Time Series Analysis in the Time Domain

CONVOLUTION
CROSS-CORRELATION
AUTOCORRELATION
CROSS SPECTRUM

O-C DIAGRAM

Time Domain Analysis: Convolution

- The convolution *h(t)* of two functions *x(t)* and *y(t)* is

$$h(t) = x(t) * y(t) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} x(\tau)y(t-\tau)\, d\tau$$

Basic properties are
commutativity: $x * y = y * x$
distributivity over addition: $x * (y+z) = x * y + x * z$

- **Convolution theorem:** The Fourier transform of the convolution is the product of the individual Fourier transforms: $x(t) * y(t) \Leftrightarrow X(\nu)\, Y(\nu)$

Time Domain Analysis: Cross-correlation

- **Correlation (Cross-correlation)** is a very similar operation to convolution. The correlation *Corr(x,y)* of two functions *x(t)* and *y(t)* is

$$Corr(x,y) = (x \star y)(\tau) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} x(t+\tau)y(t)\,dt$$

$$= (x(t) * y(-t))(\tau)$$

The correlation is a function of τ, which is called *the lag*.

Unlike for convolution, $x \star y \neq y \star x$

- **The cross-correlation theorem:** The Fourier transform of the cross-correlation of two functions is the product of the individual Fourier transforms , where one of them has been complex conjugated: *x(t) ⋆ y(t) ⇔ X(ν) Y*(ν)*

Time Domain Analysis: Autocorrelation

- The correlation of a function with itself is called its **autocorrelation**.
- The related autocorrelation theorem is also known as the ***Wiener-Khinchin theorem*** and states

$$x(t) \star x(t) \Leftrightarrow X(\nu)\, X^{*}(\nu) = |X|^{2}$$

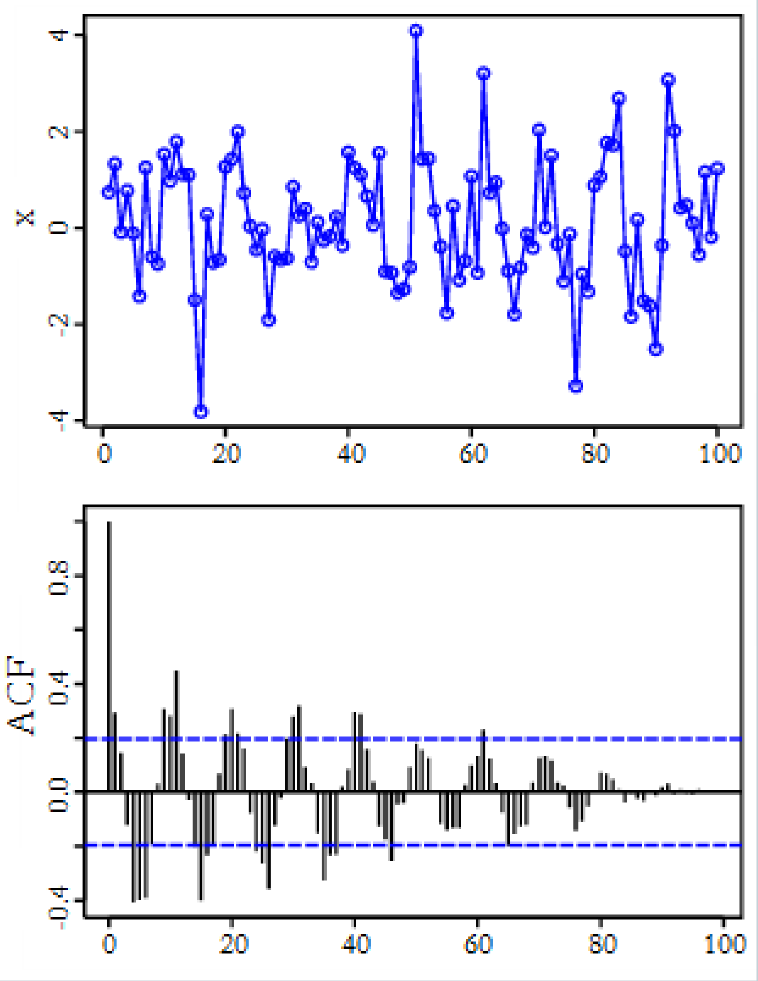
- **The Fourier transform of an autocorrelation function is the power spectrum, or equivalently, the autocorrelation is the inverse Fourier transform of the power spectrum.**

Time Series ⇔ DFT ⇔ Autocorr. ⇔ PSD

x_j (function)	$\Leftrightarrow$ DFT	X_k (transform)
$\Downarrow$		$\Downarrow$
$x_j \star x_j$ (autocorrelation)	$\Leftrightarrow$ DFT	$\lvert X_k \rvert^2$ (power spectrum)

Time Domain Analysis: Autocorrelation

- The *discrete autocorrelation* of a sampled function *x(t)* is just the discrete correlation of the function with itself.
- Obviously this is always symmetric with respect to positive and negative lags.
- 100 random numbers with a "hidden" sine function, and an autocorrelation of the series on the bottom.



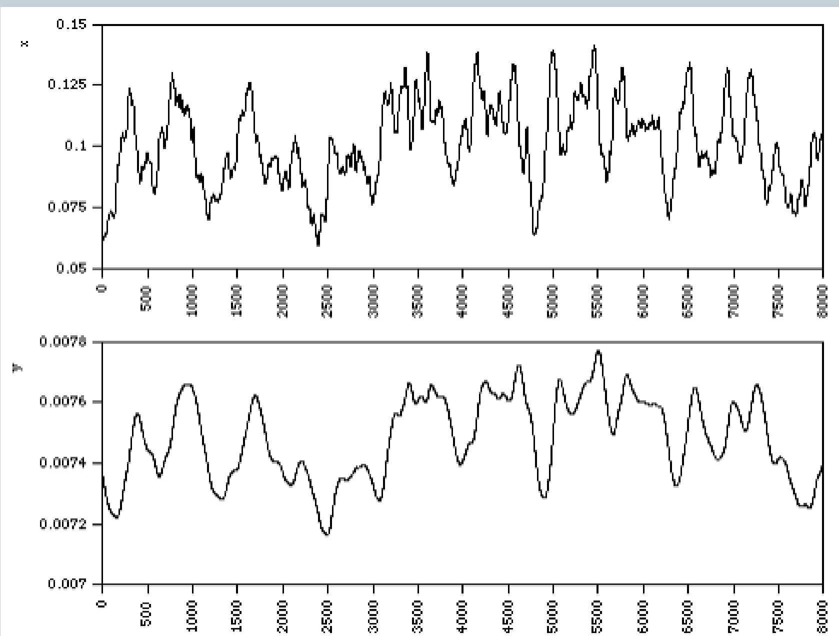
Cross Correlation

- Cross correlation is a standard method of estimating the degree to which two series are correlated.
- Consider two series *x(i)* and *y(i)* where *i*=0,1,2...*N*-1, with *mx* and *my* are the means of the corresponding series. The discrete cross correlation *r* at the lag *d* is defined as

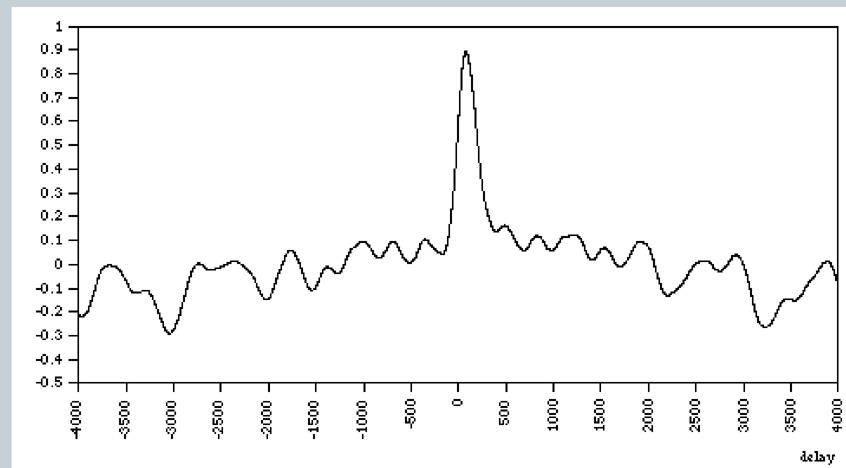
$$r(d) = \frac{\sum_i [(x(i) - mx) * (y(i-d) - my)]}{\sqrt{\sum_i (x(i) - mx)^2} \sqrt{\sum_i (y(i-d) - my)^2}}$$

Cross Correlation

Example: two time series x,y.



The cross correlation with a maximum delay of 4000.



There is a strong correlation at a delay of about 40.

Cross Spectrum

- The purpose of cross-spectrum analysis is to uncover the correlations between two series at different frequencies.
- Examples:
  - Sun spot activity may be related to weather phenomena on Earth
  - Simultaneous observations in different energy ranges
- The coherence is a measure of the correlation between two time series, at each frequency.

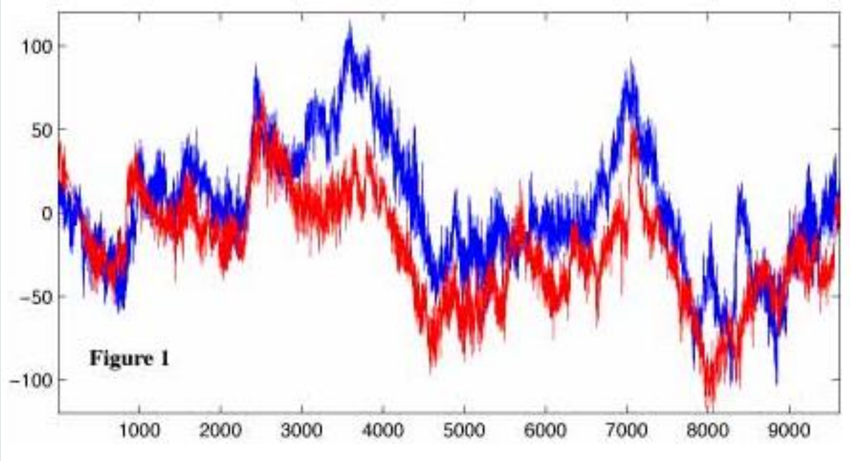
Cross Spectrum

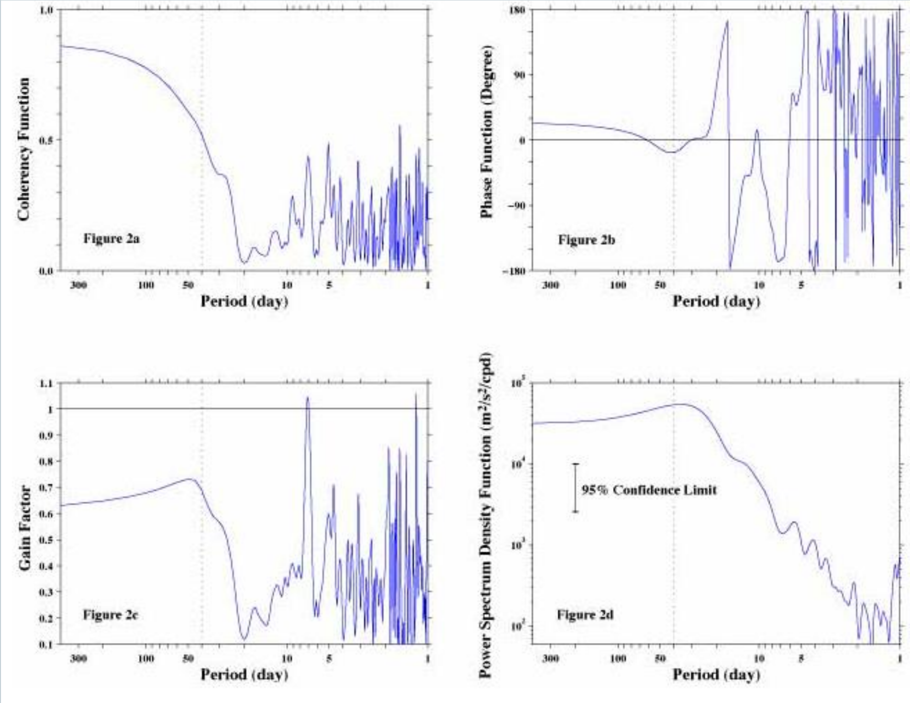
- **The coherence** is a measure of the correlation between two time series, at each frequency. In other word, it can show at which frequencies two sets of time series data are coherent and at which frequencies they are not.
- **The phase function**, which is usually computed with coherency function, shows phase difference as a function of frequency between two sets of time series data. One note about the phase difference is that it is not the same as time difference.
- **The gain factor** of the frequency response function shows the amplitude relationship between two sets of time series data as a function of frequency. The gain factor combined with coherency function and phase function would give us fairy clear picture about the relationships between two sets of time series data if they have common variations.

Cross Spectrum

2 time series shifted vertically for clarity

Coherency and Phase functions, Lag factor and PSD





O–C diagram [1]

- Basic period analysis consists of finding a reliable ephemeris of the main periodic variation and modelling of the first order effects.
- The model is usually a linear ephemeris, i.e. a prediction of eclipse times that assumes a constant period.
- ***O-C*** diagram is a powerful diagnostic tool, which compares the actual timing of an event (e.g. the mid-point of an eclipse or a pulsation cycle peak) to the moment we expect this event is occurred in a case of constant periodicity.
- ***O-C*** stands for **O**[bserved] minus **C**[alculated]

O–C diagram [2]

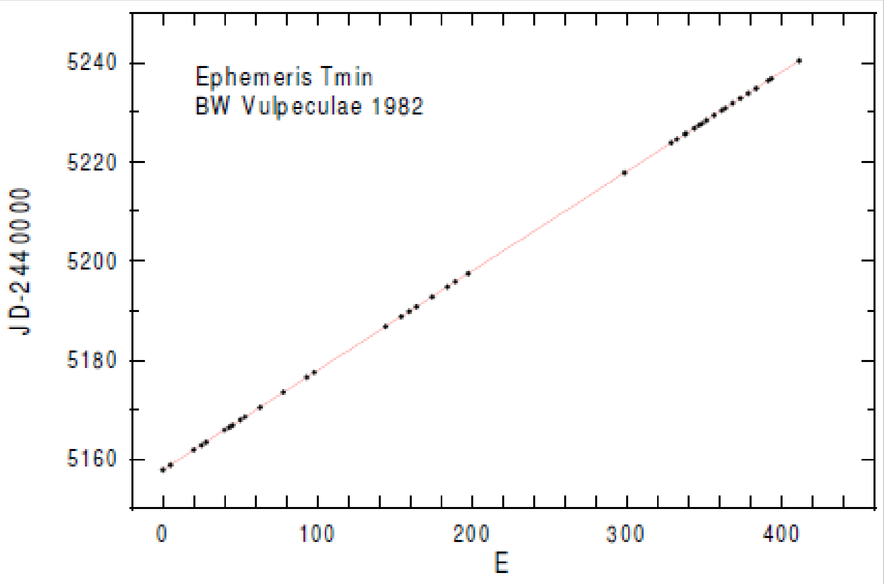
- It might appear that a period is incorrect OR variable.
- The period variations are usually delicate. By building ***O-C*** diagrams one can measure very subtle changes in the period happening with the star.
- The horizontal axis of the an **O-C** diagram most often represent time, usually expressed in days or cycles. The vertical axis is the "*O-C*" part which can expressed in days or a fraction of the period.
- Different phenomena, such as a constant but incorrect period, period increasing or decreasing at a constant rate, or sudden period changes but constant period thereafter, have distinct patterns on the ***O-C*** diagrams.

O–C diagram: constant periods [1]

If the period is constant and if its value is known, then

$$\boldsymbol{T_m = T_o + P\,E}$$

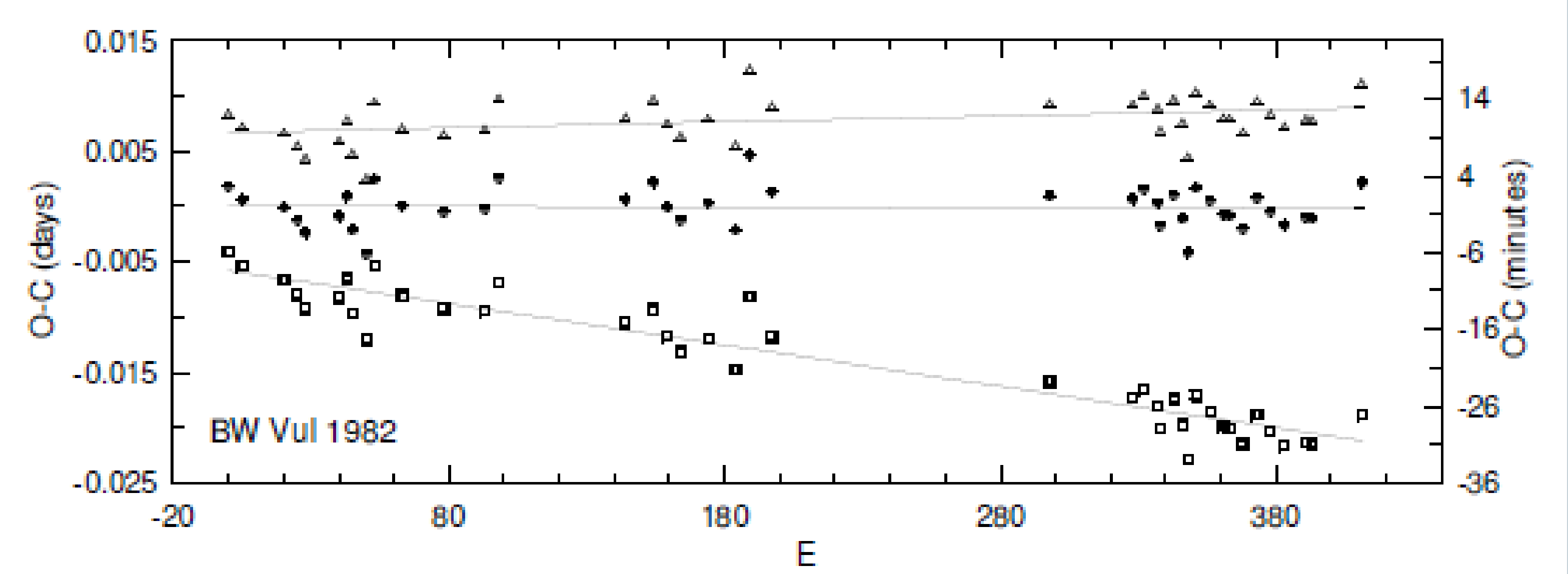
where $\boldsymbol{T_m}$ is the time of maximum or minimum light, $\boldsymbol{T_o}$ is the zero epoch and $\boldsymbol{E}$ is the number of cycles elapsed since the zero epoch.



T_o and P are obtained through a least-squares solution. The ephemeris calculation yields $T_{min} = 2445157.8072 + 0.201043E$

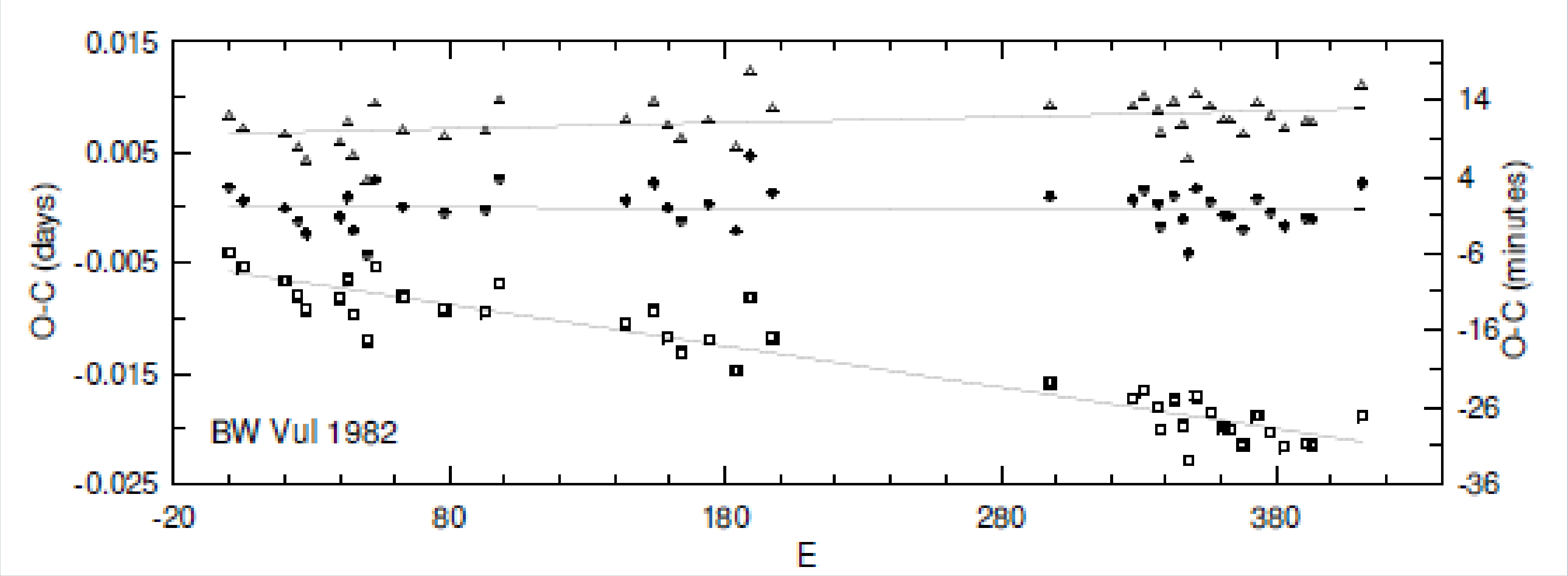
O–C diagram: constant periods [2]

- The ephemeris: $T_{\min} = 2445157.8072 + 0.201043E$



- Middle set: based on the period P=0.201043;
- Lower graph: slightly longer period P=0.201080;
- Top: using slightly shorter period *P*=0.201037.

O–C diagram: constant periods [3]



- ***O-C*** diagram showing a positive slope indicates that the real period is longer than the period used to construct the diagram;
- A negative slope points to a real period that is shorter than the assumed one.

O–C diagram: changing periods

- Changes of period could be described by any mathematical formula expressing ***P*** as a function of time.

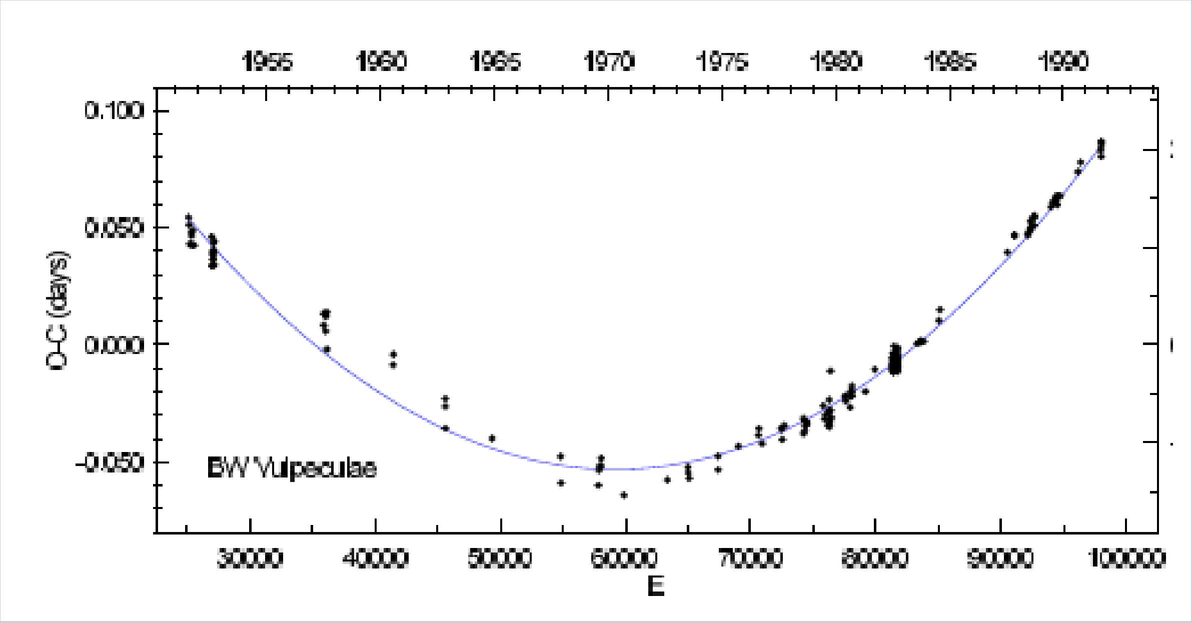
$$T_m = T_0 + \int P(t)dt \qquad \text{or} \qquad T_m = T_0 + \int P(E)dE$$

- In most cases this relation is restricted to linear variations, cyclic variations, or a combination of both.

O–C diagram: changes linear with time

- If $\bar{P}$ is the average period over the time interval, then (**show it!**)

$$O - C = \frac{1}{2}\frac{dP}{dt}\bar{P}E^2$$



CVs: Superoutbursts and superhumps

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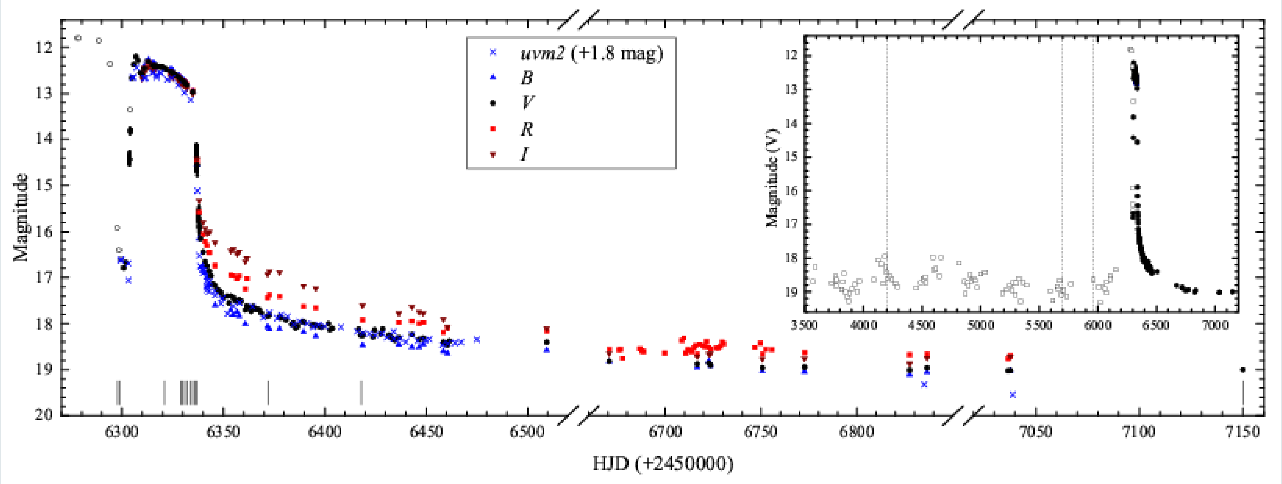
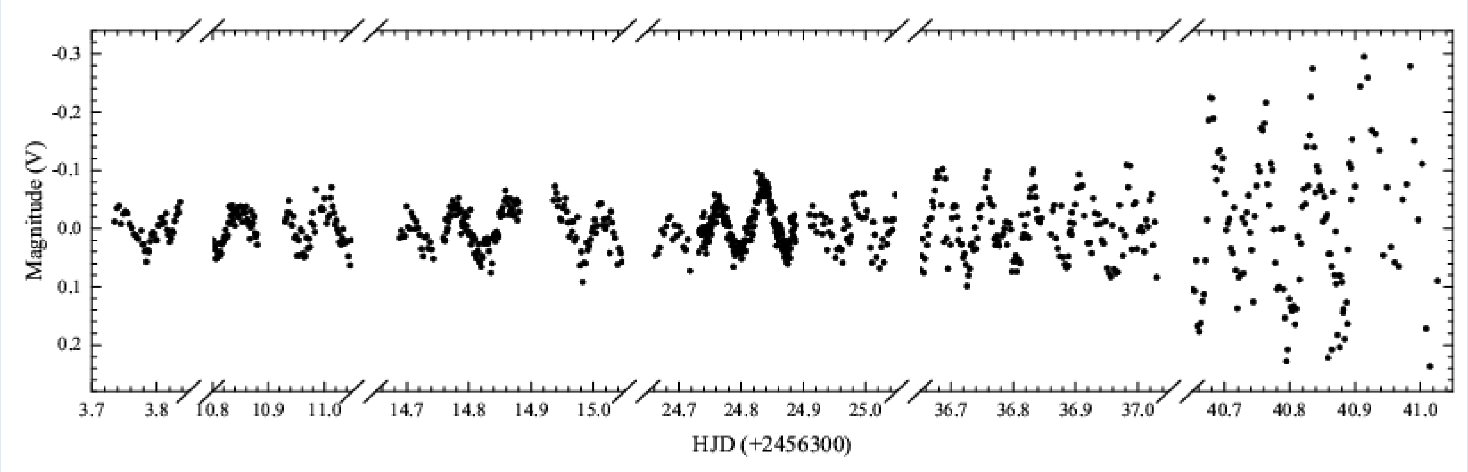
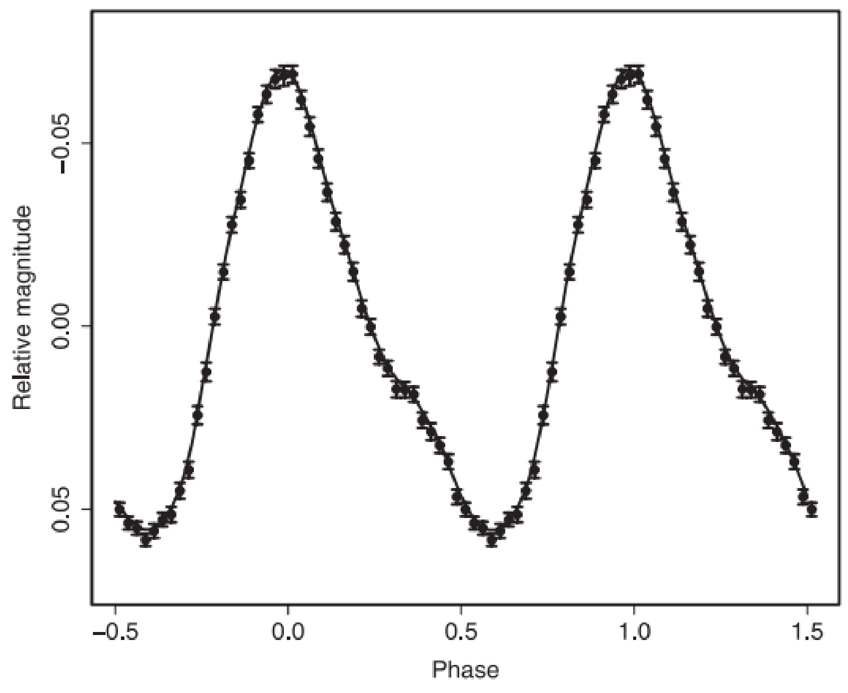




Superhumps: GW Lib

Superhump Light Curve



O-C diagram