Periodogram & Power Spectrum

- The periodogram is an estimate of the spectral density of a signal. The term was coined by Arthur Schuster in 1898 (*the Schuster Periodogram*).
- A Power Density Spectrum is computed as the squared Fourier amplitudes with **some normalization**:

$$a_j = \sum_{k=0}^{N-1} x_k \, e^{2\pi i j k/N} \qquad j = -\frac{N}{2} + 1, \ldots, \frac{N}{2}$$

$$P_j = (Normalization)|a_j|^2$$

Noise Power Distribution

- Flux measurements are **always** accompanied by noise.
- The light curve can be divided into its independent components: the deterministic signal S and the noise N. For an individual time bin, the total number of counts is composed of the sum of the signal and the noise, i.e., $x_k = s_k + n_k$.
- Noise powers follow a chi-squared distribution with 2 dof:
  - $P_j \propto A_j^2 + B_j^2$, where $A_j = \sum_k x_k \cos \omega_j t_k$ and $B_j = \sum_k x_k \sin \omega_j t_k; \quad k = 0, \ldots, N-1$
  - So, each A_j and each B_j is a linear combination of the x_k. Hence if the x_k are normally distributed then A_j and B_j are as well → $P_j \propto \chi^2$ with 2 dof by definition.
  - If x_k follow some other distribution (e.g. Poisson) then the central limit theorem ensures that A_j and B_j are still approximately normal (for large N) → P_j are still approximately χ^2 with 2 dof.
  - Exact expressions depend on the normalization of the P_j.

Power Spectrum – Leahy Normalization

- We will adopt the **Leahy** et al. (1983) normalization:

$$P_j \equiv \frac{2}{N_{tot}} |a_j|^2 \quad j = 0, \ldots, \frac{N}{2}; \; where\ N_{tot} = N_{ph} = \sum_k x_k = a_0$$

- The Leahy normalization is chosen such that if the x_k are Poisson distributed, then the P_j exactly follow the chi-squared distribution with 2 dof, χ^2.
- For the Poisson process, the variance (square of the standard deviation) is equal to the total number of counts.
- Thus, N_{tot} – dispersion of the total number of counts in the time series.

Properties of Leahy normalized PDS

- Variance in the real time series x_k:

$$Var(x_k) \equiv \sum_k (x_k - \bar{x})^2 = \sum_k {x_k}^2 - \frac{1}{N}\left(\sum_k x_k\right)^2 =$$

$$= \frac{1}{N}\sum_j |a_j|^2 - \frac{1}{N}{a_0}^2 = \frac{1}{N}\sum_{j\neq 0} |a_j|^2$$

Parseval's theorem

Leahy normalization

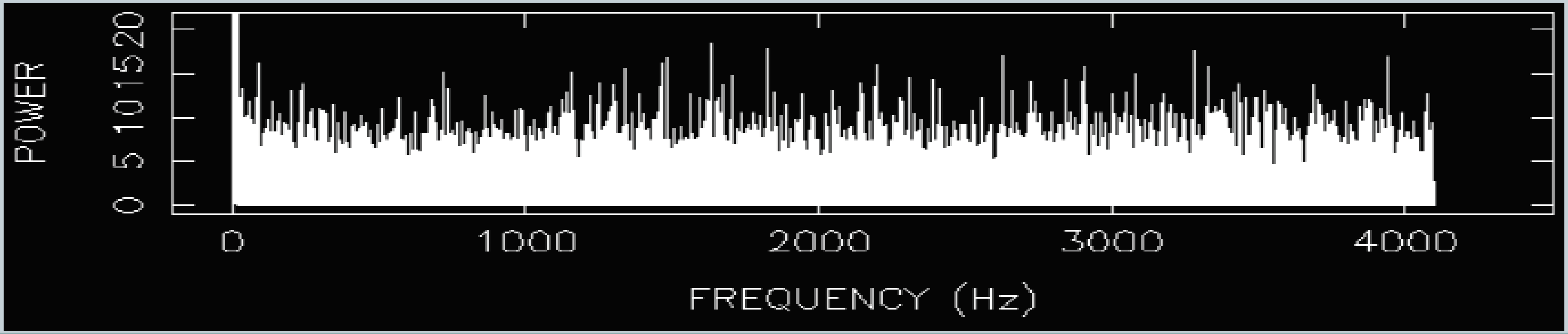
$$Var(x_k) = \frac{N_{tot}}{N}\left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2}P_{N/2}\right)$$

variance is sum of powers!

The dimension of P_j is the same as x_k and a_j: $[P_j] = [a_j] = [x_k]$

Properties of Leahy normalized PDS

- The Leahy normalization is chosen such that if the x_k are Poisson distributed, then the P_j exactly follow the chi-squared distribution with 2 dof, χ^2.
- Properties of this distribution:
  - The mean power is 2;
  - the standard deviation is 2!
- So, the power spectrum is very noisy. This does not improve with:
  - longer observation — you just get more powers
  - broader time bins — you just get a lower ν_{Ny}



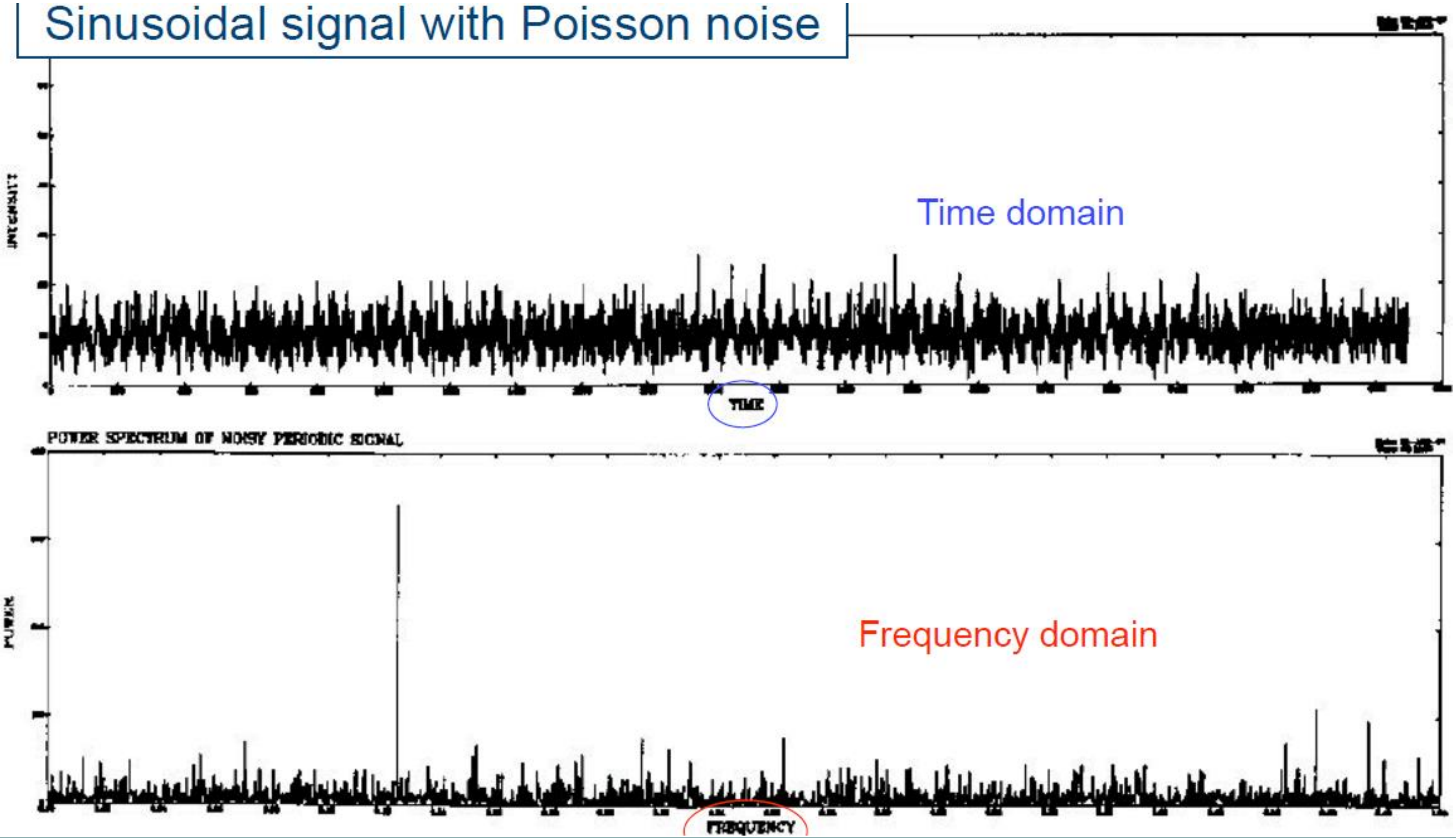
Statistics of Power Spectra

- Flux measurements are always accompanied by noise.
- The light curve can be divided into its independent components: the deterministic signal S and the noise N. For an individual time bin, the total number of counts is composed of the sum of the signal and the noise, i.e., $x_k = s_k + n_k$.
- Examples of deterministic signals:
  - a non-periodic deterministic variation, such as a nova light curve;
  - A periodic variation, such as an eclipsing binary or a RR Lyr light curve;
  - a multiply periodic variation, such as a spectroscopic triple system;
  - a modulated periodic variation where either the amplitude, frequency, or phase may vary with time - for example a pulsating system in a binary orbit.

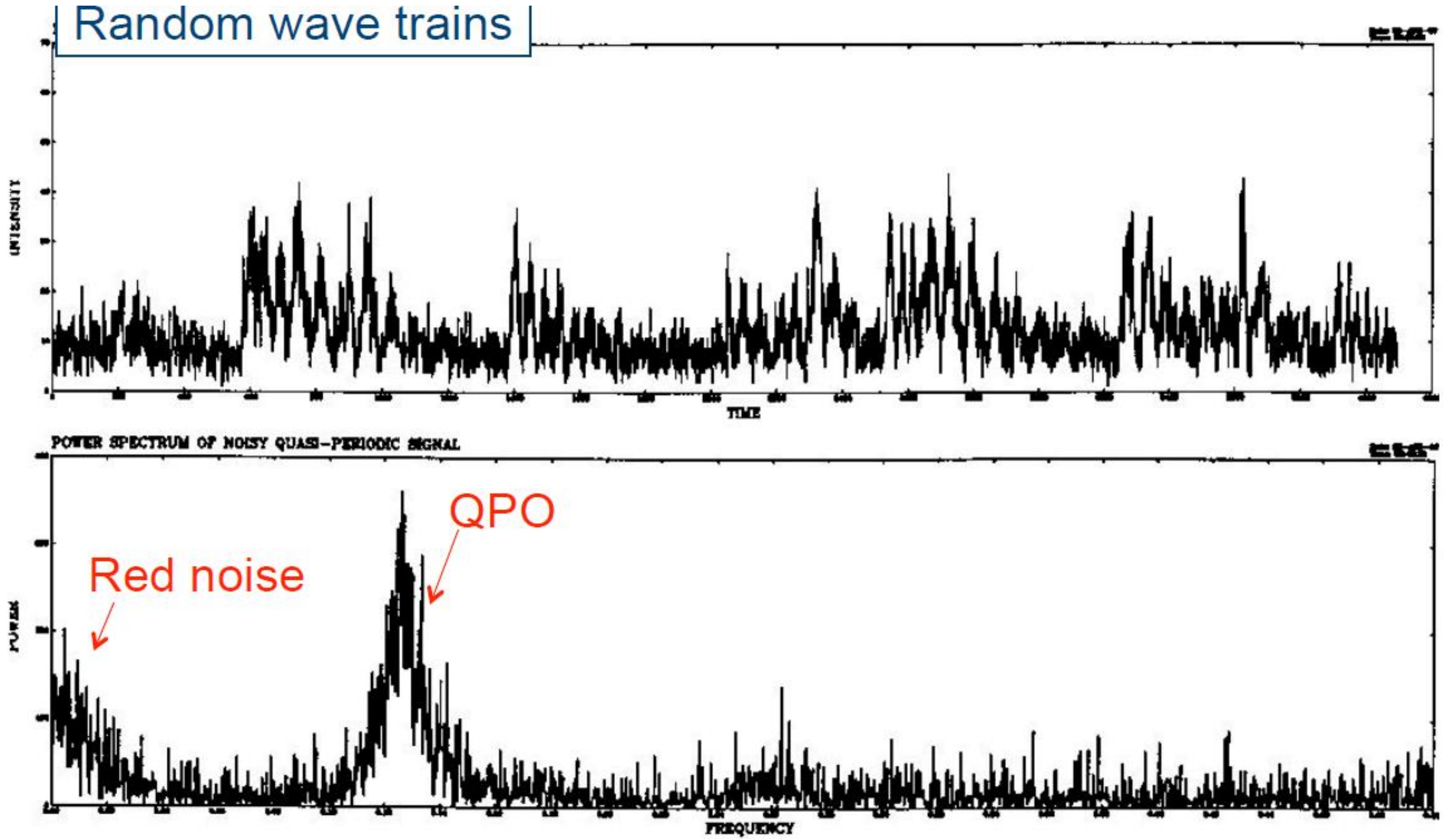
Statistics of Power Spectra

- 'Noise' (= random aka stochastic processes) in the light curve produces peaks and broad components in the power spectrum.
- Examples of noise:
  - Counting statistics noise (Poisson noise) -> white noise;
  - Poisson noise modified by instrumental effects (e.g. dead-time) and other instrumental noise;
  - Noise that is (stochastic) intrinsic source variability: QPO, band limited noise, red noise, etc.
- All these can occur at the same time, possibly together with deterministic signals.
- They can be the **background** against which you are trying to detect something else
- Or they can be the **signal** you are trying to detect.

Examples of power spectra: Periodic signal

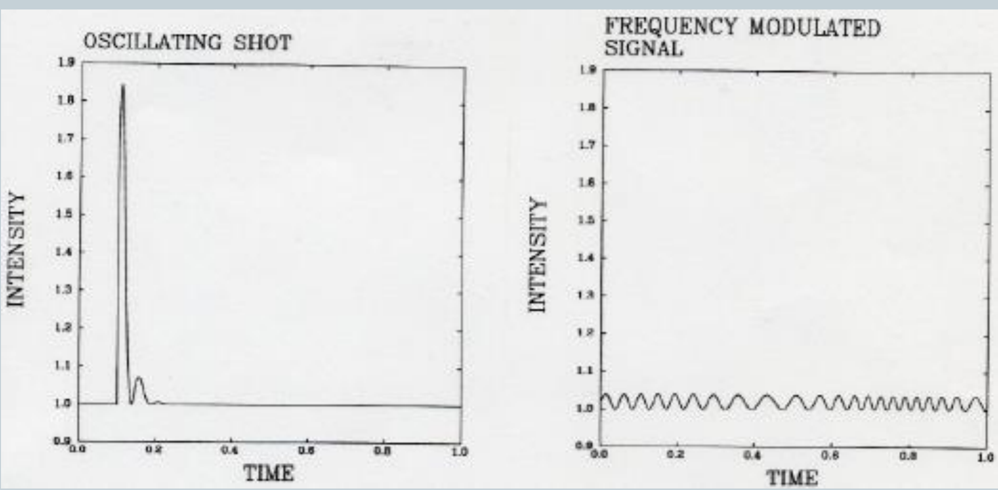


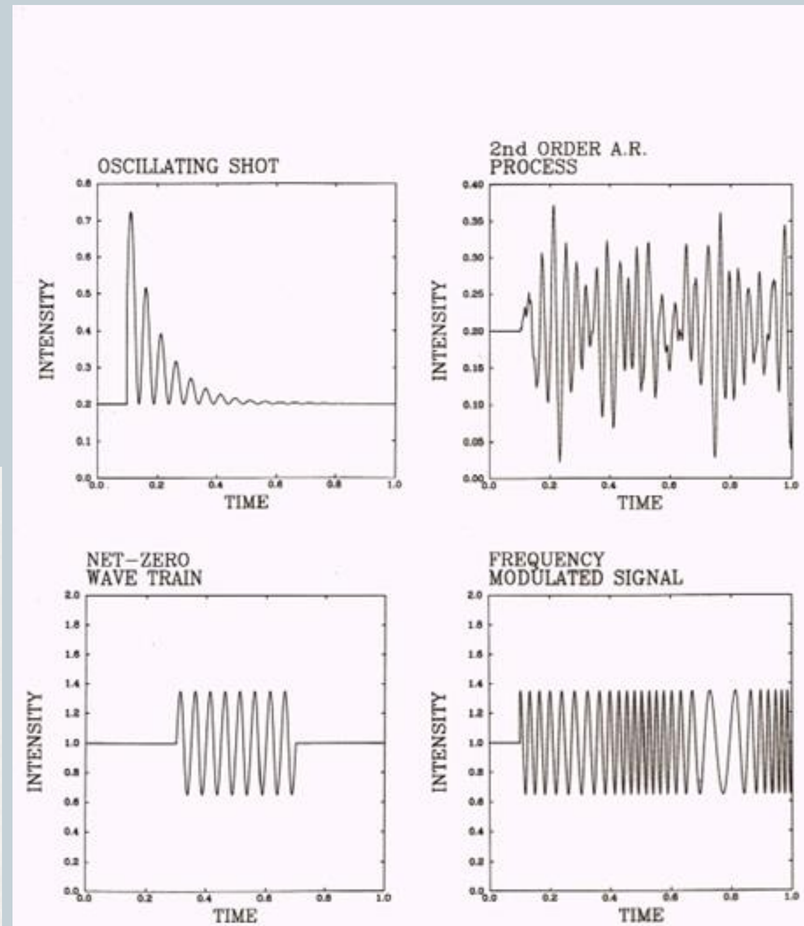
Examples of power spectra: QPO and red noise



Various possible QPO signals

Various possible time-domain signals can underlay the QPO peak we see in frequency domain





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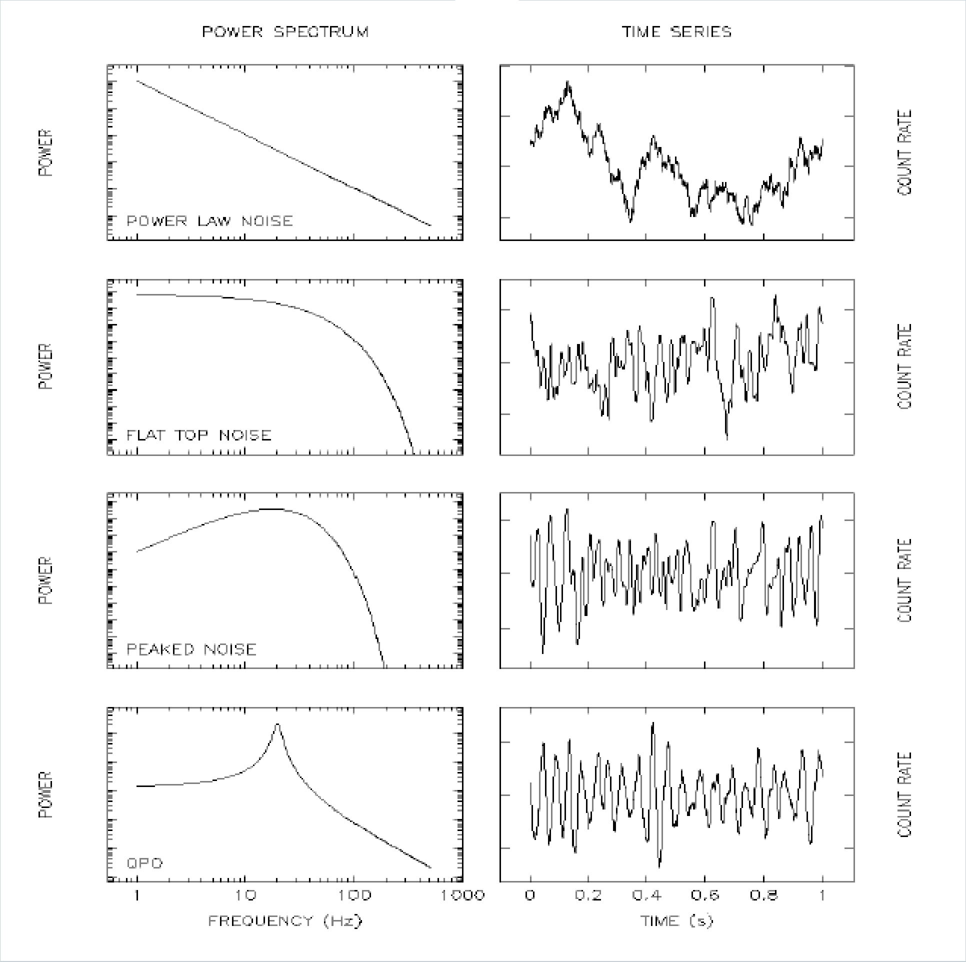
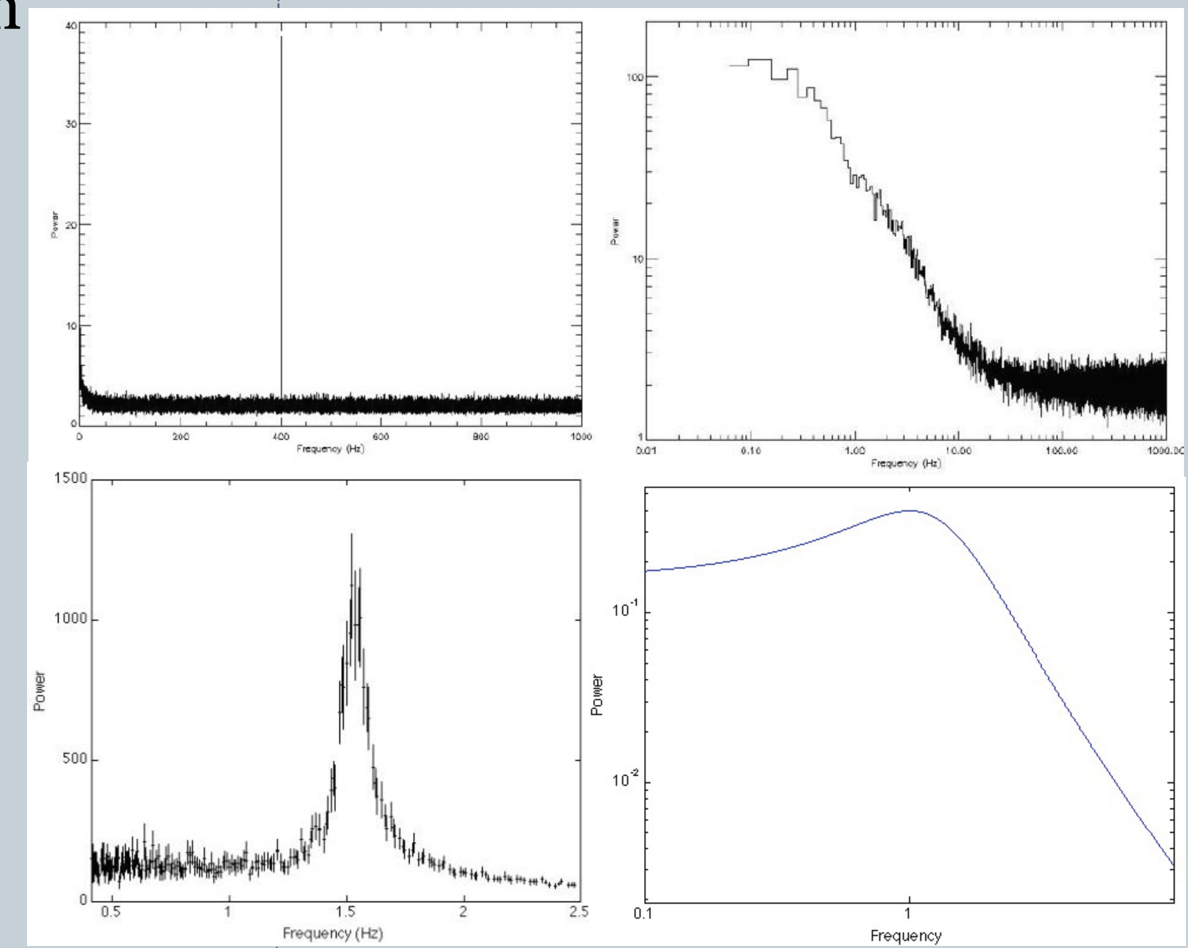


Main types of signals

- Coherent pulsation
- Broad-band noise
- Broad peak (QPO)
- Peaked-noise



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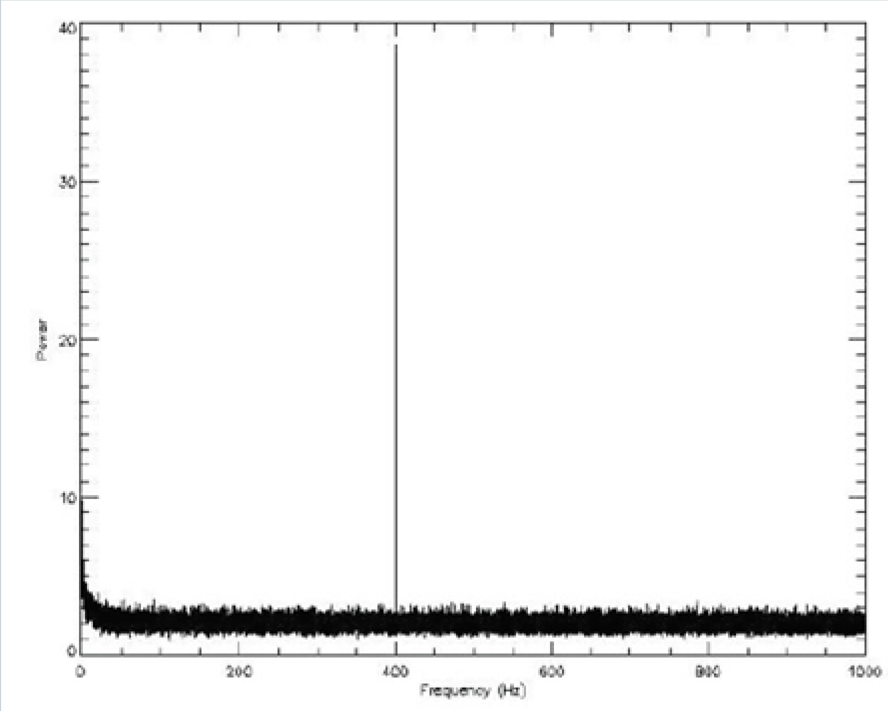

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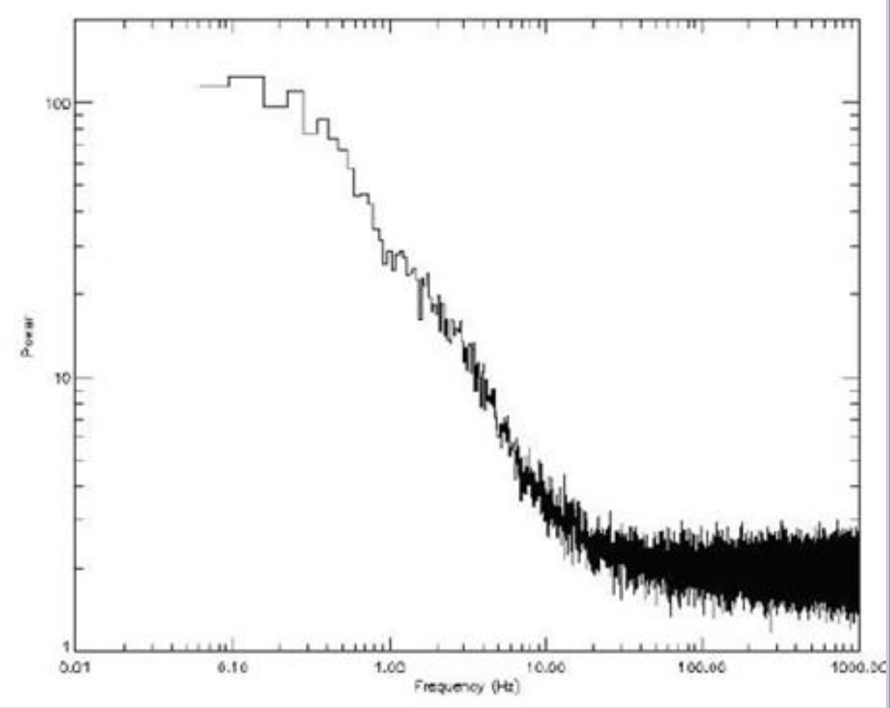

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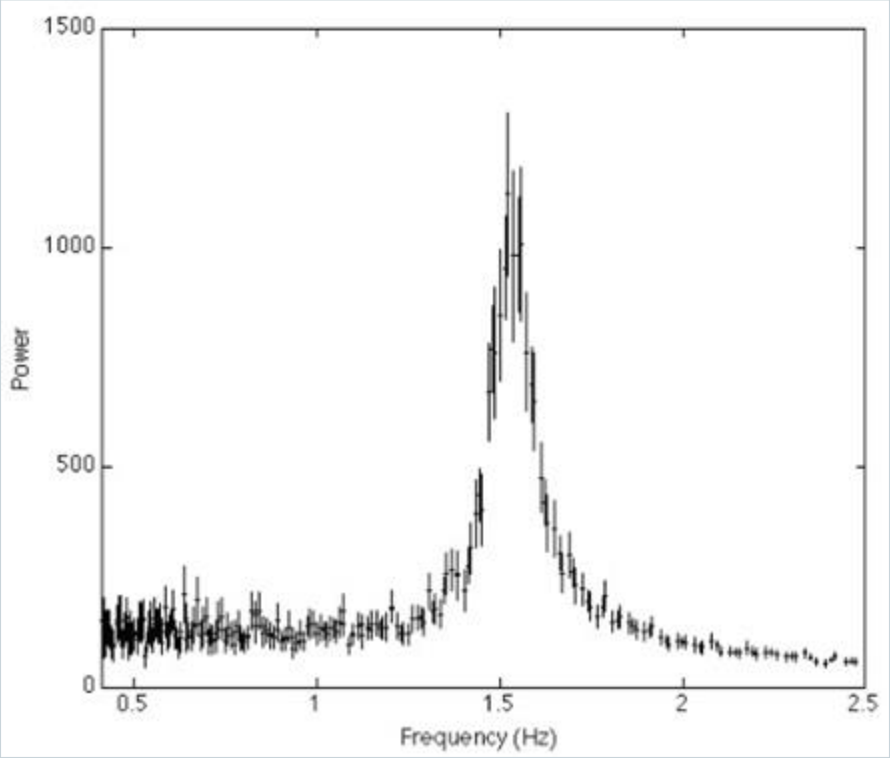

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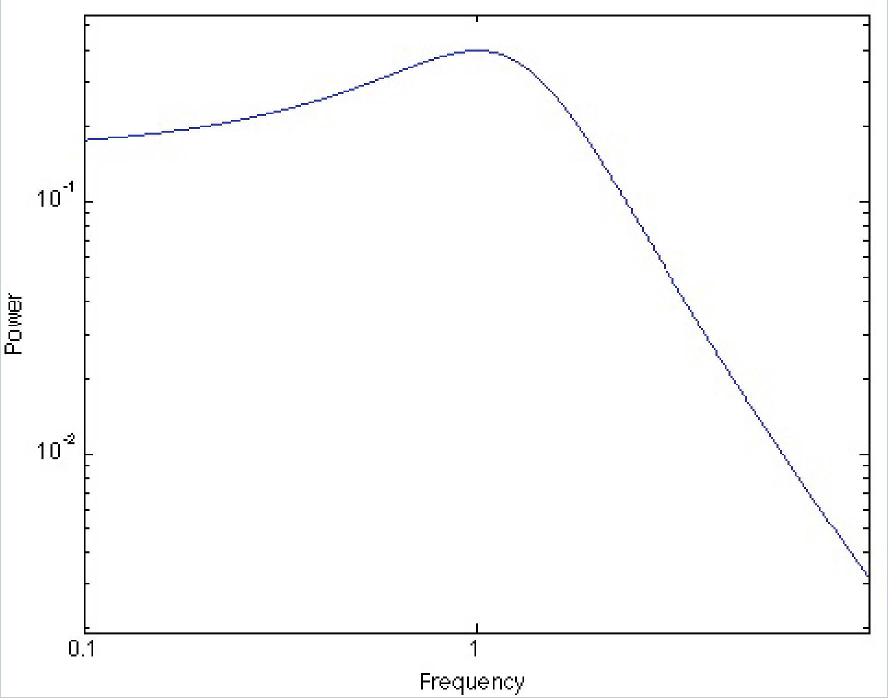
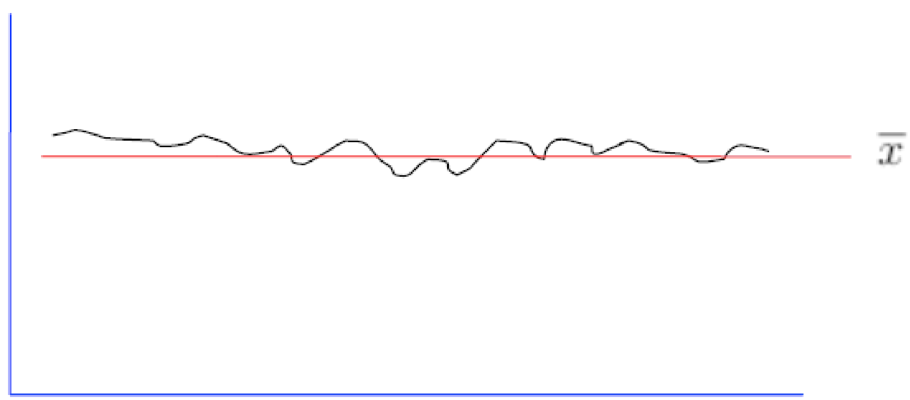


Properties of Leahy normalized PDS

Fractional rms (root-mean-square) amplitude of a signal in a time series x_k:

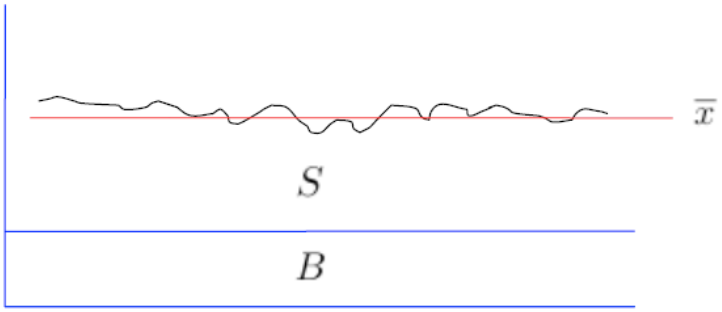
$$r \equiv \frac{\sqrt{\frac{1}{N} Var(x_k)}}{\bar{x}} = \frac{N}{N_{tot}} \sqrt{\frac{N_{tot}}{N^2}\left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2} P_{\frac{N}{2}}\right)} = \sqrt{\frac{1}{N_{tot}}\left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2} P_{N/2}\right)}$$

r is dimensionless and often expressed in % (percentage rms variation).



Properties of Leahy normalized PDS

- ”rms normalized” power density: $q(\nu_j) \equiv TP_j/N_{ph}$
  physical unit of $q(\nu_j)$ is (rms/mean)²/Hz
- ”Source” fractional rms amplitude: If the x_k are the sum of source and background: $x_k = b_k + s_k$, then the rms amplitude as a fraction of just the s_k:
  $r_s = r\frac{B+S}{S}$,
  where B and S are sums of the b_k and s_k, so $B+S = \sum_k x_k = N_{ph}$



- ”Source rms normalized” power density (“Miyamoto” normalization):
  $q_S \equiv q\left(\frac{B+S}{S}\right)^2 = TP_j\frac{B+S}{S^2}$
  the same unit as q: (rms/mean)²/Hz

Requires a model or a measurement of the background count rate