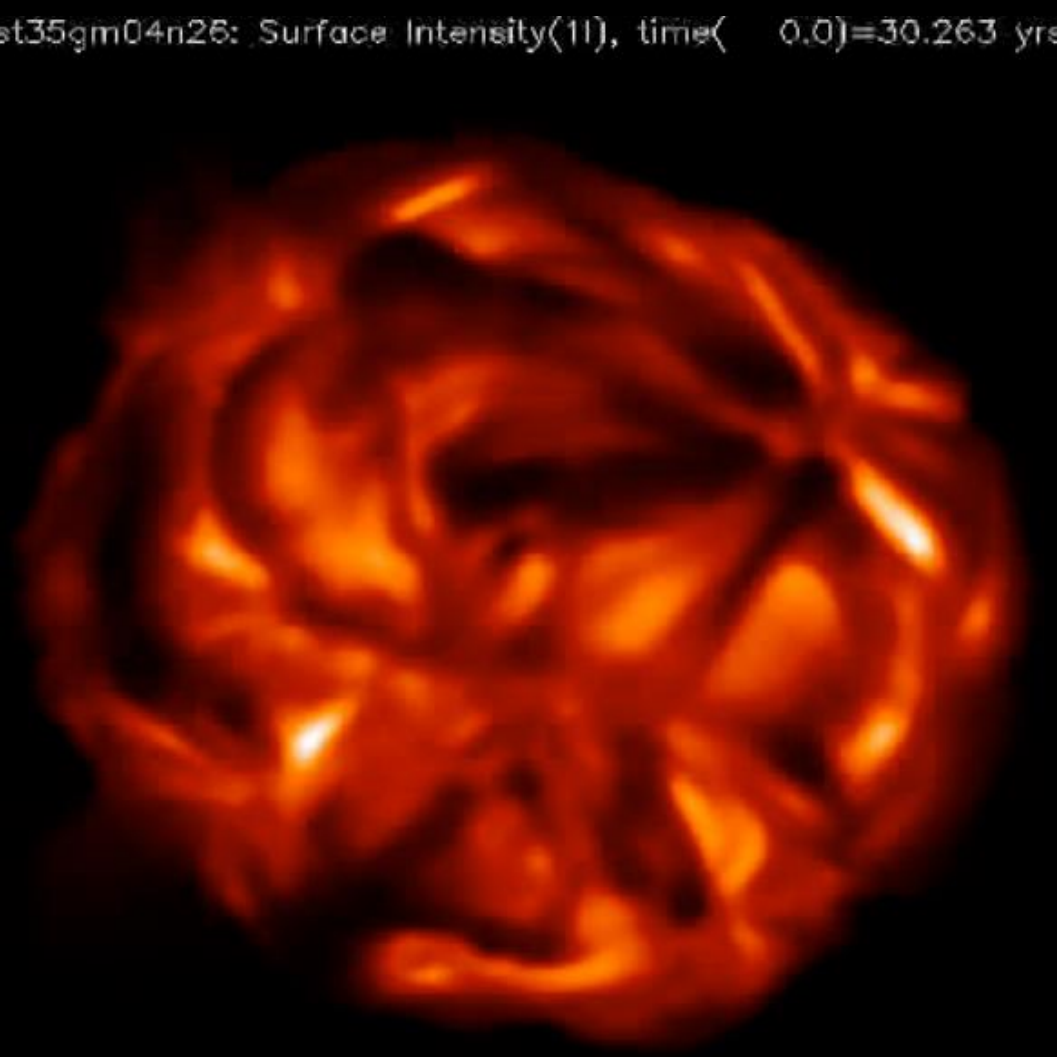
3. The light curve due to the gravitational micro-lensing of a background (generally Galactic bulge or Magellanic Cloud) star by a foreground source (e.g. PLANET team)

Limb darkening from interferometry

COAST (Cambridge Optical Aperture Synthesis Telescope) spatial resolution of 20-30 milli-arcsec) has made limb darkening observations of M supergiant Betelgeuse at different wavelengths (using filters).



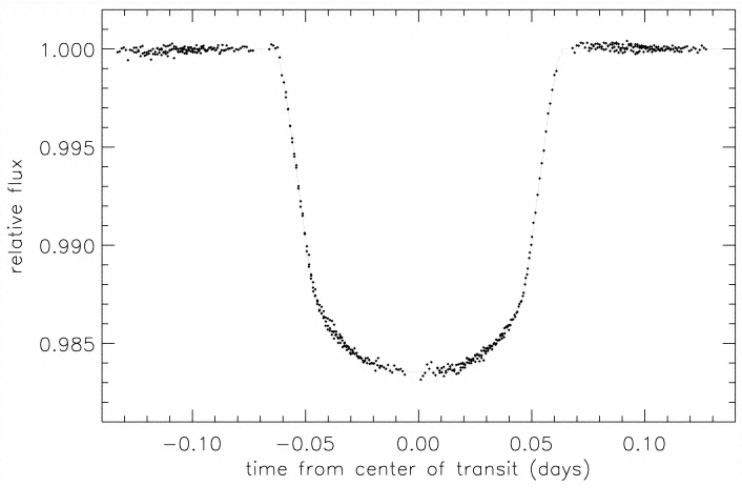
Numerical simulations of red supergiants

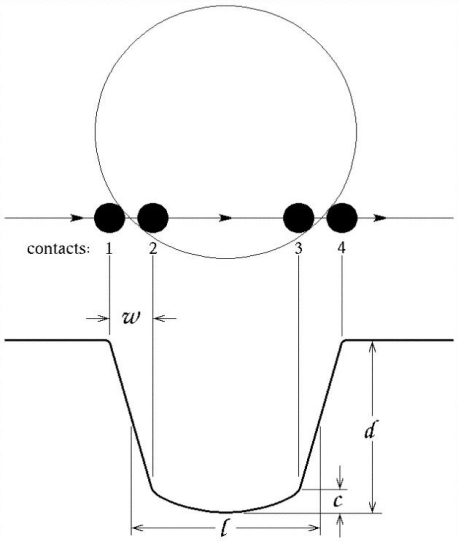


Limb darkening from eclipsing systems

- HD209458 is the first system in which extra-solar planet (P=3.5d, $0.6M_J$) has been observed to transit its (F8V) primary, allowing determination of limb darkening (Brown et al. 2001)

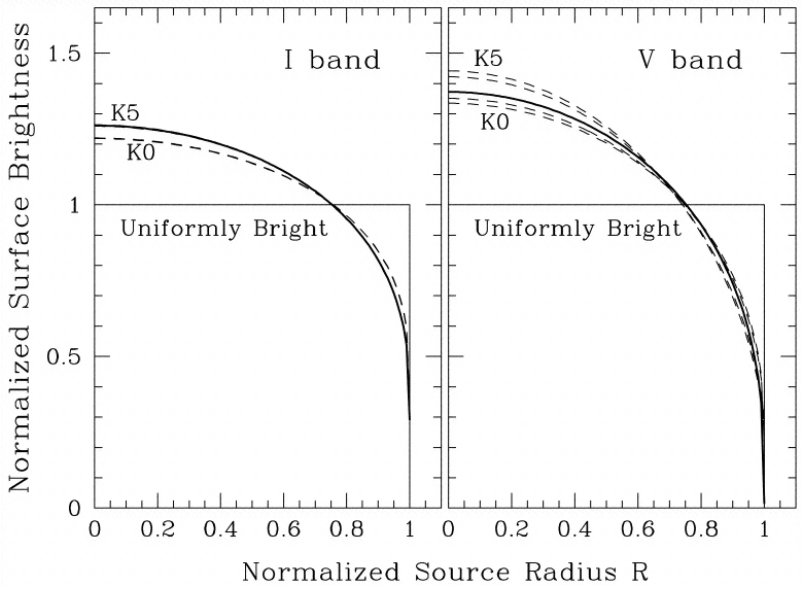
- More generally eclipsing binaries are problematic due to degeneracy with other parameters (Grygar et al. 1972). Accurate light curves needed for linear limb darkening parameters.





Limb darkening from microlensing

- Galactic gravitational micro-lensing occurs when a foreground object (lens) passes in front of a background star (source). The gravitational deflection of light by the lens causes the flux from the source to be amplified.
- Microlensing surveys (e.g. PLANET, MACHO) have identified hundreds of such events towards the Galactic bulge and Magellanic Clouds.
- One such event, MACHO 97-BLG-28 was studied to reveal limb darkening information for the background K giant (Albrow et al. 1999).



Thick lines show how much fainter the K giant becomes at its edges in the red I (left) and blue-green V filter (right). If the star emitted a uniform amount of light across its whole stellar disk, the profile would look like the straight solid black line instead

Eddington-Barbier relation

SOLUTION TO TRANSFER EQUATION
EDDINGTON-BARBIER RELATION.
GREY ATMOSPHERE.

Formal Solution to RTE (1)

The plane-parallel transfer equation (for stars with thin photospheres)

$$\cos\theta \frac{dI_\lambda(\theta)}{d\tau_\lambda} = I_\lambda(\theta) - S_\lambda$$

- The integrated form of the RTE is $$I_\nu(\tau_\nu) = -\int_c^{\tau_\nu} S_\nu(t_\nu) e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, d\tau_\nu .$$

  The integration limit, *c (which complicates the integral)*, replaces $I_\nu(0)$ in the parallel-ray transfer equation $$I_\nu(\tau_\nu) = \int_0^{\tau_\nu} S_\nu(t_\nu) e^{-(\tau_\nu - t_\nu)} \, dt_\nu + I_\nu(0) e^{-\tau_\nu}.$$

- The full intensity at the position τ_λ on the line of sight through the photosphere is

$$\begin{aligned} I_\nu(\tau_\nu) &= I_\nu^{\text{out}}(\tau_\nu) + I_\nu^{\text{in}}(\tau_\nu) \\ &= \int_{\tau_\nu}^{\infty} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu \\ &\quad - \int_0^{\tau_\nu} S_\nu e^{-(t_\nu - \tau_\nu)\sec\theta} \sec\theta \, dt_\nu \end{aligned}$$

Formal Solution to RTE (2)

- An important special case occurs at the stellar surface. In this case

$$I_\nu^{\text{in}}(0) = 0$$

$$I_\nu^{\text{out}}(0) = \int_0^\infty S_\nu \mathrm{e}^{-t_\nu \sec\theta} \sec\theta \, \mathrm{d}t_\nu$$

  where we assume that the radiation from other stars, galaxies, and so forth is completely negligible compared to the star's own radiation. This Equation is the expression we need to compute the spectrum.
- However, since the disks of most stars are spatially unresolved, we must deal with flux rather than intensity, so we will not deal with this equation any further [you will be relieved this will not be examined].

Emergent Flux

From our 2nd lecture, the flux is (If there is no azimuthal (ϕ) dependence in I_λ):

$$F = 2\pi \int_{-1}^{1} I(\mu)\, \mu\, d\mu \qquad \mu=\cos\theta$$

Netto = Outwards – Inwards, **but notice, now we are considering the opposite direction!**

Decomposition into two half-spaces:

$$= 2\pi \int_{0}^{1} I(\mu)\mu\, d\mu + 2\pi \int_{-1}^{0} I(\mu)\, \mu\, d\mu$$

$$= 2\pi \int_{0}^{1} I(\mu)\, \mu\, d\mu - 2\pi \int_{0}^{1} I(-\mu)\mu\, d\mu = \boldsymbol{F^{+} - F^{-}}$$

Eddington-Barbier relation

At the surface, negligible radiation will come from the outside, so
$I_\lambda(0,\theta)=0$ for $\pi/2<\theta<\pi$, i.e. $0<\mu<-1$

$$F_\lambda(0) = 2\pi\int_0^1 I_\lambda(0,\theta)\mu d\mu$$

From earlier, assuming a linear source function $S_\lambda(\tau_\lambda) = a_\lambda + b_\lambda\tau_\lambda$ yields

$$I_\lambda(0,\theta) = a_\lambda + b_\lambda \cos\theta = a_\lambda + b_\lambda\mu$$

In this case we obtain the "Eddington-Barbier" relation

The emergent flux from the stellar surface is π times the Source function at an optical depth of 2/3

$$F_\lambda(0) = \pi(a_\lambda + 2/3 b_\lambda) = \pi S_\lambda(\tau_\lambda = 2/3)$$

Grey atmosphere

If we assume LTE,

$$F_\lambda(0) = \pi S_\lambda(\tau_\lambda = 2/3) = \pi B_\lambda(T(\tau_\lambda = 2/3))$$

Let us assume the opacity is independent of λ, i.e. $\kappa_\lambda = \kappa$. We call such a (hypothetical) atmosphere a grey atmosphere. Then

$$F_\lambda(0) = \pi B_\lambda(T(\tau = 2/3))$$

The energy distribution of F_λ is that of a blackbody corresponding to the temperature at the optical depth $\tau = 2/3$. Integrating over λ

$$F(0) = \int_0^\infty F_\lambda(0) d\lambda = \pi \int_0^\infty B_\lambda(T(\tau = 2/3)) d\lambda = \sigma T^4(\tau = 2/3)$$

From Stefan-Boltzmann, $F(0) = \sigma T_{eff}^4$, by definition, we find $T_{eff} = T(\tau = 2/3)$. *The "surface" of a star, which has temperature* T_{eff} *(by definition) is not at the very top of the atmosphere (where* $\tau = 0$*), but lies deeper down,* at $\tau = 2/3$.

This can be considered as an *average* point of origin from the observed photons.

Summary

- Solution to plane-parallel transfer equation at surface explains limb darkening in Sun.
- Limb darkening in other stars can be estimated from interferometry, eclipsing binaries, microlensing.
- Eddington-Barbier relation.
- Grey atmosphere.