ACTA ASTRONOMICA
Vol. **19** (1969) No 2

On the Rotational Velocities of Gaseous Rings in Close Binary Systems

by

J. Smak

ABSTRACT

It is shown that in the case of gaseous rings of finite width the observed rotational velocity (as determined from radial velocities of the emission lines) is systematically smaller than the true, mean velocity of rotation. The difference depends on the width and the structure of the ring; for example, when the width of the ring is comparable to its radius, it amounts to about 20 percent.

The $K_1/V_r \sin i$ versus mass-ratio relation (Kruszewski 1967) is rediscussed. It is suggested that the discrepancy between the theory and observations can be due to (a) non-Keplerian motion of large rings, and (b) underestimated rotational velocities of rings of finite width.

1. Introduction

Rotating gaseous rings are known to be a fairly common phenomenon in close binary systems (*cf., e. g.,* Sahade 1960). Their presence is best manifested in systems with the orbital inclination close to $i = 90°$, when the emission lines originating in the ring are characteristically double. The investigations are seriously limited, however, by the fact that in most cases (except for some nova-type and U Gem-type binaries) the emission lines can be seen only during the primary eclipse; besides, all the objects in question are rather faint to permit high dispersion observations with an appropriate resolution in time. Primarily for these reasons the observational data are restricted to the rotational velocities which refer to the observed separations of the components. There are only very few cases with the relative dimensions of the ring being determined directly from observations; the best such case is probably RZ Oph (Hiltner 1946). No major effort has also been made to study the observed profiles, except for a recent paper on RW Tau by Plavec (1968) and rather general discussion by Gorbatzky (1964).

The rotational velocity of the ring V_r and its radius R are related by

$$V_r = \left(\frac{G\mathfrak{M}_1}{R}\right)^{1/2}, \tag{1}$$

where $\mathfrak{M}_1$ is the mass of the primary component. This relation is widely used either for determining R (when $\mathfrak{M}_1$ is known), or for evaluating $\mathfrak{M}_1$ (when some estimates of R are available). It should be kept in mind, however, that Eq. (1) is strictly correct only when the mass of the secondary component is negligible. Otherwise, the relation between V_r and R must include also the mass-ratio and the *relative* dimensions of the ring. This was discussed recently by Huang (1967).

The observed values of the rotational velocities can also be used to test various theories of the ring formation (see, *e. g.*, Kruszewski 1967). In the case when the angular momentum considerations are involved, Eq. (1) also enters implicitly, its correctness being an important factor too. We shall return to the problem of Kruszewski's (1967) relation between $K_1/V_r \sin i$ and the mass-ratio in Section 4.

With all these applications in mind, it will be the aim of the present paper to see whether the observed rotational velocity (as measured on the spectrograms) can be considered as being representative for the *mean* rotational velocity of a ring of finite width.

2. Profiles of the Emission Lines

The observed profile of an emission line originating in the ring depends on several factors, like: (a) the density distribution within the ring, as a function of at least two coördinates (R and Z); (b) ionization and excitation conditions; (c) the dependence of the rotational velocity on the distance from the central star; (d) velocity dispersion due to thermal motions and turbulence; and (e) the instrumental profile. For observations made during an eclipse we have an additional dependence on its photometric phase. Any general predictions involving so many free parameters would be rather difficult to compare with the observational data. It is clear that much more meaningful results can be obtained for more particular cases (see, for example, a discussion of RW Tau by Plavec (1968)).

In what follows we shall limit ourselves to a much simpler situation, by making the following assumptions:

1. The density distribution depends only on R, while the thickness

of the ring (in the Z-coordinate) is small. The width of the ring is determined by its outer and inner radii, R_2 and R_1.

2. We neglect the absorption effects within the ring.

3. The rotational velocity is given by Eq. (1), *i. e.* we assume that the width of the ring is large as compared with the macro-scale of the turbulent motions.

4. We neglect the velocity dispersion due to random motions as well as the influence of the instrumental profile.

5. We are interested in profiles outside of eclipse.

We introduce the following units: R_2 for the unit of length, and $V_2 = (\mathfrak{M}_1/R_2)^{1/2}$ for the unit of velocity. In these units the outer and inner radii of the the ring are

$$r_2 = 1 , \quad \text{and} \quad r_1 = R_1/R_2 , \tag{2}$$

and the corresponding rotational velocities are

$$v_2 = 1 , \quad \text{and} \quad v_1 = r_1^{-1/2} . \tag{3}$$

The lines of constant radial velocity, u are given by (Fig. 1):

$$u = v \cos \varphi = r^{-1/2} \cos \varphi . \tag{4}$$

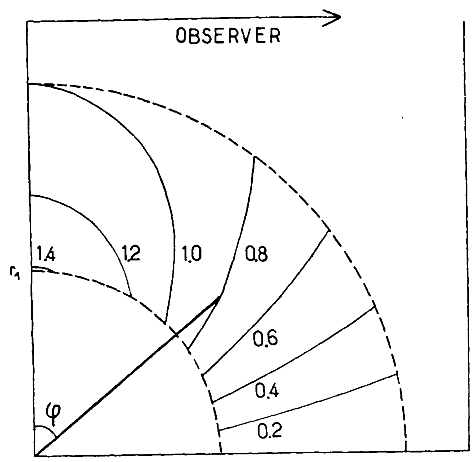

Fig. 1. Lines of constant radial velocity u determined by Eq. (4). In this diagramme $r_1 = 0.5$.

Let the density along the r-coordinate by denoted by $f(r)$. Then the observed frequency function $F(u)$ is given by

$$F(u) = \int_{r_1}^{r_z} f(r) \frac{d\varphi}{du} dr = \int_{r_1}^{r_z} \frac{f(r) r^{1/2}}{(1 - u^2 r)^{1/2}} dr , \tag{5}$$

where use has been made of Eq. (4). The integration limit r_z depends on u, namely $r_z = 1$ for $u \leqslant 1$, and $r_z = u^{-2}$ for $u > 1$ (this results from Eq. (4) and Fig. 1).

Computations have been made for two cases:

Case I. $f(r) = \text{const.}$, *i. e.* the density distribution within the ring is uniform.

Case II. $f(r) = \text{const} \times (1 - r)/(1 - r_1)$, *i. e.* the density decreases linearly from the inner to the outer edge with $f(1) = 0$.

The details of computations are given in Appendix 1. Their results — for $r_1 = 0.3, 0.5$, and 0.7 — are shown in Fig. 2. The values of $F(u)$ have been arbitrarily normalized in these diagrams. The horizontal scale can be transformed into Ångstroms by the relation

$$\Delta\lambda \,(\text{in Å}) = \lambda \frac{V_2}{c} \times u\,, \tag{6}$$

where V_2 is the rotational velocity of the outer edge of the ring and λ is the wavelength of the line. In our discussion, however, we shall use consistently only the velocity units.

Some of the observational implications will be discussed in the next section. Here we wish to comment briefly on the validity and consequences of our initial assumptions. First, it is clear from Fig. 2 that the observed profiles depend on $f(r)$; with the density being a decreasing function of r the maximum of $F(u)$ shifts towards larger values of u. The z-distribution is obviously unimportant as long as the absorption effects can be neglected. If the absorption cannot be neglected, then major changes in $F(u)$ will occur for those values of u, for which the lines of constant radial velocity (see Fig. 1) are — at least partly — parallel to the line of sight; we can see from Fig. 1 that this will happen for very small and very large values of u. The position of maximum will not be much affected, or shifted slightly towards smaller values of u. The velocity dispersion and the instrumental profile will tend to smooth out the profiles; the observed maximum may be shifted in either direction depending on the „asymmetry" of the original maximum. Finally, partial eclipse of the ring may change the profile to a considerable extent. Again, however, major changes in $F(u)$ will occur for small values of u, without any major shift of the maximum (see the profiles computed by Plavec (1968) for RW Tau), except for phases of nearly total eclipse of the ring; needless to say such observations would be of great importance for studying the structure of the ring.

To summarize, it is felt that in spite of our simplifying assumptions the profiles shown in Fig. 2 give the positions of maxima of $F(u)$

with a reasonable accuracy and that most factors neglected in the computations (as discussed above) would tend to shift them rather towards smaller values of u.

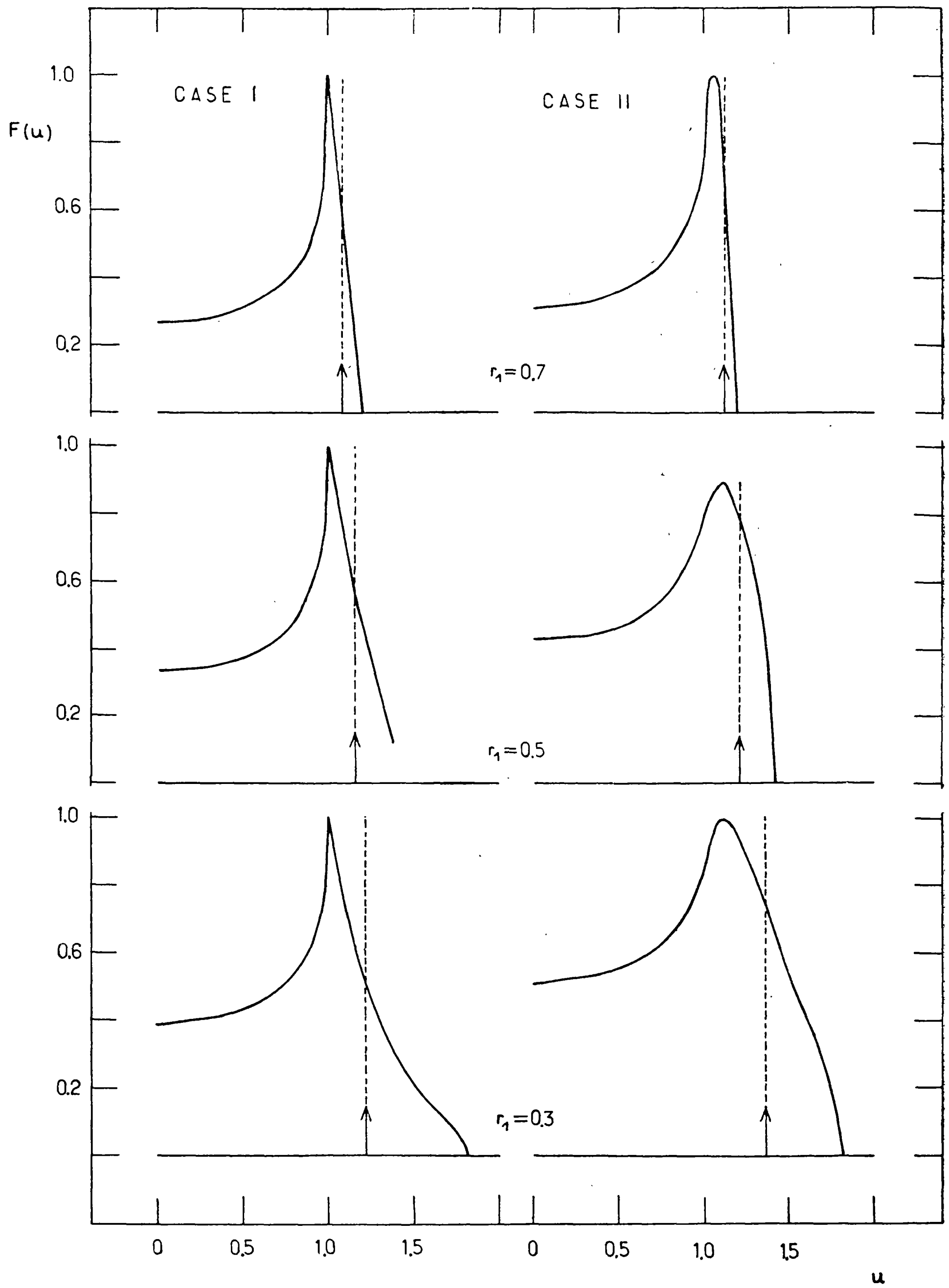

Fig. 2. Theoretical profiles for Case I (left panel) and Case II (right panel) and for three different values of r_1. All the profiles have been arbitrarily normalized. Vertical dotted lines with arrows indicate $u = \langle v \rangle_m$.

3. The Observed and Mean Rotational Velocities

The observed velocities obtained from the positions of the emission components correspond roughly to the maxima of $F(u)$. The degree of coincidence, however, is difficult to estimate, primarily because the results of measurements depend on several factors, such as the plate density, the asymmetry of the maximum, and the observer's habits. We assume, for simplicity, that what is measured does correspond exactly to the maximum of $F(u)$. As can be inferred from Fig. 2, this assumption cannot affect but quantitatively our basic conclusions to be made below in a qualitative way.

From Fig. 2 we determine the values of u_{max}, as being representative for the observed rotational velocity. For Case I we have $u_{max} = 1$, regardless of r_1, while for Case II we have $u_{max} \cong 1.13$, 1.10, and 1.06 for $r_1 = 0.3$, 0.5, and 0.7, respectively. These are to be compared with the mean rotational velocities.

Having in mind different applications of the rotational velocity we can introduce the following three definitions of the mean rotational velocity. First, we can compute the ordinary mean value, weighted by the distribution of mass:

$$\langle v \rangle_m = \frac{\int_{r_1}^{1} v\, 2\pi r f(r)\, dr}{\int_{r_1}^{1} 2\pi r f(r)\, dr}, \tag{7}$$

with $v = r^{-1/2}$. Second, we can compute the mean radius (weighted by the distribution of mass)

$$\langle r \rangle = \frac{\int_{r_1}^{1} 2\pi r^2 f(r)\, dr}{\int_{r_1}^{1} 2\pi r f(r)\, dr}, \tag{8}$$

and the corresponding value of the rotational velocity will be:

$$\langle v \rangle_r = (\langle r \rangle)^{-1/2} \tag{9}$$

Finally, we can compute the mean rotational momentum (weighted by the distribution of mass)

$$\langle h \rangle = \frac{\int_{r_1}^{1} h\, 2\pi r f(r)\, dr}{\int_{r_1}^{1} 2\pi r f(r)\, dr}, \tag{10}$$

with $h = r^{1/2}$, and the corresponding value of the rotational velocity will be:

$$\langle v \rangle_h = (\langle h \rangle)^{-1}. \tag{11}$$

The results are collected in Table 1. The important parameter is the ratio of the mean to the observed rotational velocity. In Case I, where $v_{obs} = u_{max} = 1$, this ratio is equal simply to the mean velocity.

Table 1.

Mean Rotational Velocities.

Inner radius of the ring, r_1 [1]	0.3	0.5	0.7
Case I:			
Mean radius, $\langle r \rangle$	0.713	0.778	0.859
Mean momentum, $\langle h \rangle$	0.836	0.878	0.926
Mean velocities: $\langle v \rangle_m$	1.224	1.149	1.083
$\langle v \rangle_r$	1.184	1.134	1.079
$\langle v \rangle_h$	1.196	1.139	1.080
Case II:			
Mean radius, $\langle r \rangle$	0.584	0.688	0.806
Mean momentum, $\langle h \rangle$	0.756	0.826	0.897
Mean velocities: $\langle v \rangle_m$	1.354	1.210	1.117
$\langle v \rangle_r$	1.309	1.206	1.114
$\langle v \rangle_h$	1.323	1.211	1.115
Observed velocity, v_{obs}	1.13	1.10	1.06
Mean-to-observed ratios: $\langle v \rangle_m / v_{obs}$	1.19	1.10	1.05
$\langle v \rangle_r / v_{obs}$	1.16	1.09	1.05
$\langle v \rangle_h / v_{obs}$	1.17	1.10	1.05

[1] In units of the outer radius of the ring.

For Case II we obtain values which are listed in Table 1. As can be seen, the values of $\langle v \rangle_m$, $\langle v \rangle_r$, and $\langle v \rangle_h$ do not differ much from each other. There is also no much difference between Case I and Case II. Fig. 3. shows the values of $\langle v \rangle / v_{obs}$ plotted against r_1. Taking into account all the uncertainties involved we can conclude that $\langle v \rangle / v_{obs}$ increases with the increasing width of the ring, being equal to about 1.1 at $r_1 = 0.6$ and to about 1.2 at $r_1 = 0.3$. Unfortunately, very little is known about typical values of the width of the ring. In the best documented case of RZ Oph Hiltner (1946) estimated that the outer radius is about 50 percent larger than the inner radius; this would correspond to about $r_1 = 0.6$. Smaller values of r_1 (*i. e.* wider rings) seem entirely possible in the case of some other systems and the possibility of having a disk-like structure rather than a ring cannot be excluded a priori as a limiting case. While all such questions are of quantitative character, it is quite safe to make the following qualitative conclusion:

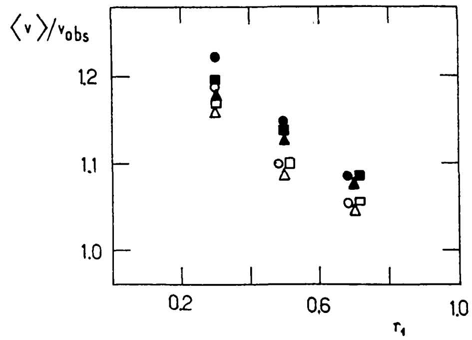

Fig. 3. The ratio $\langle v\rangle/v_{obs}$ plotted against r_1 (*i. e.* the inner radius of the ring expressed as a fraction of the outer radius). Filled symbols are for Case I, open symbols are for Case II. Circles correspond to $\langle v\rangle_m$, squares — to $\langle v\rangle_h$, and triangles — to $\langle v\rangle_r$.

The mean rotational velocity of the ring of finite width is always larger than the observed velocity determined from the radial velocity measurements. The difference depends on the width and the structure of the ring and can be as large as 10 or 20 percent.

4. The $(K_1/V_r \sin i)$ versus Mass-ratio Relation

Kruszewski (1967) showed that the observed relation between the ratio $K_1/V_r \sin i$ (where K_1 is the semi-amplitude of the radial velocity curve of the primary component and V_r — the rotational velocity of the ring) and the mass-ratio, $q = \mathfrak{M}_1/\mathfrak{M}_2$ [1] can be reproduced, at least qualitatively, on the basis of the angular momentum considerations connected with the mass-exchange and the ring formation. The quantitative agreement, however, is not so good, the observed values of $K_1/V_r \sin i$ being systematically larger than the theoretical ones. At least two plausible explanations of this discrepancy can now be added to those suggested by Kruszewski.

As mentioned in the Introduction, Eq. (1) is only approximately correct. When the mass of the secondary is not negligible and when the relative radius of the ring is large, then the rotational velocity of the

[1] Note that in Kruszewski's notation symbols "1" and "2" were used in the opposite sense. To avoid misunderstandings: q is the ratio of the mass of the ring possessing component to the mass of its companion.

ring of a given radius is *smaller* than that given by Eq. (1). This effect should enter into Kruszewski's considerations in the following way. His Eq. (7) which gives the radius of the ring a function of the mass--ratio and the angular momentum and which was obtained by using the Keplerian approximation (our Eq. (1)), should be modified. In general, the „non-Keplerian" radius of the ring should be *larger* than that given by Kruszewski's Eq. (7). As a consequence his final expression for $K_1/V_r \sin i$ (Eq. (8)) should also be modified in the same sense. That means that the theoretical values of this ratio corrected for the effect of non-Keplerian motion would be *larger* than those obtained by Kruszewski.

The second explanation is connected directly with the results presented in Section 3. If the observed value of $V_r \sin i$ is too small, as compared with the mean rotational velocity of the ring, then the observed ratio $K_1/V_r \sin i$ must obviously be too large. It may be noted that the majority of points in Kruszewski's Fig. 4 deviate from the theoretical relation by about 10—40 percent. For example, if we consider only the five stars with q being determined spectroscopically (exluding the strange case of U Sge), we obtain that they deviate from the theoretical relation, on the average, by 23 percent. This is comparable with the predictions made in Section 3.

The author is indebted to Professor S. L. Piotrowski for critical reading of the preliminary draft of the present paper.

Appendix 1

Integration of Eq. (5)

By introducing a new variable: $x = u\, r^{1/2}$ we obtain the following expressions for $F(u)$ for Case I and Case II, respectively,

$$F_{\rm I}(u) = \frac{\rm const}{u^3} \int\limits_{x_1}^{x_z} \frac{2\, x^2}{(1 - x^2)^{1/2}}\, dx\,, \tag{A1}$$

$$F_{\rm II}(u) = {\rm const} \left[\frac{1}{u^3} \int\limits_{x_1}^{x_z} \frac{2\, x^2}{(1 - x^2)^{1/2}}\, dx - \frac{1}{u^5} \int\limits_{x_1}^{x_z} \frac{2\, x^4}{(1 - x^2)^{1/2}}\, dx \right], \tag{A2}$$

with $x_z = u$ for $u \leqslant 1$ and $x_z = 1$ for $u > 1$, and with $x_1 = u\, r_1^{1/2}$.

The two integrals can easily be computed:

$$I_1(x) = \int \frac{2\,x^2}{(1-x^2)^{1/2}}\,dx = -\,x\,(1-x^2)^{1/2} + \arcsin x\,, \tag{A3}$$

$$I_2(x) = \int \frac{2\,x^4}{(1-x^2)^{1/2}}\,dx = -\left(\frac{x^3}{2} + \frac{3\,x}{4}\right)(1-x^2)^{1/2} + \frac{3}{4}\arcsin x\,. \tag{A4}$$

Numerical results are given in Table A1.

Table A1.

Integrals $I_1(x)$ and $I_2(x)$.

x	$I_1(x)$	$I_2(x)$	x	$I_1(x)$	$I_2(x)$
0·0	0·00000	0·000000	0·85	0·5682	0·2644
0·1	0·00067	0·000004	0·90	0·7275	0·3867
0·2	0·00540	0·000130	0·92	0·8075	0·4530
0·3	0·01851	0·001005	0·94	0·9019	0·5347
0·4	0·04507	0·004488	0·96	1·0182	0·6398
0·5	0·0906	0·01381	0·98	1·1754	0·7880
0·6	0·1635	0·03623	0·99	1·2896	0·8986
0·7	0·2755	0·08416	0·995	1·3714	0·9795
0·8	0·4473	0·18187	1·000	1·5708	1·1781

REFERENCES

Gorbatzky, V. G. . . . . . . 1964 *Astr. Zu.*, **41**, 849.

Hiltner, W. A. . . . . 1946 *Ap. J.*, **104**, 396.

Huang, S. S. . . . . 1967 *Ap. J.*, **148**, 793.

Kruszewski, A. . . . . . . 1967 *Acta Astr.*, **17**, 297.

Plavec, M. . . . . 1968 *B. A. C.*, **19**, 11.

Sahade, J. . . . . 1960 *Stellar Atmospheres*, ed. J. L. Greenstein (Chicago: University of Chicago Press), Chap. 12.

Institute of Astronomy
Polish Academy of Sciences
Warsaw, October 8, 1968

J. Smak