Properties of Leahy normalized PDS

- Fractional rms (root-mean-square) amplitude of a signal in a time series x_k:

$$r \equiv \frac{\sqrt{\frac{1}{N} Var(x_k)}}{\bar{x}} = \frac{N}{N_{tot}} \sqrt{\frac{N_{tot}}{N^2}\left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2} P_{\frac{N}{2}}\right)}$$

$$= \sqrt{\frac{1}{N_{tot}}\left(\sum_{j=1}^{N/2-1} P_j + \frac{1}{2} P_{N/2}\right)}$$

r is dimensionless and often expressed in % (percentage rms variation).

Properties of Leahy normalized PDS

- ”rms normalized” power density: $q(\nu_j) \equiv TP_j/N_{ph}$
  physical unit of $q(\nu_j)$ is (rms/mean)²/Hz

- ”Source” fractional rms amplitude: If the x_k are the sum of source and background: $x_k = b_k + s_k$, then the rms amplitude as a fraction of just the s_k:
  $r_s = r\frac{B+S}{S}$,
  where B and S are sums of the b_k and s_k, so $B+S = \sum_k x_k = N_{ph}$

- ”Source rms normalized” power density (“Miyamoto” normalization):
  $q_S \equiv q\left(\frac{B+S}{S}\right)^2 = TP_j\frac{B+S}{S^2}$
  the same unit as q: (rms/mean)²/Hz

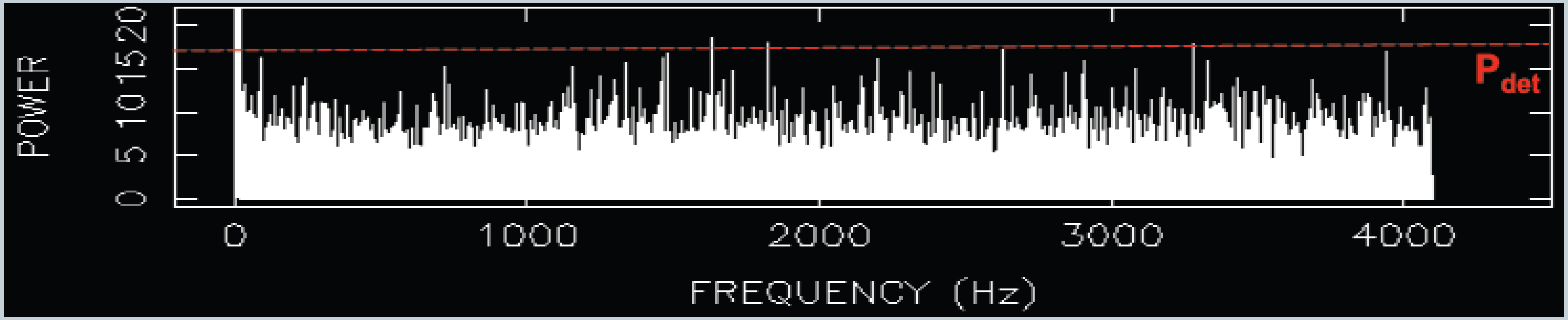
  Requires a model or a measurement of the background count rate.

Coherent Signals

- Much analysis involves "coherent" signals, i.e. periodic signals whose phase is constant over the relevant duration
  - $Q = \nu/\Delta\nu >> 1000$
- Examples:
  - Pulses from rotating pulsars;
  - Orbital modulation or eclipses;
  - Precession periods.

Statistics of Power Spectra

- How to determine the significance of peaks found in power spectra? How big must a power be to constitute a significant excess over the noise?
- Let's define ε as the probability that a noise fluctuation exceeds P_{det}. The (1- ε) confidence detection level P_{det} is a level that has a false alarm probability of ε. If there is just noise, Prob($P_j > P_{det}$) = ε. We want ε to be small, e.g., ε =1% for 99% confidence.
- If $P_j > P_{det}$ then with 99% confidence there is something else than just noise, a source signal.



Statistics of Power Spectra

- To determine P_{det}, we need to know **the noise power distribution**.

- **Warning:** Because in High-Energy Astrophysics we are counting individual photons, the relevant statistics are ***Poisson***, not ***Gaussian***.
- The Leahy normalization is chosen such that if the x_k are Poisson distributed, then the P_j exactly follow the chi-squared distribution with 2 degrees of freedom, χ^2. This is actually an exponential distribution:

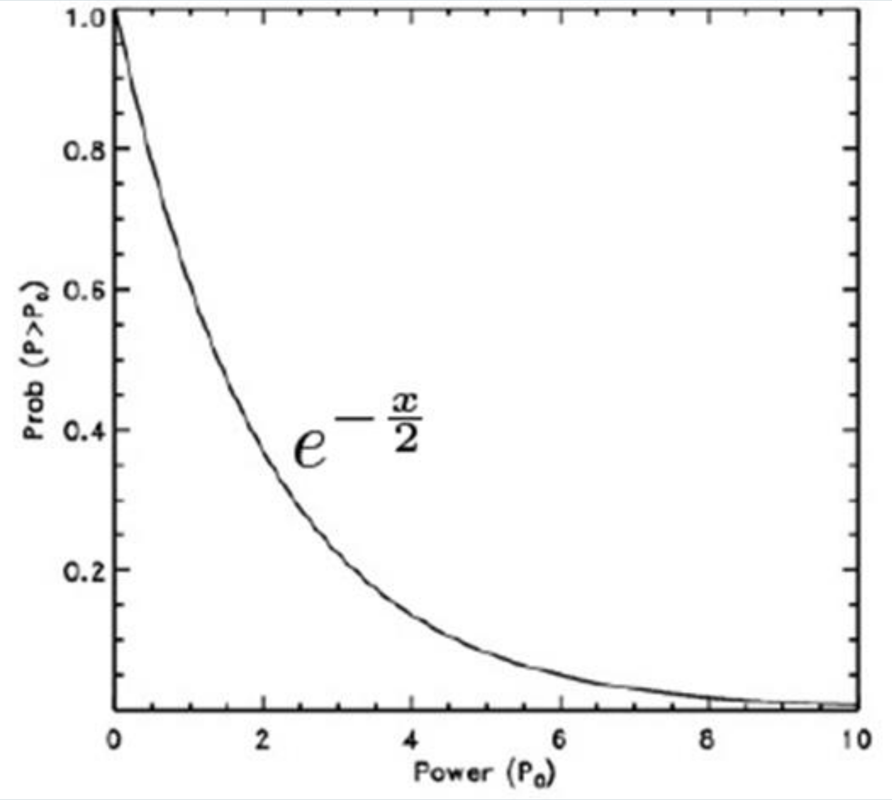
$$\varepsilon = Prob_{single}(P_j > P_{det}) = e^{-P_{det}/2} \longrightarrow P_{det} = -2\ln\varepsilon$$

- Properties of this distribution: $\langle P_{noise} \rangle = 2$; $\mathrm{Var}(P_{noise}) = 4$

Statistics of Power Spectra

- Examples:
  - ε=1% corresponds to P_{det}=9.2;
  - a power of 40 has a probability of $e^{-40/2}=2\times10^{-9}$ of being noise.
- Since a large number of independent frequencies N_{trial} are examined, the detection threshold has to be defined as that power that has an ε (small) probability to be exceeded in one frequency bin out of the N_{trial} examined.
  - One should divide ε by the number of trials.

$$\varepsilon = N_{trial}\, e^{-P_{det}/2}$$



Statistics of Power Spectra

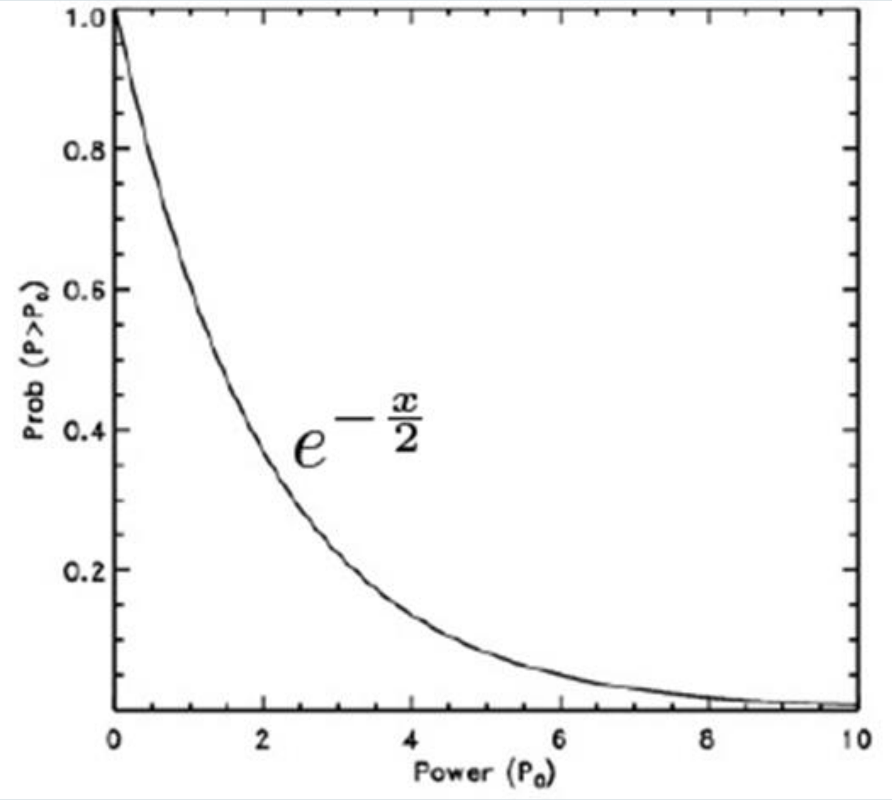
Important! The number of trial powers N_{trial} over which the search has been carried out:

N_{trial} = to the powers in the PSD if all the Fourier frequencies are considered;

N_{trial} < than the powers in the PSD if a smaller range of frequencies has been considered.

- Examples (cont.): N_{trial}=10 000
  - ε=1% corresponds to P_{det}=27.6;
  - a power of 40 has a probability of $e^{-40/2}=2\times10^{-5}$ of being noise.

Still significant!!



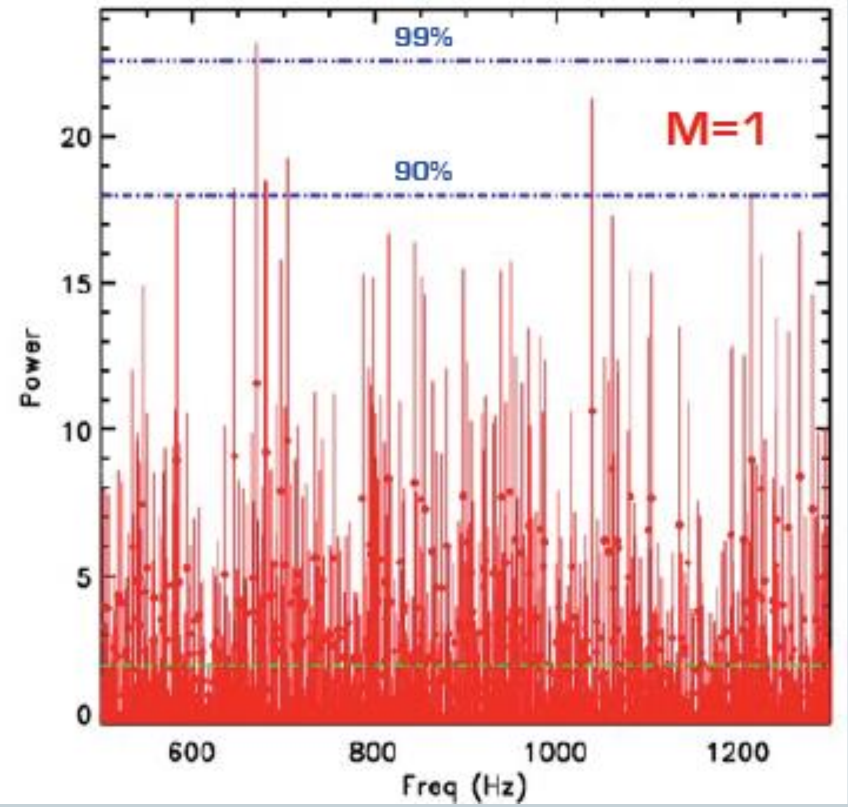
Rebinning and Averaging

- The power spectrum is very noisy. Smoothing methods:
  - Average several power spectra of subsegments of the time series;
  - Average adjacent bins in a power spectrum: rebinning;
  - Windowing is also possible.
- Averaged power distribution:
  - Individual P_j follow the chi-squared distribution with 2 dof.
  - Additive property of χ^2 distribution: sum of M powers is distributed as χ^2_{2M}
- M – the number of the time series, W – Frequency rebinning factor:

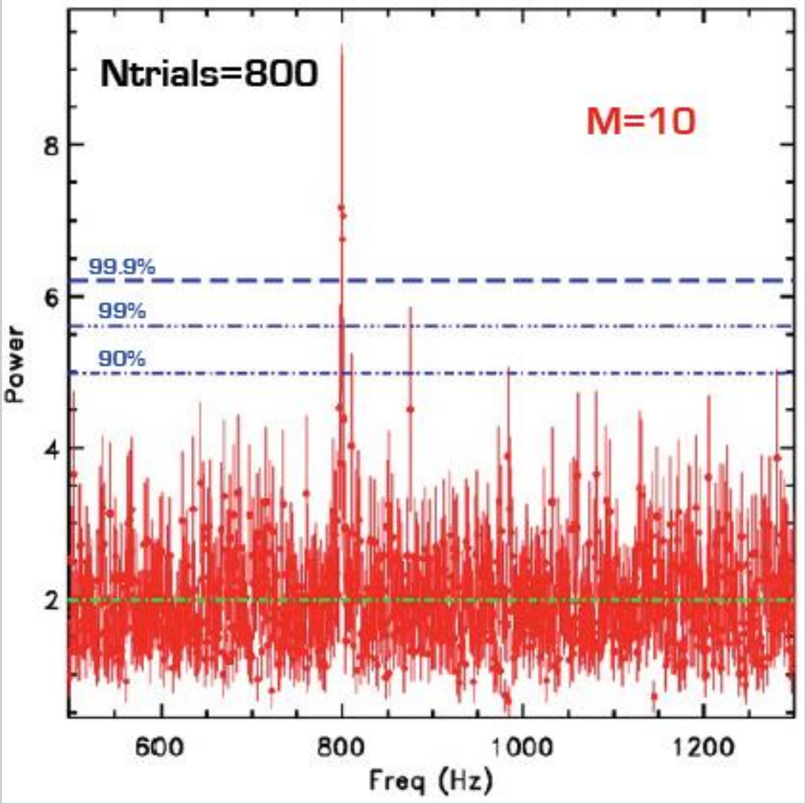
  $\langle P_{noise} \rangle = 2$; Var $(P_{noise}) = 4/MW$ (the number of trials decreases)
- **Central limit theorem:**
  for large MW the distribution of $\overline{P_{WM}}$ tends to normal (Gaussian), with mean 2 and standard deviation $2/\sqrt{MW}$

Signal Detection

M=1,
Noisy PDS



M=10,
A signal is clearly detected



A note about rebinning

- **Coherent peak:** narrow power distribution – the longer the observation span, the better. The signal power to decrease by 1/MW.
  Is it worth to average or rebin? No.
  - The signal power decreases faster than the threshold power when averaging/rebinning;
  - If the frequency varies (orbital motion) is even worse as you average signal with noise.

- **Broad peak:** broad power distribution - length of observation not crucial - rebinning helps.

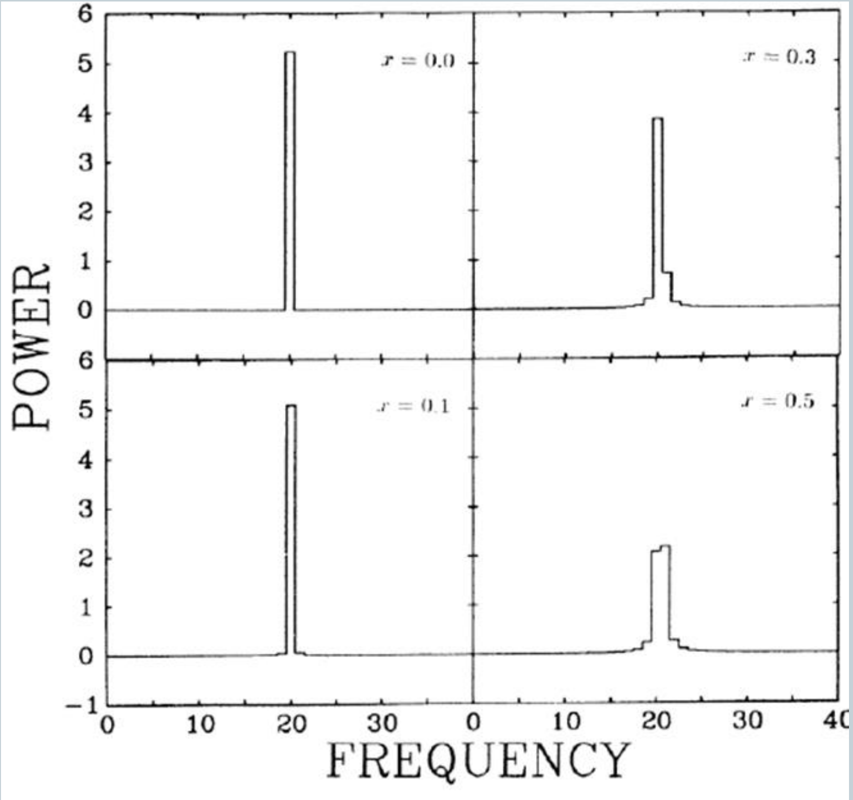
Signal detection optimization

- The power spectrum of a sinusoidal signal $x_k = A \cos(2\pi \nu_{sine} t_k + \varphi)$:

$$|a_j|^2 \approx \frac{1}{4} A^2 N^2 \left(\frac{\sin \pi x}{\pi x} \right)^2$$

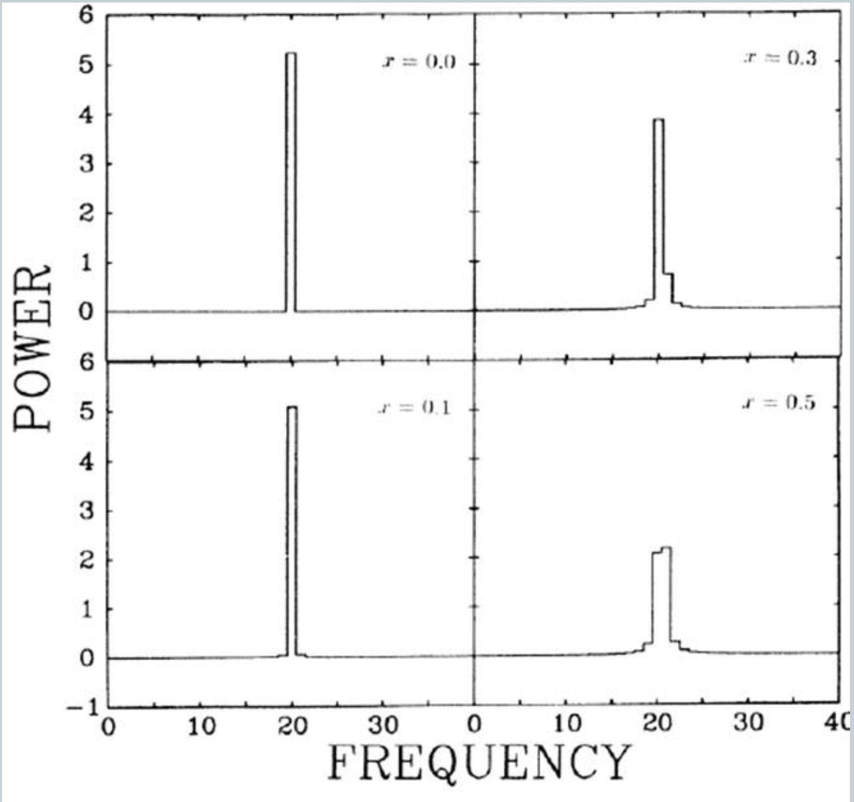
where $x = (\nu_{sine} - \nu_j)T$

- The highest power in the signal power spectrum will be obtained at the Fourier frequency ν_j closest to ν_{sine}. Normalized to a power of 1 for $\nu_{sine} = \nu_j$ (x = 0), this power varies between 0.405 and 1, with an average value of 0.773



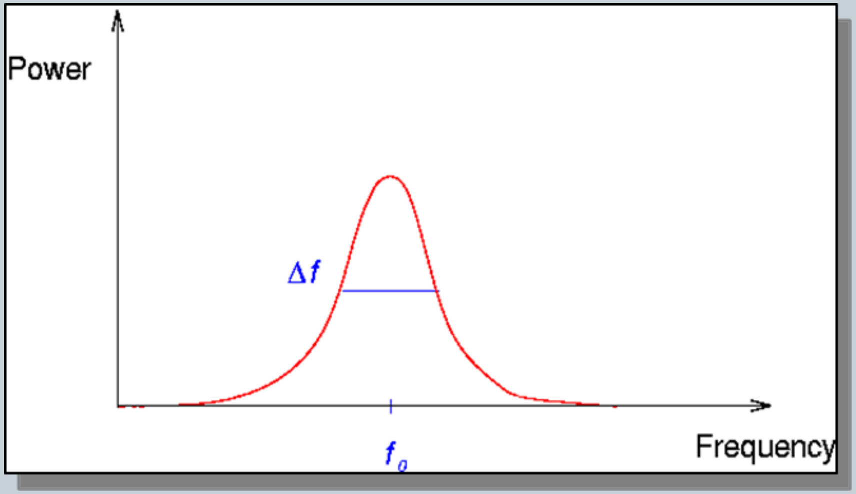
Signal detection optimization

- **Implications:** When searching for strictly coherent signals it is important to rely upon the original/maximum Fourier resolution (1/T).



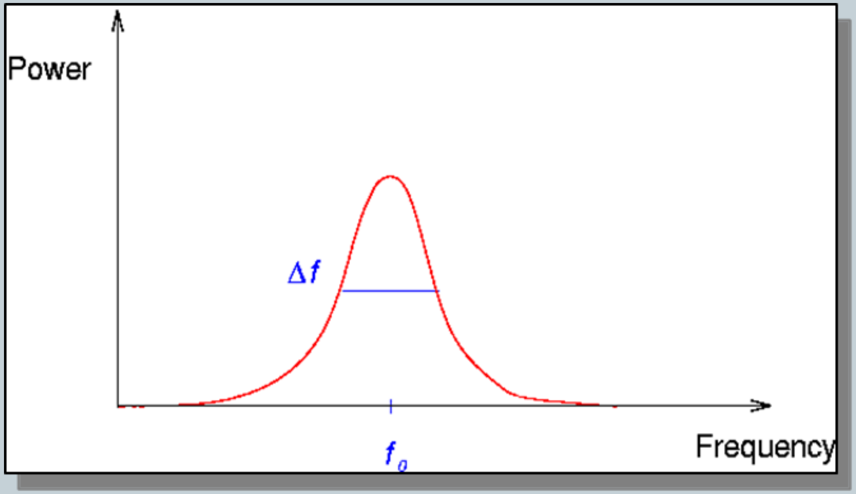
Signal detection optimization

- Similar reasoning shows that the signal power for a feature with finite width $\Delta\nu$ drops proportionally to $1/MW$ when degrading the Fourier resolution. However, as long as feature width exceeds the frequency resolution, $\Delta\nu > MW/T$, the signal power in each Fourier frequency within the feature remains approx. constant. When $\Delta\nu < MW/T$ the signal power begins to drop.



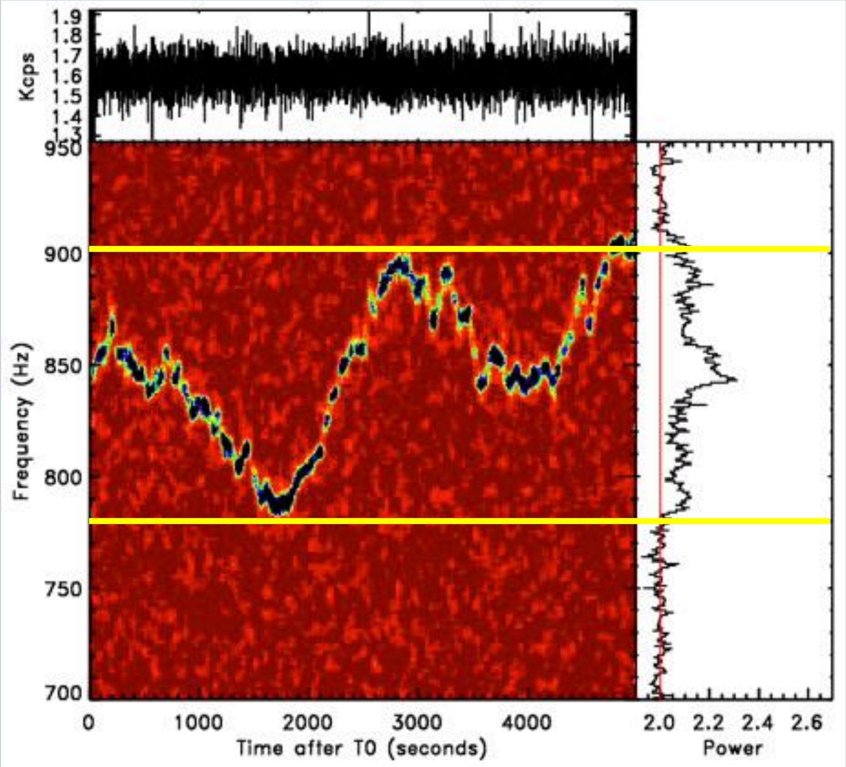
Signal detection optimization

- **Implications:** The search for QPOs is a three step interactive process.
- Firstly, estimate (roughly) the feature width.
- Secondly, run again a PSD by setting the optimal value of MW equal to ~Δν T. Two or three iterations are likely needed.
- Finally, use χ^2 hypothesis testing to derive significance of the feature, its centroid and r.m.s.



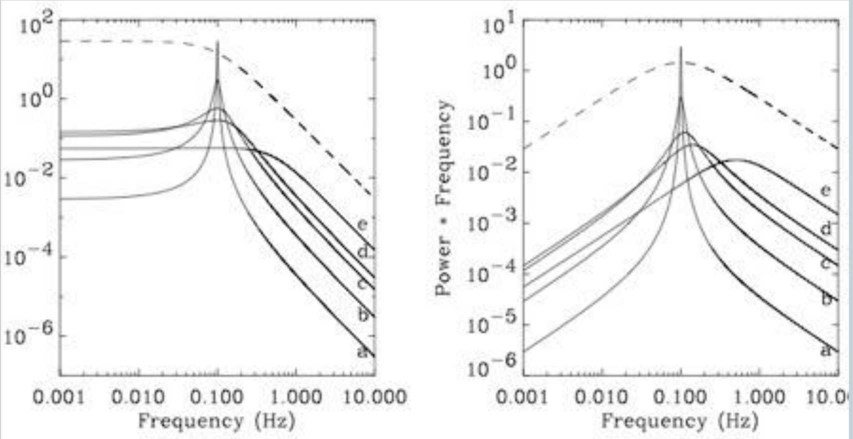
Measuring narrow features in PSD

- **The QPO frequency varies with time (on short timescales).**
- To minimize the pollution of the frequency drift to the measured QPO parameters, PDS must be integrated on the shortest possible timescales
- **Useful tip:** Produce a dynamical PSD
  - Smooth it in time and frequency
  - Restrict the frequency range to where you see the QPO



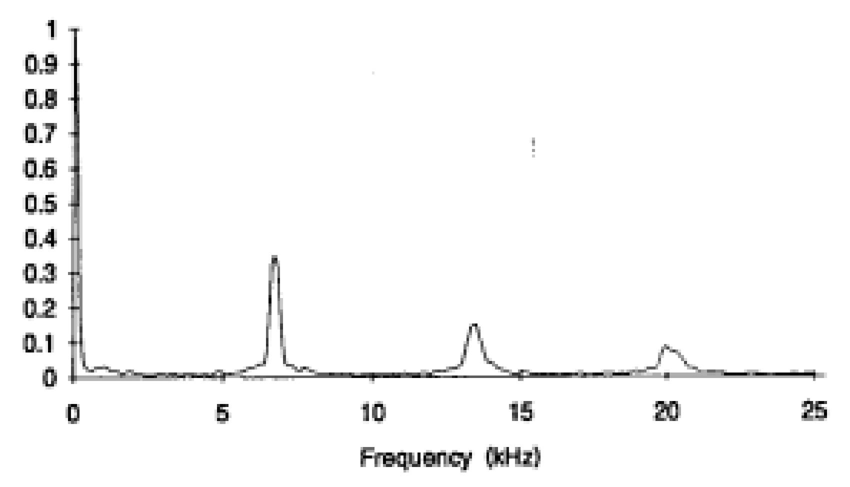
Power spectrum plots

- Multiply the power spectrum by the frequency
- Obtain a νP_ν representation
- Useful to see where the power per decade peaks
- Characteristic frequencies are peaks in νP_ν



Periodic Non-sinusoidal Signals

- Power for Periodic Nonsinusoidal Signals is spread over harmonics of the modulation frequency: Confidence lower.



Summary: Detecting something in a power spectrum

The process of detecting something in a power spectrum against the background of noise has several steps:

- knowledge of the probability distribution of the noise powers;
- knowledge of the interaction between the noise and the signal powers (determination of the signal upper limit);
- The detection level: Number of trials (frequencies and/or sample);
- Specific issues related to the intrinsic source variability (non Poissonian noise);
- Specific issues related to a given instrument/satellite (spurious signals – spacecraft orbit, wobble motion, large data gaps, etc.).