Stellar Atmospheres
Lecture 3

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Basics about radiative transfer II

RADIATION DENSITY & PRESSURE

Intensity & Flux

From the previous lecture:

- The (specific) intensity $\boldsymbol{I_\lambda}$ is a measure of brightness:

$$I_\lambda = \frac{E_\lambda}{\cos\theta \, d\lambda \, d\sigma \, d\omega \, dt}$$

- Flux $\boldsymbol{F_\lambda}$, is the projection of the specific intensity in the radial direction (integrated over all solid angles):

$$F_\lambda = \oint I_\lambda \cos\theta \, d\omega$$

- I_λ is ***independent of distance*** from the source, $\boldsymbol{F_\lambda}$ obeys the inverse square law.

Why?

Intensity

Let's try in another way:

- The (specific) intensity $\boldsymbol{I_\lambda}$ is a measure of brightness:

$$I_\lambda = \frac{dE_\lambda}{\cos\theta \; d\lambda \; d\sigma \; d\omega \; dt}$$

$d\lambda$, $d\sigma$, $d\omega$, $dt \rightarrow 0$ **dE diminishes to zero as well**

In this way, we define the specific intensity at a "point" on the surface, at a given time, in a direction θ, at a wavelength λ ("*brightness*").

Flux

Flux is related to the intensity ("specific" is often omitted):

- Flux $\boldsymbol{F}_{\lambda}$, is a measure of the net energy flow across an area $d\sigma$, over a time dt, in a $d\lambda$. The only directional significance is whether the energy crosses $d\sigma$ from the top or from the bottom. Then we can write:

The solid angle $d\omega$ appears for $\boldsymbol{I_\lambda}$ but not for $\boldsymbol{F_\lambda}$

$$F_\lambda = \frac{\oint dE_\lambda}{d\lambda\, d\sigma\, dt}$$

Integrated over all directions.

substitute

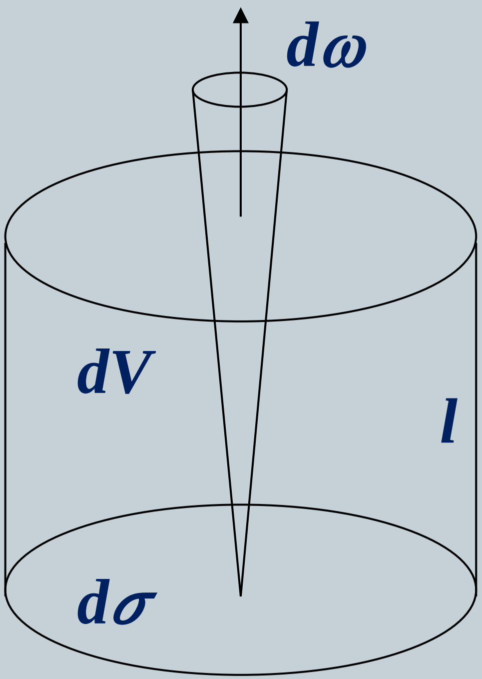
$$I_\lambda = \frac{dE_\lambda}{\cos\theta\, d\lambda\, d\sigma\, d\omega\, dt}$$

$$F_\lambda = \oint I_\lambda \cos\theta\, d\omega$$

$$\left[\frac{erg}{\text{Å}\ cm^2\ s}\right]$$

Mean intensity and Energy density

$$J_\lambda = \frac{1}{4\pi}\oint I_\lambda d\omega$$



- The *mean* intensity $\boldsymbol{J_\lambda}$ is related to the energy density $\boldsymbol{u_\lambda}$:
- Energy radiated through area element $d\sigma$ during time dt:

$$dE_\lambda = I_\lambda \, d\lambda \, d\sigma \, d\omega \, dt$$

$$l = c\,dt \Rightarrow dV = l\,d\sigma = c\,dt\,d\sigma$$

- Hence, the energy contained in volume element dV per wavelength interval is:

$$u_\lambda dV d\lambda = \oint I_\lambda \, d\omega \, d\lambda \, d\sigma \, dt = 4\pi J_\lambda \frac{dV}{c} d\lambda$$

$$u_\lambda = \frac{4\pi}{c} J_\lambda \quad \left[\frac{erg}{cm^3 \text{Å}}\right]$$

$$u = \int_0^\infty u_\lambda \, d\lambda = \frac{4\pi}{c}\int_0^\infty J_\lambda \, d\lambda \quad \left[\frac{erg}{cm^3}\right]$$

Total radiation energe in volume element

K-integral and radiation pressure

- K-integral is related to the radiation pressure: $$K_\lambda = \frac{1}{4\pi}\oint I_\lambda \cos^2\theta \, d\omega$$
- A photon has momentum $p_\lambda = E_\lambda / c$
- Consider photons transferring momentum to a solid wall. Force:

$$F = \frac{dp_{\lambda\perp}}{dt} = \frac{1}{c}\frac{dE_\lambda}{dt}\cos\vartheta$$

- Pressure: $$dP_\lambda = \frac{F}{d\sigma} = \frac{1}{c}\frac{dE_\lambda \cos\vartheta}{dt \quad d\sigma} = \frac{1}{c} I_\lambda \cos^2\vartheta \;\, d\omega \, d\lambda$$

$$I_\lambda = \frac{dE_\lambda}{\cos\theta \, d\lambda \, d\sigma \, d\omega \, dt}$$

$$P(\lambda) = \frac{1}{c}\oint_{4\pi} I_\lambda \cos^2\vartheta \;\, d\omega = \frac{4\pi}{c} K_\lambda$$

Interaction radiation – matter

CHANGE OF INTENSITY ALONG PATH ELEMENT

ABSORPTION AND EMISSION COEFFICIENTS

OPTICAL DEPTH, SOURCE FUNCTION

Interaction radiation – matter

Energy can be removed from, or delivered to, the radiation field. Classification by physical processes:

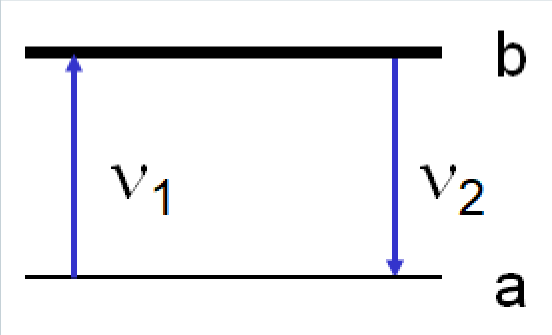
- True absorption: photon is destroyed, energy is transferred into kinetic energy of gas; photon is thermalized
- True emission: photon is generated, extracts kinetic energy from the gas
- Scattering: photon interacts with a scatterer
  - $\rightarrow$ direction changed, energy slightly changed
  - $\rightarrow$ no energy exchange with gas

Examples: true absorption and emission

- photoionization (bound-free) excess energy is transferred into kinetic energy of the released electron $\rightarrow$ effect on local temperature
- photoexcitation (bound-bound) followed by electron collisional de-excitation; excitation energy is transferred to the electron $\rightarrow$ effect on local temperature
- photoexcitation (bound-bound) followed by collisional ionization
- reverse processes are examples for true emission

Examples: scattering processes

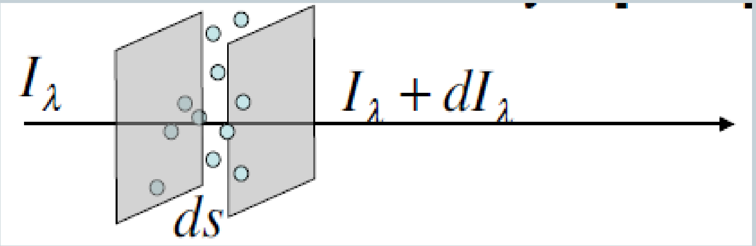
- 2-level atom absorbs photon with frequency ν_1, re-emits photon with frequency ν_2; frequencies not exactly equal, because
  - levels a and b have non-vanishing energy width
  - Doppler effect because atom moves



- Scattering of photons by free electrons: Compton- or Thomson scattering, (inelastic or elastic) collision of a photon with a free electron

Absorption coefficient

- Consider radiation shining through a layer of material (e.g., a stellar atmosphere)



- The intensity of light is found experimentally to decrease by an amount dI_λ where

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds$$

Here κ_λ is the so-called (mass) absorption coefficient (alias opacity) [cm^2 g^{-1}], ρ is the density (in mass per unit volume), and ds is a length. It can also be represented as $\alpha_\lambda = \kappa_\lambda \rho$, where α_λ is the absorption coefficient [cm^{-1}]. The photon mean free path is inversely proportional to α_λ.

Optical Depth

- Two physical processes contribute to the opacity κ_λ (note that subscript λ just means that absorption is photon-wavelength dependent);
  - (i) true absorption where the photon is destroyed and the energy thermalized;
  - (ii) scattering where the photon is shifted in direction and removed from the solid angle under consideration.

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds \qquad \ln I_\lambda = -\int_0^s \kappa_\lambda \rho ds + \ln C$$

$$I_\lambda = Ce^{-\int_0^s \kappa_\lambda \rho ds} = Ce^{-\tau_\lambda}$$

If s=0, then $C = I_\lambda{}^0$ $\quad = I_\lambda{}^0 e^{-\tau_\lambda}$ the usual simple extinction law

$$\boldsymbol{\tau_\lambda = \int_0^s \kappa_\lambda \rho ds}$$

the "optical depth"

Importance of optical depth

- We can write the change in specific intensity over a path length as

$$dI_\lambda = -I_\lambda d\tau_\lambda$$

This is a "passive"situation where no emission occurs and is the simplest example of the radiative transfer equation.

- An optical depth of τ=0 corresponds to no reduction in intensity (i.e. the top of photosphere for a star).
- An optical depth of τ=1 corresponds to a reduction in intensity by a factor of e=2.7.
- If the optical depth is large (τ >>1) negligible intensity reaches the observer.
- In stellar atmospheres, typical photons originate from τ=2/3 (the proof will follow later on).

Emission coefficient and Source function

- We can also treat emission processes in the same way as absorption via a (volume) emission coefficient ε_λ [erg/s/cm^3/str/Å] , or a (mass) emission coefficient j_λ [erg/s/g/str/Å]

$$dI_\lambda = \varepsilon_\lambda ds = j_\lambda \rho ds$$

- Physical processes contributing to ε_λ , are

  (i) real emission – the creation of photons;

  (ii) scattering of photons into a given direction from other directions.

- The ratio of emission to absorption coefficients is called the Source function

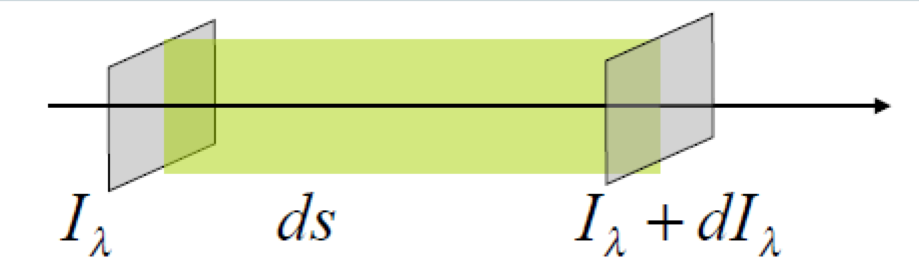
$$S_\lambda = j_\lambda / \kappa_\lambda$$

Radiative transfer I

PARALLEL-RAY TRANSFER EQUATION

Radiative transfer equation (1)

- The primary mode of energy transport through the surface layers of a star is by radiation.
- The radiative transfer equation describes how the physical properties of the material are coupled to the spectrum we ultimately measure.
- Recall, energy can be removed from (true absorption or scattered), or delivered to (true emission or scattered) a ray of radiation:



- The rate of change of (specific) intensity is:

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds + j_\lambda \rho ds$$

Radiative transfer equation (2)

$$dI_\lambda = -\kappa_\lambda \rho I_\lambda ds + j_\lambda \rho ds$$

$$d\tau_\lambda = \kappa_\lambda \rho ds$$
$$S_\lambda = j_\lambda / \kappa_\lambda$$

- We can re-write this equation in terms of the optical depth τ_λ and the source function S_λ

$$dI_\lambda / d\tau_\lambda = -I_\lambda + S_\lambda$$

- **This is the (parallel-ray) equation of radiative transfer (RTE).** It will need a small modification before it is applicable to stars, but we can already gain some insight from its solution.
- If $S_\lambda < I_\lambda$, the intensity will decrease with increasing τ_λ, it will stay constant if $S_\lambda = I_\lambda$ and increase if $S_\lambda > I_\lambda$. *When* $\tau_\lambda \rightarrow \infty$, $I_\lambda \rightarrow S_\lambda$

Thermodynamic Equilibrium (TE)

Physical interpretation of S_λ

$$dI_\lambda/d\tau_\lambda = -I_\lambda + S_\lambda$$

- If $S_\lambda < I_\lambda$, the intensity will decrease with increasing τ_λ
- The intensity will increase if $S_\lambda > I_\lambda$
- It will stay constant if $S_\lambda = I_\lambda$

- Thermodynamic Equilibrium (TE) → nothing changes with time
- A beam of light passing through such a gas volume will not change either

$$dI_\lambda/d\tau_\lambda = 0 \longrightarrow S_\lambda = I_\lambda = B_\lambda$$

in TE, the source function equals the Planck function

$$\kappa_\lambda B_\lambda = j_\lambda \quad \text{or} \quad \alpha_\lambda B_\lambda = \varepsilon_\lambda$$

The law of Kirchhoff

- In the theory of stellar atmospheres we make the assumption of **local thermodynamic equilibrium**, abbreviated by **LTE**. $\longrightarrow$ $\boldsymbol{S_\lambda = B_\lambda}$
- In LTE, it doesn't necessary mean that $S_\lambda = I_\lambda$, it still can be $S_\lambda \neq I_\lambda$

Radiative transfer equation (3)

- One can formally solve this form of the RTE with an integrating factor e^{τ_λ}, i.e.

$$\frac{d}{d\tau_\lambda}\left(e^{\tau_\lambda} I_\lambda\right) = e^{\tau_\lambda} I_\lambda + e^{\tau_\lambda}\frac{dI_\lambda}{d\tau_\lambda} \qquad so \qquad \frac{d}{d\tau_\lambda}\left(e^{\tau_\lambda} I_\lambda\right) = e^{\tau_\lambda} S_\lambda$$

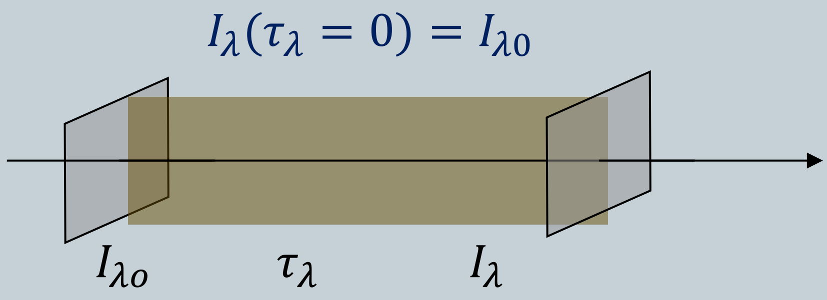
$$\int_0^{\tau_\lambda} \frac{d}{d\tau_\lambda}\left(e^{\tau_\lambda} I_\lambda\right) d\tau_\lambda = \left[e^{\tau_\lambda} I_\lambda\right]_0^{\tau_\lambda} = \int_0^{\tau_\lambda} e^{\tau_\lambda} S_\lambda \, d\tau_\lambda = \left[e^{\tau_\lambda} S_\lambda\right]_0^{\tau_\lambda}$$

S_λ is constant along the path – **that is a very rude assumption!**
See D. Gray (pp. 127-129) for more accurate integration.

Radiative transfer equation (4)

Inserting boundary conditions:

$$I_\lambda e^{\tau_\lambda} - I_{\lambda 0} = S_\lambda(e^{\tau_\lambda} - 1)$$



Rearrange:

$$I_\lambda = S_\lambda(1 - e^{-\tau_\lambda}) + I_{\lambda 0} e^{-\tau_\lambda}$$

The second term of the RHS describes the amount of radiation left over from the intensity entering the box, after it has passed through an optical depth τ, the first term gives the contribution of the intensity from the radiation emitted along the path.

$$I_\nu(\tau_\nu) = \int_0^{\tau_\nu} S_\nu(t_\nu) \mathrm{e}^{-(\tau_\nu - t_\nu)} \, \mathrm{d}t_\nu + I_\nu(0) \mathrm{e}^{-\tau_\nu}.$$

This is the formal solution of the RTE which assumes that the source function is known (D. Gray)

Solution to transfer equation

Imagine first the case in which $I_{\lambda 0}$=0, i.e. solely emission from the volume of gas. $I_\lambda = S_\lambda(1 - e^{-\tau_\lambda})$ We have two limiting cases:

- Optically thin case (τ_λ<<1)

$$e^{-\tau_\lambda} \approx 1 - \tau_\lambda \Rightarrow I_\lambda = \tau_\lambda S_\lambda$$

EXAMPLE:

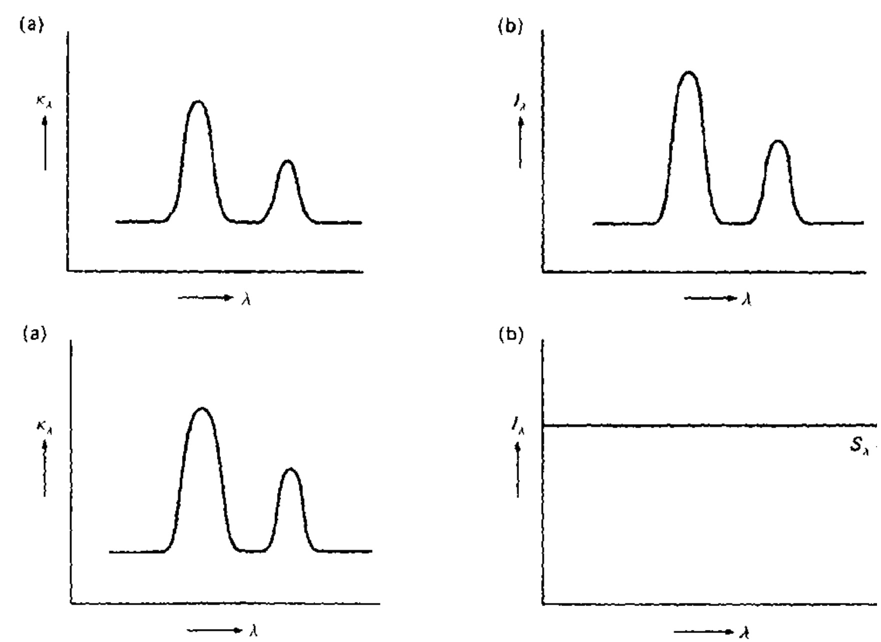
Hot, low density nebula

- Optically thick case (τ_λ>>1)

$$e^{-\tau_\lambda} \approx 0 \Rightarrow I_\lambda = S_\lambda$$

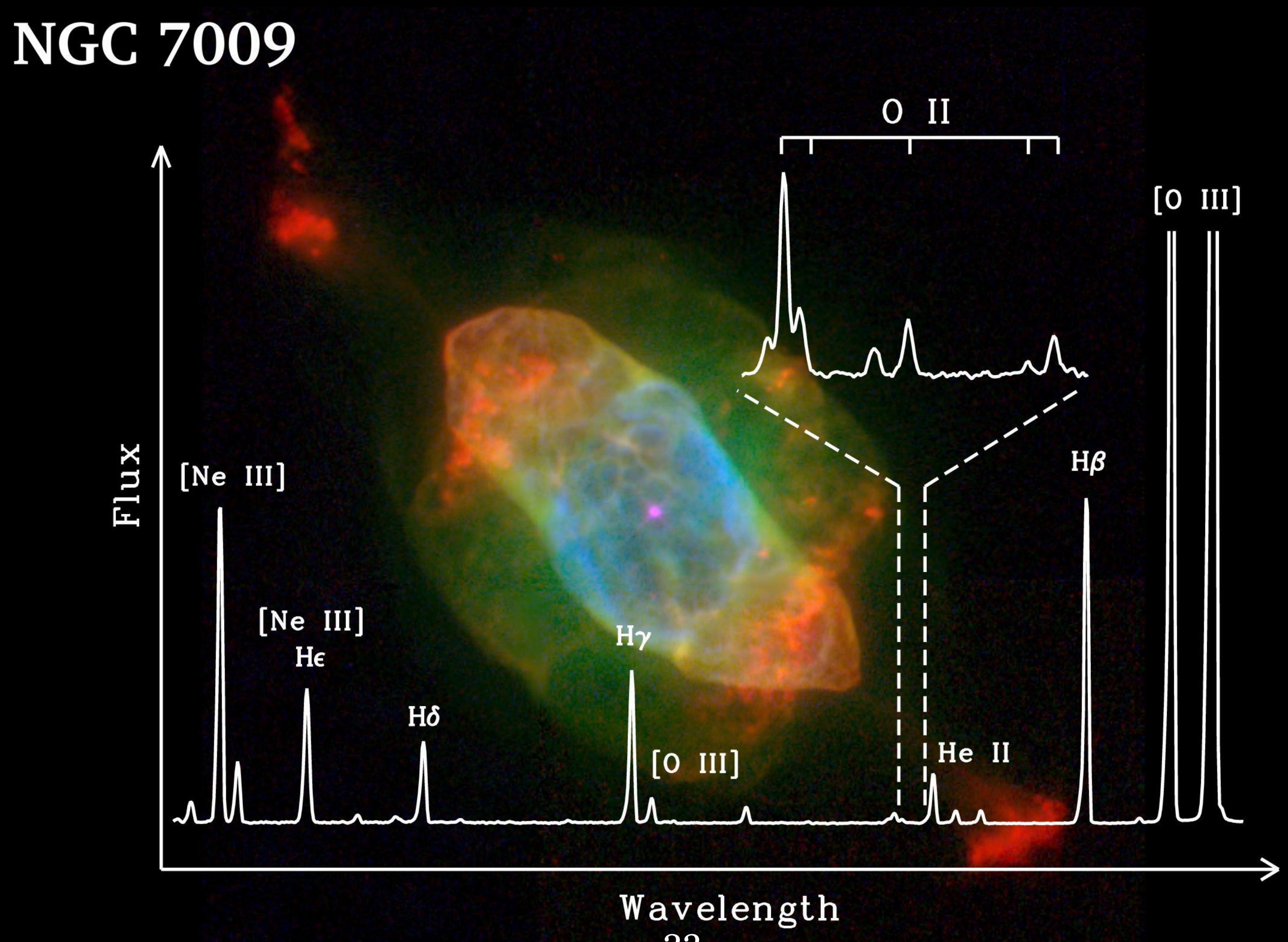
EXAMPLE:

Black body, $S_\lambda=B_\lambda(T)$



Opacity κ versus λ ➔Intensity versus λ

Hot nebular gas: emission lines –optically thin



Absorption versus emission

Imagine now $I_{\lambda 0} \neq 0$, again with two extreme cases:

$$I_\lambda = I_{\lambda 0} e^{-\tau_\lambda} + S_\lambda (1 - e^{-\tau_\lambda})$$

$$I_\lambda = I_{\lambda 0}(1 - \tau_\lambda) + \tau_\lambda S_\lambda = I_{\lambda 0} + \tau_\lambda (S_\lambda - I_{\lambda 0})$$

- Optically thin case ($\tau_\lambda \ll 1$)

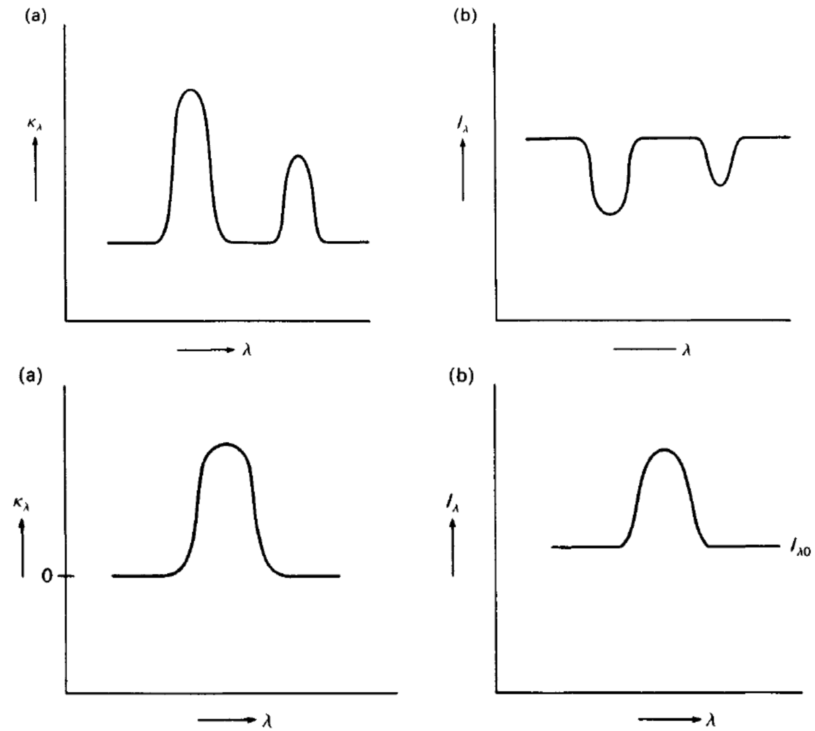
(a) If $I_{\lambda 0} > S_\lambda$, so there is something subtracted from the original intensity which is proportional to the optical depth – we see absorption lines on the continuum intensity I_λ.
EXAMPLE: [stellar photospheres]

(b) If $I_{\lambda 0} < S_\lambda$, we will see emission lines on top of the background intensity.

Example: [Solar UV spectrum]

- Optically thick case ($\tau_\lambda \gg 1$): $I_\lambda = S_\lambda$

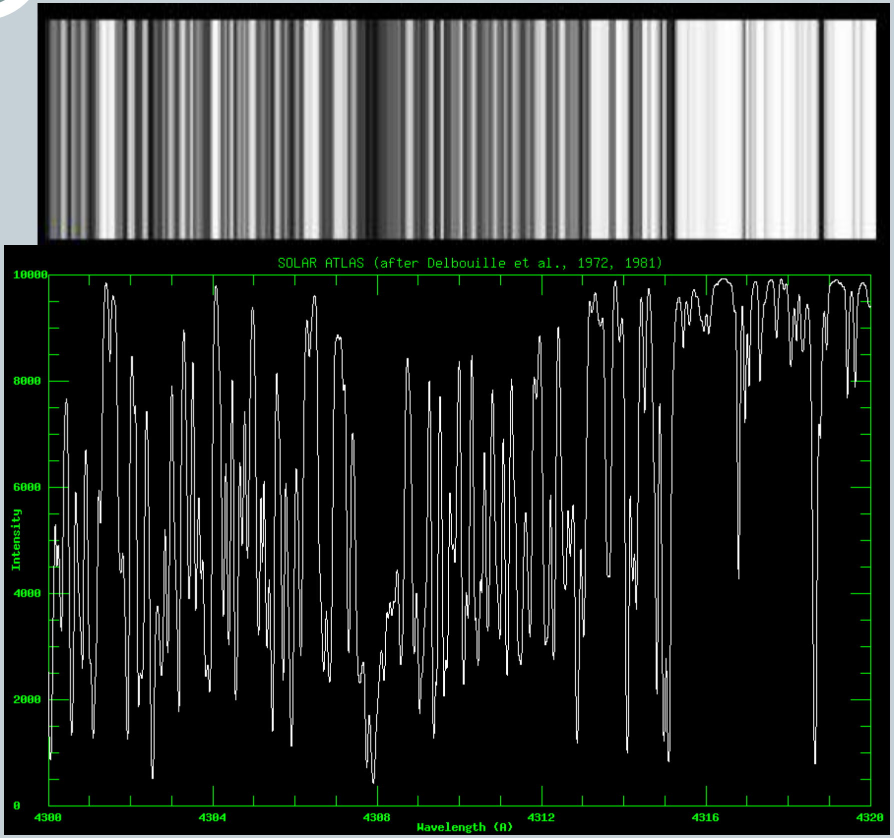
Planck function as before.



Opacity κ versus λ ➔ Intensity versus λ

Outward decreasing temperature

- In a star absorption lines are produced if $I_{\lambda o} > S_{\lambda}$ i.e. the intensity from deep layers is larger than the source function from top layers.
- In LTE, the source function is $B_{\lambda}(T)$, so the Planck function for the deeper layers is larger than the shallower layers. Consequently the deeper layers have a higher temperature than the top layers (since the Planck function increases at all wavelengths with T).
- (Instances occur where LTE is not valid, and the source function declines outward in parallel with an increasing temperature).



Solar Spectrum (4300-4320Ang)

Absorption versus emission lines

Emission line spectra:

- Optically thin volume of gas with no background illumination (emission nebula)
- Optically thick gas in which the source function increases outwards (UV solar spectrum)

Absorption line spectra:

- Optically thin gas in which source function declines outward, generally T decreases outwards (Stellar photospheres)
- Optically thin gas penetrated by background radiation (ISM between us and the star)

Things we learned about

- Radiation density & pressure are defined.
- The optical depth τ_λ is a dimensionless quantity related to the absorption coefficient or opacity.
- The source function is defined as the ratio of emission to absorption coefficients.
- The source function equals the Planck function in LTE.
- Radiative transfer equation and its formal solution.
- Absorption and emission line spectra.