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FOURIER ANALYSIS WITH UNEQUALLY-SPACED DATA*

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Abstract. The general problems of Fourier and spectral analysis are discussed. A discrete Fourier transform $F_N(\nu)$ of a function $f(t)$ is presented which (i) is defined for arbitrary data spacing; (ii) is equal to the convolution of the true Fourier transform of $f(t)$ with a spectral window. It is shown that the 'pathology' of the data spacing, including aliasing and related effects, is all contained in the spectral window, and the properties of the spectral windows are examined for various kinds of data spacing. The results are applicable to power spectrum analysis of stochastic functions as well as to ordinary Fourier analysis of periodic or quasiperiodic functions.

1. Introduction

This section serves as a general, non-technical, introduction to the technical part of the paper which follows, and may be omitted by readers familiar with Fourier analysis techniques. It contains an account of why we are interested in Fourier analysis and what the problems are, and it summarizes the results of the following sections.

The kind of data which we consider analysing by Fourier techniques nearly always consists of the observed variation of something with time – perhaps the light variations of a variable star or quasi-stellar object, or the output of a microphotometer tracing across a spectral line, or across a photograph of solar granulation. We shall therefore speak in terms of time as our independent variable, although our considerations apply equally well to situations with spatial or other independent variables.

Practical sets of data are necessarily limited in length, since we cannot observe for an infinite time, and are also usually obtained only at a set of N discrete times t_k within a total data length T. Both the data length and the data spacing have important limiting effects on the accuracy with which we can perform Fourier analysis. We are especially concerned in this paper with formulating the analysis of these effects in a way which does not depend upon the specific data spacing, so that it is as valid for unequally as for equally spaced data.

There are at least two distinct philosophies regarding the application of Fourier analysis techniques to real data. On the one hand, one may adopt a totally phenomenological point of view in which one simply *defines* the complex Fourier transform $F(\nu)$ of a function $f(t)$ as

$$F(\nu) = \int_{-\infty}^{+\infty} f(t) e^{i2\pi\nu t}\, \mathrm{d}t, \tag{1}$$

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without necessarily saying why one might want to do so. The question is then how to estimate $F(\nu)$, which depends on an integral of $f(t)$ over $(-\infty, +\infty)$ from observations in a limited section $(-T/2, +T/2)$ at discrete times t_k. One thus thinks of the Fourier analysis process as simply a transformation of the data to a new representation – in frequency instead of time. Such an approach is perfectly legitimate and equivalent things are done frequently in other areas of data analysis. Furthermore, it can be a very powerful investigative tool, particularly if the Fourier transform turns out to be concentrated in certain regions of frequency, suggesting certain characteristic time scales present in the data. As a practical detail, since $F(\nu)$ is complex, it is usual to examine its amplitude $|F(\nu)|$, or more commonly $|F(\nu)|^2$, in such an investigative program.

A second philosophy is an almost inevitable outcome of the first, although it may have an independent origin, the approach of interpreting the data in terms of a *model*. Here we mean a *mathematical model* for the variation which is a representation of the *physical model* we have in mind. One of the most important distinctions between mathematical model types is that between deterministic and non-deterministic models. In a deterministic model we assume a basic predictability which is connected with the deterministic character of classical, non-quantum physical processes. Some of the possibilities for deterministic models are: (i) a *non-periodic deterministic variation*, such as a nova or supernova light curve, or a line profile in a spectral scan, (ii) a *periodic variation*, such as a simple eclipsing or spectroscopic binary, or an RR Lyrae light curve, (iii) a *multiply periodic variation*, such as a star pulsating in an overtone mode as well as a fundamental, or a spectroscopic triple system, (iv) a *modulated periodic variation* where either the amplitude, frequency, or phase may vary with time – for example a pulsating system in a binary orbit.

Non-deterministic models are *noise* sources of various kinds, and are usually called *stochastic processes* by statisticians. The classic example of a non-deterministic process is that of radioactive decay, where the lack of determinism is a product of the fundamental indeterminism of the laws of quantum mechanics. Astrophysically, apparent non-determinism usually arises through complexity – for example in the brightness fluctuations of solar granulation originating in the complex turbulence of the solar convection zone. In addition, we may have to deal with fluctuations due to photon statistics, variable sky transparency and, of course, simple observational error. The essence of a non-deterministic process is that it can only be handled statistically. Definite predictions cannot be made about a non-deterministic variation $f(t)$ at time t. Instead we can make probability statements about $f(t)$. Thus we can perhaps give the probability distribution $P(f(t))$ of the possible values of $f(t)$, or we may give the moments of the distribution, such as the mean, variance, etc. In many practical cases, the probability distribution of $f(t)$ at time t will depend upon the value of f at an earlier time $t-\tau$, that is, there is a *correlation* between the value of $f(t)$ and $f(t-\tau)$. Since definite predictions cannot be made about $f(t)$, it follows that definite predictions cannot be made about its Fourier transform $F(\nu)$ either, we can only discuss its probability

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distribution, or its mean, variance, etc. Actually, it turns out that it is rather difficult to define a full Fourier transform in the sense of Equation (1) for a non-deterministic process. Instead, one has to resort to a different approach which we shall discuss below.

It is important to realise that even a *pure noise* model, i.e. a pure stochastic process, does not necessarily contain variation on all possible time scales. If it did, and if there were, on average, the same amplitude of variation at all time scales, we would have *white noise*. However, white noise represents both a mathematical and a physical singularity since to produce truly white noise a system would have to be able to change infinitely fast in order to have power at the shortest possible time scales. Also, if it were to vary on extremely long time scales, it would have to have something like an infinite 'memory'. All practical noise sources are therefore limited in characteristic time scales or in characteristic frequencies. This kind of thing is called *band limited noise* by engineers. It is not generally realized by non-statistically oriented astronomers that almost all the standard treatments of spectral analysis such as, for example, the classic text of Blackman and Tukey (1958), deal exclusively with stochastic models and not with deterministic models. The more recently developed methods of maximum entropy spectral analysis are also framed in terms of stochastic models.

The important point is that deterministic and non-deterministic models behave differently under a Fourier analysis. In fact, even within deterministic models, it is necessary to distinguish between periodic (or multiply periodic) models and non-periodic models. This is because the defining integral of the Fourier transform Equation (1), exists only if the function $f(t)$ is absolutely integrable, i.e. if $\int |f(t)|\, \mathrm{d}t$ exists. Thus periodic functions are excluded from this definition because they are not absolutely integrable, and something special is needed to handle them. It is possible to extend the domain of Fourier transforms to include functions such as the Dirac delta function and its derivatives (see Lighthill, 1958), so that the Fourier transform of the complex periodic function $e^{-i2\pi\nu_0 t}$ is $\delta(\nu-\nu_0)$. It is also possible to consider only finite or discrete versions of the Fourier transform where the integration (or summation) limits are finite rather than infinite.

In the following section of this paper, the *finite Fourier transform* $F_T(\nu)$ is defined as

$$F_T(\nu) = \int_{-T/2}^{+T/2} f(t) e^{i2\pi\nu t}\, \mathrm{d}t. \tag{2}$$

In addition, a *discrete Fourier transform* $F_N(\nu)$ is defined as

$$F_N(\nu) = \sum_{k=1}^{N} f(t_k) e^{i2\pi\nu t_k}. \tag{3}$$

The analogy of this expression with $F_T(\nu)$ should be clear, although it must be realized that $F_N(\nu)$ is dimensionally different from $F_T(\nu)$, because it does not contain a multiplying factor Δt. This discrete transform is the one we are primarily interested in because we want to handle discrete data. Note that there is no restriction on the data

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spacing in this definition. We may now examine the result of applying these limited transforms to two of the various model types mentioned above – periodic and stochastic.

1.1. Periodic functions

The basic result of Fourier analysis as historically developed and as presented in the later part of this paper is that, if $f(t)$ is a pure cosine wave of frequency ν_0, then the Fourier transforms $F(\nu)$ and $F_T(\nu)$ have amplitudes that are significantly different from zero only in the immediate vicinity of $\nu=\nu_0$ and $\nu=-\nu_0$. This is also true, with a qualification due to aliasing, which is discussed below, for the discrete transform $F_N(\nu)$. Thus a Fourier analysis is able to detect the presence of a frequency in the data and, with some care in the normalization, to determine its amplitude. In the case of a multiply periodic function with frequencies $\nu_1, \nu_2, \ldots$, etc., the transform will be large in the vicinity of $\nu= \pm\nu_1, \pm\nu_2, \ldots$ etc., and ideally the analysis can detect the presence of each of these frequencies independently, and determine their amplitudes. In practice, this ideal cannot quite be realized because of the finite data length and the discrete data sampling. The full Fourier transform would consist of a series of delta functions at the frequencies $\pm\nu_1, \pm\nu_2, \ldots$, but the observed $F_T(\nu)$ or $F_N(\nu)$ will differ from $F(\nu)$. This difference can be described in the frequency domain as an interference between frequencies. It is normally possible to recognize two types of such interference; (i) interference from nearby frequencies, which is usually described by a *spectral window* (see below), and is primarily a product of the finite *length* of the data, and (ii) interference from distance frequencies, which is usually called *aliasing*, and is a product of the data *spacing*. For continuously recorded data, aliasing does not exist; while for equally spaced data, it exists in its most extreme form. A physical analogy of aliasing in this case is in the side-lobes of an interferometric array, or the various orders of a diffraction grating. For general arbitrary data spacing, it is not possible to make such a clean separation of the two effects which, indeed, are really just different manifestations of the same effect, which we shall lump together in the term spectral window, now to be explained.

A fundamental result of this paper is that this interference effect can be expressed by the statement that, for determinate processes (not necessarily periodic), the observed Fourier transform, $F_N(\nu)$, is the *convolution* of the true Fourier transform $F(\nu)$ with a *spectral window*, $\delta_N(\nu)$ – i.e.,

$$F_N(\nu) = F(\nu)*\delta_N(\nu) \equiv \int_{-\infty}^{+\infty} F(\nu-\nu')\delta_N(\nu')\,\mathrm{d}\nu', \tag{4}$$

where the spectral window $\delta_N(\nu)$ is obtainable as a function only of ν and the times of observation t_k from

$$\delta_N(\nu) = \sum_{k=1}^{N} e^{i2\pi\nu t_k}. \tag{5}$$

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The symbol $\delta_N(\nu)$ is used for this quantity because in the limit of a completely filled time interval $(-T/2, +T/2)$, i.e. a continuous record, the corresponding quantity $\delta_T(\nu)$ in fact tends to the Dirac delta-function as $T \to \infty$. Thus $\delta_T(\nu)$ is, so to speak, a 'finite' version of $\delta(\nu)$. The limit of $\delta_N(\nu)$ as $N \to \infty$ is similar to the delta function in its *locating* property, but not in its *normalization*. In practical computations it is most convenient to work instead with the quantity $N^{-1}F_N(\nu)$, and with a corresponding spectral window $\gamma_N(\nu) = N^{-1}\delta_N(\nu)$ because this yields a spectral window normalized to $\gamma_N(0) = 1$. Thus

$$N^{-1}F_N(\nu) = \gamma_N(\nu)^* F(\nu). \tag{6}$$

This choice of normalization is largely a matter of taste and computational convenience. Therefore, if $F(\nu)$ is a delta function at frequency ν_0, say, the quantity $N^{-1}F_N(\nu)$ will reproduce the shape of the normalized spectral window, $\gamma_N(\nu)$, centered on ν_0 [i.e. $\gamma_N(\nu - \nu_0)$]. If $F(\nu)$ is a series of delta functions, corresponding to a multiply periodic function, then $N^{-1}F_N(\nu)$ will consist of a series of γ_N centered on the various frequencies present. Since $\gamma_N(\nu)$ may well be significantly different from zero at frequencies other than $\nu = 0$, there may well be interference between the different frequencies present in the data. The important point, however, is this: *The pathology of the data distribution is all contained in the spectral window* $\gamma_N(\nu)$, which can be calculated from the data spacing alone, and does not depend directly on the data themselves. The spectral window is therefore all-important. Typically, a plot of the amplitude of $\gamma_N(\nu)$ vs frequency shows (i) a reasonably well defined central peak at $\nu = 0$, with a width of the order of T^{-1} in frequency and (ii) some subsidiary peaks corresponding to peculiarities in the data spacing. For example, Figure 1 shows the spectral window corresponding to equal data spacing (top graph) illustrating the large subsidiary peaks found in this special case, and the lower graphs show the effect of changing the data spacing from equally spaced. As a practical example, Figure 2 shows the spectral window obtained for a long series of observations of the QSO 3C345. Any such set of observations necessarily has a one year periodicity in its data spacing because a given object can only be observed when the sun is not too close in the sky. Hence a moderate peak is present in the spectral window at an annual frequency, $\nu_A = 1$ yr^{-1}. This warns us that if a strong frequency were present in the data at ν_0, say, we would see interference between this frequency and the 1 yr^{-1} frequency in the data spacing.

Thus there should be subsidiary peaks ('aliases') in the transform at frequencies $\nu = \nu_A + \nu_0$ and $\nu = \nu_A - \nu_0$. These are in fact observed. Furthermore, by comparing the shape of any peak in $F_N(\nu)$ with the shape of the spectral window $\gamma_N(\nu)$, it is possible to judge whether a peak corresponds to a well-defined delta function in $F(\nu)$.

1.2. Stochastic functions

A similar situation holds for stochastic functions, but with differences in detail. Here, since we have a non-deterministic function, the characteristic time scales of the physical system manifest themselves in the way in which the probability distribution of $f(t)$ at

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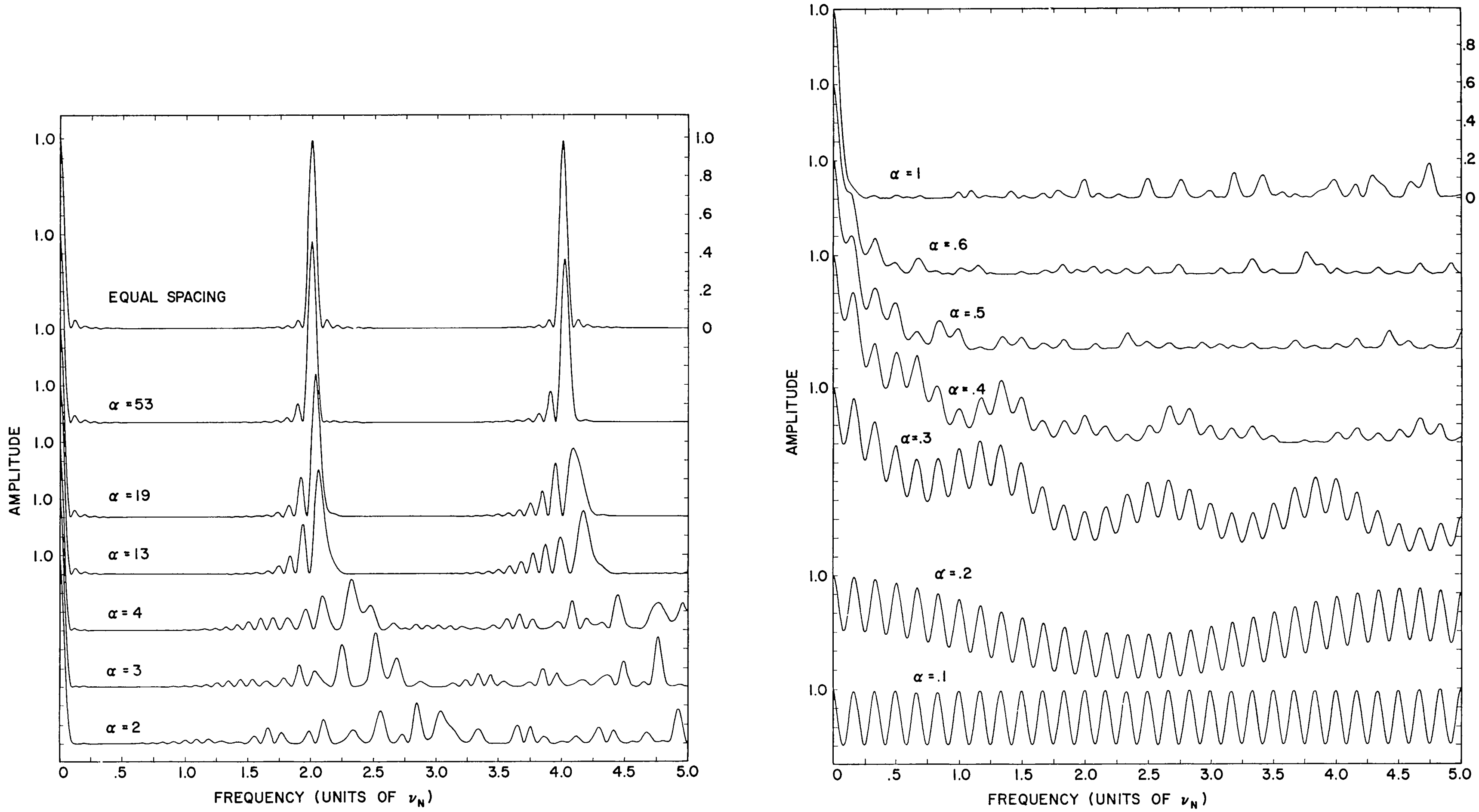

Fig. 1. The variation of spectral window shape for 25 data points distributed over a fixed time interval T, with data spacing governed by the parameter α in

$$t \sim k^{-1/\alpha}, \quad (k = 1 \ldots 12), \qquad t \sim (25 - k)^{-1/\alpha}, \quad (k = 13 \ldots 24)$$

Decreasing values of α represent increasing concentration of data points to the center of the time interval T, ranging from equal data spacing ($\alpha = \infty$) to a distribution ($\alpha = 0.1$) with one point at each end and the rest almost together at the center. The frequency unit is the Nyquist frequency for the equal spacing case, i.e. $12.5/T$. The amplitude scale is the same for all diagrams and is given at the right-hand side. Since each spectral window has the value unity at zero frequency, the origin of each curve is marked with a 1 at the left-hand side of the diagram.

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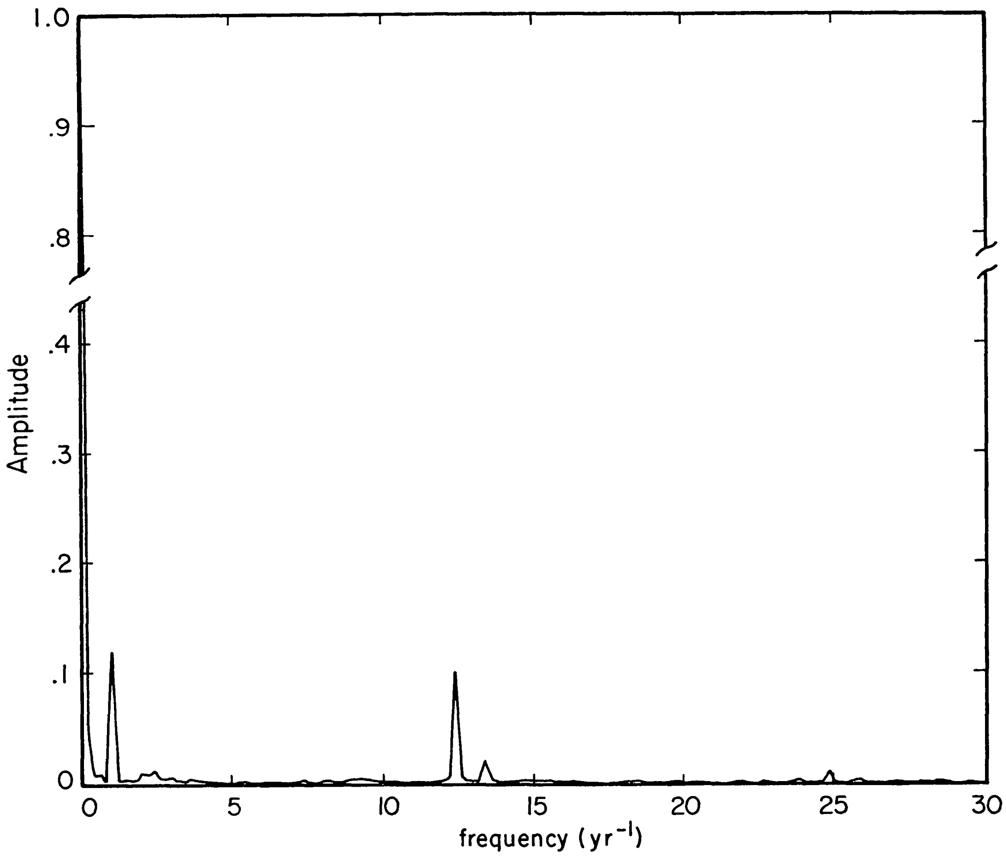

Fig. 2. Power spectral window obtained for photometric observations of 3C345 by Kinman. Note the clean central peak, the appearance of three smaller peaks at periods of one year, one synodic month, and one calendar month, and the very low 'noise' level at other frequencies.

time t depends on the value of f at time $t-\tau$. This dependence may often come about through the action of a filter or other form of 'memory' which causes the system to be influenced at time t by its state at time $t-\tau$. In practice, one almost always restricts one's attention to situations in which this dependence is characterized by only one parameter, the *correlation coefficient* between $f(t)$ and $f(t-\tau)$. Furthermore, since one can only make progress if the data sample can be taken to be a 'typical' data sample, such as might have been obtained wherever the actual time interval $(-T/2, +T/2)$ is located in the interval $(-\infty, +\infty)$, one must assume that this correlation coefficient is a function only of τ and not of t. This property is called *stationarity*, and we have a *stationary stochastic process*. The correlation coefficient expressed as a function of τ is called the *autocorrelation function*, $r(\tau)$. This, then, is the important quantity which characterizes the time-dependence properties of the noise source. A Fourier transform of $r(\tau)$ will then reveal characteristic time scales in much the same way as it did for a deterministic process, although the interpretation of these time scales in terms of a physical model will, of course, be different. In a stochastic model, we will probably look for filters or other forms of memory having the characteristic time scales found. The

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Fourier transform of the autocorrelation function is known as the *power spectrum* $P(\nu)$. Since $r(\tau)$ must necessarily be symmetric – i.e. $r(-\tau)=r(\tau)$ – the power spectrum is purely real: namely,

$$P(\nu) = \int_{-\infty}^{+\infty} r(\tau)e^{i2\pi\nu t}\,\mathrm{d}t = 2\int_{0}^{\infty} r(\tau)\cos 2\pi\nu t\,\mathrm{d}t. \tag{7}$$

The power spectrum so defined is normalized so that its integral is unity. Power spectra generally show broad features, rather than the sharply defined peaks in the transforms of periodic functions.

Suppose we calculate the transforms $F_T(\nu)$ or $F_N(\nu)$ for a stochastic function. Since $f(t)$ is not predictable, neither are $F_T(\nu)$ or $F_N(\nu)$. We can only make probability statements about them, which usually take the form of the parameters of their probability distributions. In the following section of this paper, we show that there is a convolution relationship between the *expectation* (statistical mean value) of the *power* $|F_N(\nu)|^2$, and the power spectrum $P(\nu)$ which is essentially the same as that for periodic functions: namely,

$$\langle|F_N(\nu)|^2\rangle = \mathrm{var}\,(f)N^2P(\nu)*|\gamma_N(\nu)|^2, \tag{8}$$

where var(f) is just the variance (mean squared value) of $f(t)$ and $|\gamma_N(\nu)|^2$ is the *power spectral window*. This is just the square of the same spectral window as previously, so *the same remarks about interference, aliasing, and the fact that the pathology of the data spacing is all contained in the spectral window*, apply here also. Again, it is most convenient computationally to work with $N^{-1}F_N(\nu)$, so we have

$$\langle|N^{-1}F_N(\nu)|^2\rangle = \mathrm{var}(f)P(\nu)*|\gamma_N(\nu)|^2. \tag{9}$$

The power spectral window is thereby normalized to unity at zero frequency.

1.3. Combined Functions

It is shown in the next part of this paper that the dependence of the amplitude of a finite transform on the data length, T, is different for periodic, non-periodic, and stochastic functions. In fact,

$$|F_T(\nu)| \propto \begin{cases} T^0 & \text{non-periodic} \\ T^1 & \text{periodic} \\ T^{1/2}\ (\text{rms}) & \text{stochastic.} \end{cases} \tag{10}$$

If we have an $f(t)$ consisting of a combination of these processes, the contribution of each to the transform will be a function of the data length. In other words, since the variation is T for periodic functions and $T^{1/2}$(rms) for noise functions, it is possible to detect a periodic signal in the presence of noise by observing for a long enough time. It may, therefore, be possible in principle to distinguish between different model types

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if several series of observations of different lengths are available, on the basis of the way in which the amplitudes of the transform change as a function of data length.

It is this different variation with T for non-periodic, periodic and stochastic functions that makes difficult the definition of the ordinary Fourier transform for stochastic functions mentioned earlier. This is why we normally use $r(\tau)$, which has a well-defined Fourier transform, to define the power spectrum. Nevertheless, provided we allow ourselves to renormalize by dividing by the appropriate power of T, it is sometimes useful to think of a stochastic function as the limit of a multiply periodic function when variations of all possible frequencies are present with different amplitudes $A(\nu)$, and phases distributed at random. The distribution of these amplitudes with frequency, or rather $|A(\nu)|^2$, is then proportional to the power spectrum $P(\nu)$. However, because of the mathematical difficulties, it is preferable to define $P(\nu)$ in a purely statistical context as the Fourier transform of the autocorrelation function.

The following Sections 2 through 5 give the details leading to Equations (6) and (9) and discuss the properties and normalizations of the spectral windows. The Appendix contains explicit instructions for calculating the transform $F_N(\nu)$ and the spectral windows $\gamma_N(\nu)$ and $|\gamma_N(\nu)|^2$.

2. Fourier Transforms, Full, Finite and Discrete

We want to determine the Fourier transform of a function $f(t)$ whose values are observed only at certain discrete times t_k. (We shall refer to t as a 'time', although in many cases it may be a space or other variable.) The literature in this field deals almost exclusively with the case where the data points are *equally spaced* – i.e. if

$$t_k = t_0 + k\Delta t, \tag{11}$$

where k is an integer, and Δt is the *data spacing*. We present here a version of the theory which is valid for arbitrary data spacing, including equal spacing as a special case. We include a parallel discussion of the theory for *continuously* observed functions, $f(t)$, for comparison purposes.

We define the full Fourier transform of a function $f(t)$ as the complex expression

$$F(\nu) = \int_{-\infty}^{+\infty} f(t)e^{i2\pi\nu t}\,\mathrm{d}t. \tag{12}$$

Minor variations in the definition exist – usually in the location of various factors of 2π, and in the sign of the exponent in (12). With this definition, the inverse transform is

$$f(t) = \int_{-\infty}^{+\infty} F(\nu)e^{-i2\pi\nu t}\,\mathrm{d}\nu. \tag{13}$$

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In proving the Fourier inversion theorem, it is necessary to evaluate an expression

$$\delta(\nu) = \int_{-\infty}^{+\infty} e^{i2\pi\nu t}\,\mathrm{d}t. \tag{14}$$

Its value may be taken as the limit of the corresponding finite integral

$$\delta_T(\nu) = \int_{-T/2}^{+T/2} e^{i2\pi\nu t}\,\mathrm{d}t = T\left(\frac{\sin \pi\nu T}{\pi\nu T}\right) \tag{15}$$

as $T \to \infty$; and is of course the Dirac delta function, with properties

$$\int_{-\infty}^{+\infty} \delta(\nu)\,\mathrm{d}\nu = 1; \qquad \int_{-\infty}^{+\infty} f(x)\delta(x-\xi)\,\mathrm{d}x = f(\xi). \tag{16}$$

We shall assume in this note that the scope of Fourier theory includes generalized functions such as the delta-function (Lighthill, 1958). The convolution theorem from standard Fourier theory will be referred to without proof. This states that if

$$z(t) = x(t)y(t), \tag{17}$$

then

$$Z(\nu) = X(\nu) * Y(\nu) = \int_{-\infty}^{+\infty} X(\nu - \nu')Y(\nu')\,\mathrm{d}\nu', \tag{18}$$

where $Z(\nu)$ is the Fourier transform of $z(t)$, etc. The inverse of this theorem is also true.

The quantity $F(\nu)F^*(\nu)$, where the asterisk denotes the complex conjugate, is sometimes referred to as the *power*. It must be carefully distinguished from the *power spectrum* of a stochastic process which we shall always denote as $P(\nu)$.

Most procedures for estimating $F(\nu)$ from finite amounts of data give rise to a result which is expressible as the convolution of the true $F(\nu)$ with a *spectral window*, $W(\nu)$, which is determined by the particular procedure used. That is, the quantity obtained is

$$F(\nu) * W(\nu) = \int F(\nu')W(\nu - \nu')\,\mathrm{d}\nu'. \tag{19}$$

Some rather awkward problems arise with the normalization of spectral windows. In practice, the choice of normalization will depend on the type of function, $f(t)$, being analysed. Ideally, one would like to have

$$\int W(\nu)\,\mathrm{d}\nu = 1; \tag{20}$$

but this is not always possible and in practice the choice $W(0) = 1$ is more useful. This is discussed further in Section 4.

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We now introduce two new expressions: the *finite Fourier transform* over a data length T

$$F_T(\nu) \equiv \int_{-T/2}^{+T/2} f(t) e^{i2\pi\nu t}\,\mathrm{d}t, \tag{21}$$

and the *discrete Fourier transform* over a set of N data points

$$F_N(\nu) \equiv \sum_{k=1}^{N} f(t_k) e^{i2\pi\nu t_k}. \tag{22}$$

These expressions are defined by analogy with $F(\nu)$, and are *not* to be regarded as attempts to evaluate $F(\nu)$ by some method of numerical integration. In particular, note that even for equal data spacing, $F_N(\nu)$ differs from a trapezium rule integration formula by (i) not having the factor $\frac{1}{2}$ at the end ordinates and (ii) omitting the multiplying factor Δt. Thus $F_N(\nu)$ is *dimensionally different* from $F(\nu)$ and $F_T(\nu)$. $F_N(\nu)$ is similar to functions frequently used in periodogram analysis.

Notice that we may write

$$F_{T,N}(\nu) = \int_{-\infty}^{+\infty} w_{T,N}(t) f(t) e^{i2\pi\nu t}\,\mathrm{d}t, \tag{23}$$

where

$$\begin{aligned} w_T(t) &= \begin{cases} 1; & (-T/2 \leqslant t \leqslant T/2) \\ 0; & \text{otherwise} \end{cases} \\ w_N(t) &= \sum_{k=1}^{N} \delta(t - t_k). \end{aligned} \tag{24}$$

The functions $w_{T,N}(t)$ are usually known as *data windows*. Additional data windows, which are sometimes introduced in practice to smooth the resulting transforms, are not discussed here.

It follows from Equation (23) and the convolution theorem, that

$$F_{T,N}(\nu) = F(\nu) * \bar{W}_{T,N}(\nu), \tag{25}$$

where

$$W_T(\nu) = \int_{-T/2}^{+T/2} e^{i2\pi\nu t}\,\mathrm{d}t = \delta_T(\nu) = T\left(\frac{\sin \pi\nu T}{\pi\nu T}\right), \tag{26}$$

$$W_N(\nu) = \sum_{k=1}^{N} e^{i2\pi\nu t_k} = \delta_N(\nu). \tag{27}$$

Thus both $F_T(\nu)$ *and* $F_N(\nu)$ *are the convolutions of the true Fourier transform with a*

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spectral window. This result is valid, regardless of the data spacing in the discrete transform.

In Equation (27) we have defined a new function

$$\delta_N(\nu) = \sum_{k=1}^{N} e^{i2\pi\nu t_k} \tag{28}$$

by analogy with $\delta_T(\nu)$. Note, however, that while $\delta_T(\nu)$ becomes an ordinary delta function as $T \to \infty$, so that $F_T(\nu)$ becomes identical with $F(\nu)$, this is not true of $\delta_N(\nu)$, which generally does *not* tend to a simple limit as $N \to \infty$. The form of $\delta_N(\nu)$ is specific to a given set of *times* t_k, and generally cannot be analytically simplified further, except in special cases, such as that of equal data spacing.

The spectral windows $W_T(\nu)$ and $W_N(\nu)$ suffice for the discussion of Fourier transforms of most functions, for continuous sampling over an interval $(-T/2, +T/2)$ or discrete sampling at (arbitrary) times t_k. However, an extension of the discussion is required for *stochastic functions*. If $f(t)$ is a realization of a stochastic process, then the expectation of $F(\nu)$ is zero at all frequencies, assuming that the stochastic process has a zero mean. That is

$$\langle F(\nu)\rangle = \left\langle \int_{-\infty}^{+\infty} f(t) \right\rangle e^{i2\pi\nu t} =$$

$$= \int_{-\infty}^{+\infty} \langle f(t)\rangle e^{i2\pi\nu t}\,\mathrm{d}t = 0. \tag{29}$$

Similarly the expectation of $F_T(\nu)$ and $F_N(\nu)$ is zero at all frequencies. Nevertheless, $F_T(\nu)F_T^*(\nu)$ and $F_N(\nu)F_N^*(\nu)$ have non-zero expectations which are related to the *power spectrum* of the stochastic process. We have

$$\langle F_T(\nu)F_T^*(\nu)\rangle = \left\langle \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(t)f(t')e^{i2\pi\nu t'}e^{-i2\pi\nu t}\,\mathrm{d}t\,\mathrm{d}t' \right\rangle =$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \langle f(t)f(t')\rangle e^{i2\pi\nu(t-t')}\,\mathrm{d}t\,\mathrm{d}t' =$$

$$= \sigma_f^2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} r_f(t-t')e^{i2\pi\nu(t-t')}\,\mathrm{d}t\,\mathrm{d}t', \tag{30}$$

where $r_f(\tau)$ is the *autocorrelation function* of the stochastic process of which $f(t)$ is a realization and σ_f^2 is the *variance* of the stochastic process.

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The *power spectrum* of a stochastic process is defined as the Fourier transform of its autocorrelation function:

$$P_f(\nu) \equiv \int_{-\infty}^{+\infty} r(\tau) e^{i2\pi\nu\tau}\, \mathrm{d}\tau = 2 \int_0^{\infty} r(\tau) \cos 2\pi\nu\tau; \tag{31}$$

and by the Fourier inversion theorem, $r(\tau)$ is given by

$$r_f(\tau) = \int_{-\infty}^{+\infty} P_f(\nu) e^{-i2\pi\nu\tau}\, \mathrm{d}\nu. \tag{32}$$

Hence,

$$\langle F_T(\nu) F_T^*(\nu) \rangle = \sigma_f^2 \int_{-\infty}^{+\infty} P_f(\nu') \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{2\pi(\nu-\nu')(t-t')}\, \mathrm{d}t\, \mathrm{d}t'\, \mathrm{d}\nu' =$$

$$= \sigma_f^2 P_f(\nu) * V_T(\nu), \tag{33}$$

where the *power spectral window* $V_T(\nu)$ is given by

$$V_T(\nu) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} e^{i2\pi\nu(t-t')}\, \mathrm{d}t\, \mathrm{d}t' = \delta_T(\nu) \delta_T^*(\nu) =$$

$$= T^2 \frac{\sin^2(\pi\nu T)}{(\pi\nu T)^2}. \tag{34}$$

In a similar way, one may show that

$$\langle F_N(\nu) F_N^*(\nu) \rangle = \sigma_f^2 P_f(\nu) * V_N(\nu), \tag{35}$$

where

$$V_N(\nu) = \sum_{j,k} e^{i2\pi\nu(t_j - t_k)} = \delta_N(\nu) \delta_N^*(\nu) = |\delta_N(\nu)|^2. \tag{36}$$

Thus the *observed power*, $FF_{T,N}^*$, is proportional to the convolution of the power spectrum of the stochastic process with a power spectral window. (For generality, the variance σ_f^2 of the stochastic process is usually factored out of the power spectrum as we have done here. However, the reader should be aware that sometimes the power spectrum of a stochastic process is defined as the Fourier transform of the *auto-covariance* function rather than the *autocorrelation* function, in which case the factor of σ_f^2 is absorbed into $P_f(\nu)$.)

It should be carefully noted that this result for stochastic functions is *not* generally true for all functions; that is, generally $FF_{T,N}^*$ is *not* given by the convolution of the true power FF^* with the spectral window $\delta_{N,T}(\nu)^2$. This is because convolution and multiplication are *not* mutually associative, i.e.

$$a*(bc) \neq (a*b)c. \tag{37}$$

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We clearly have

$$F_{T,N}(\nu)F^*_{T,N}(\nu) = [F(\nu)*\delta_{T,N}(\nu)]F^*(\nu)*\delta^*_{T,N}(\nu) \neq \neq F(\nu)F^*(\nu)*[\delta^*_{T,N}(\nu)\delta_{T,N}(\nu)]. \tag{38}$$

It should also be pointed out that neither the inversion theorem nor the convolution theorem is true for the finite or discrete transforms.

Finally, it should be remarked that the definition of the power spectrum of a stochastic process as the Fourier transform of its autocorrelation function was for many years used as the basis for numerical computation of the power spectrum from a given data set, since fewer computations were required than to do a direct Fourier transform, with a resulting saving in computer time. However, such a method is necessarily restricted to equally spaced (or continuous) data, since there is no direct way of calculating an autocorrelation function with unequally spaced data. Recently the Cooley-Tukey algorithm (the Fast Fourier Transform) has allowed the rapid calculation of direct Fourier transforms, so that this older method of computing power spectra is used less now. Unfortunately, the Fast Fourier Transform also requires that data be equally spaced, so it cannot be used with unequally spaced data without interpolation.

Results similar to Equations (33) and (35) may be obtained for the cross-spectra of two stochastic processes, $f(t)$ and $g(t)$. For example one may show that

$$\langle F^*_T(\nu)G_T(\nu)\rangle = \sigma_f\sigma_g P_{fg}(\nu)*V_T(\nu), \quad V_T(\nu) = |\delta_T(\nu)|^2, \tag{39}$$

and

$$\langle F^*_N(\nu)G_N(\nu)\rangle = \sigma_f\sigma_g P_{fg}(\nu)*V_{N(fg)}(\nu), \quad V_{N(fg)}(\nu) = \delta^*_{N(f)}(\nu)\delta_{N(g)}(\nu), \tag{40}$$

where $P_{fg}(\nu)$ is the cross power spectrum between f and g, and $\delta_{N(f)}(\nu)$ and $\delta_{N(g)}(\nu)$ are the functions $\delta_N(\nu)$ calculated at the times $f(t)$ is observed, and the times $g(t)$ is observed, respectively. If these are the same, then $V_{N(fg)}(\nu)$ reduces to $V_N(\nu)$ defined in Equation (36).

3. Spectral Windows

In the previous section, we have shown that the finite transform $F_T(\nu)$ and the discrete transform $F_N(\nu)$ are given by the convolution of the true $F(\nu)$ with spectral windows $W_T(\nu)$ and $W_N(\nu)$. We have also shown that for stochastic processes, $F_T(\nu)F^*_T(\nu)$ and $F_N(\nu)F^*_N(\nu)$ are given, in the mean, by the convolution of the power spectrum of the stochastic process with power spectral windows $V_T(\nu)$ and $V_N(\nu)$. In this section, we discuss the properties of these spectral windows.

The awkward problem of spectral window normalization has been briefly referred to above. There are generally two aspects to normalization, the first one of mathematical convenience, so that inessential features of a function (such as its amplitude) are eliminated while essential features (such as its shape) are retained, and the second

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one of physical convenience, so that the 'right' answer is produced in the most convenient form. It is not always obvious what should be the 'right' answer; for example, if one is analysing a periodic function, so that the Fourier transform will show peaks at the fundamental frequency and its harmonics, does one wish to have the amplitude of each harmonic given by the *magnitude* of the peak in $F_T(\nu)$, or by the *area* under the peak? The normalization will be different in each case, and will be different again if a non-periodic function or a stochastic function is being analysed.

For convenience in discussing the *shape* of the spectral windows, we shall consider in this section the functions $\gamma_T(\nu)$ and $\gamma_N(\nu)$ which are defined by

$$\gamma_T(\nu) = \frac{1}{T}\delta_T(\nu) = \frac{\sin \pi\nu T}{\pi\nu T}, \tag{41}$$

$$\gamma_N(\nu) = \frac{1}{N}\delta_N(\nu) = \frac{1}{N}\sum e^{i2\pi\nu t_k}. \tag{42}$$

These functions clearly have the same shape as $\delta_T(\nu)$ and $\delta_N(\nu)$ but have the normalization property that $\gamma_{T,N}(0)=1$. (Note that the maximum possible value for $\gamma_{T,N}(\nu)$ is unity, whatever the data spacing.) In the following section, we shall discuss the physical aspects of the normalization problem for different types of function, and we shall conclude that this mathematical normalization to $\gamma(0)=1$ is the most convenient in practice.

The sinc function shape of $\gamma_T(\nu)$ is well known and is discussed in the standard texts (e.g. Blackman and Tukey, 1958). It has the property that its integral is $1/T$, and hence that of $\delta_T(\nu)$ is unity. The sinc^2 function, $\gamma_T(\nu)^2$, appears in the power spectral window for stochastic processes; its integral is also $1/T$, and hence that of $\delta_T(\nu)^2$ is T.

In the more interesting case of discrete data, we have noted already that the shape of $\delta_N(\nu)$ and hence of $\gamma_N(\nu)$ is determined by the distribution of the times t_k. We may immediately note, however, that both $\gamma_N(\nu)$ and $\delta_N(\nu)$ have divergent integrals

$$\int_{-\infty}^{+\infty} \gamma_N(\nu)\,\mathrm{d}\nu = \frac{1}{N}\sum_k \int_{-\infty}^{+\infty} e^{i2\pi\nu t_k}\,\mathrm{d}\nu = \frac{1}{N}\sum_k \delta(t_k) \to \infty, \tag{43}$$

so that we may expect difficulties with a physical normalization which depends on the integral of the spectral window, in the discrete data case. We also note that, although the shape of the spectral window depends on the exact distribution of times t_k, if these points are distributed across the time interval $(-T/2, +T/2)$ according to some distribution function $\phi(t)$, then $\gamma_N(\nu)$ can be written, in the limit for large N, as

$$\gamma_N(\nu) \simeq \int_{-T/2}^{+T/2} \phi(t)e^{i2\pi\nu t}\,\mathrm{d}t. \tag{44}$$

Thus the spectral window is approximately the finite Fourier transform of the distribution of data points. In particular, if the data points are randomly distributed across

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the time interval $(-T/2, +T/2)$ according to a uniform distribution, we should expect a spectral window which approximates that of the continuous case – the sinc function $\gamma_T(\nu)$.

The case of equal data spacing represents perhaps the most non-random possible distribution of times, t_k. If we substitute Equation (11) into the expression for $\gamma_N(\nu)$, we find that

$$\gamma_N(\nu) = \frac{1}{N}\sum_k e^{i2\pi\nu t_0} e^{i2\pi\nu k\Delta t} = e^{i2\pi\nu t_0} e^{i\pi\nu(N+1)\Delta t} \frac{\sin \pi\nu N\Delta t}{N \sin \pi\nu\Delta t}. \tag{45}$$

It is always possible to choose t_0 so that the exponential terms in this expression vanish – this will be the case, for instance, if t_0 is taken as the mid-point of the total time covered – and in this case $\gamma_N(\nu)$ becomes purely real, and we may refer to it as a *phase-adjusted spectral window*

$$\gamma_N(\nu) = \frac{\sin \pi\nu N\Delta t}{N \sin \pi\nu\Delta t}. \tag{46}$$

The most important properties of this function are:

(i) it is *symmetric*: i.e.,

$$\gamma_N(-\nu) = \gamma_N(\nu); \tag{47}$$

(ii) it is *periodic*, with period Δt^{-1}: i.e.,

$$\gamma_N(\nu) = \gamma_N(\nu + n\Delta t^{-1}); \tag{48}$$

(iii) for small values of ν, it is approximately the same as $\gamma_T(\nu)$ with $T = N\Delta t$: i.e.,

$$\gamma_N(\nu) \simeq \frac{\sin \pi\nu N\Delta t}{N\pi\nu\Delta t} = \frac{\sin \pi\nu T}{\pi\nu T}. \tag{49}$$

(iv) Combining its property of symmetry and periodicity, we see that

$$\gamma_N(\nu) = \gamma_N(n\Delta t^{-1} + \nu). \tag{50}$$

In brief, $\gamma_N(\nu)$ is approximately like an infinite row of sinc functions spaced $(\Delta t)^{-1}$ apart. As N tends to infinity, $\delta_N(\nu)$ tends to an infinite row of delta functions spaced $(\Delta t)^{-1}$ apart, sometimes called an *infinite Dirac comb*. The corresponding power spectral window for stochastic processes in $|\gamma_N(\nu)|^2$ and is clearly approximately a row of sinc^2 functions spaced $(\Delta t)^{-1}$ apart. It has the same symmetry properties, (47), (48), (50), as $\gamma_N(\nu)$.

We see that for equally spaced data, $\gamma_N(\nu)$ takes the value unity, its maximum value, at an infinite set of frequencies $\nu_n = n(\Delta t)^{-1}$. For unequally spaced data, the periodicity/symmetry property of $\gamma_N(\nu)$ will not generally hold; however, we may well expect that $\gamma_N(\nu)$ will take on large values, perhaps near unity, at frequencies far removed

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from $\nu=0$. Notice also that generally $\gamma_N(\nu)$ will not be purely real, so that there will be phase shifts in $F_N(\nu)$ compared to $F(\nu)$. The *power spectral window*, $V_N(\nu)=|\gamma_N(\nu)|^2$ is of course purely real by definition; it will generally also show some large values far removed from $\nu=0$.

If the spectral window has large amplitudes at frequencies far removed from $\nu=0$, then frequencies in the true Fourier transform, or the true power spectrum of a stochastic process, which are far removed from the frequency of interest will contribute significantly to $F_N(\nu)$. For example, if $W_N(\nu)$ is large at $\nu=\nu_0$, say, then since

$$F_N(\nu) = F*W_N = \int F(\nu-\nu')W_N(\nu')\,\mathrm{d}\nu', \tag{51}$$

it follows that a significant contribution to $F_N(\nu)$ comes from $F(\nu-\nu_0)$. This phenomenon is customarily described as *aliasing*, the frequencies ν, $\nu-\nu_0$ in the above example being *aliases* of each other. For the case of equal data spacing, this aliasing is *complete*, so that these frequencies are indistinguishable from one another. This is easily seen, for if a spectral window has the periodicity/symmetry property (40), then

$$\begin{aligned} F_N(\nu) &= \int_{-\infty}^{+\infty} F(\nu')W(\nu-\nu')\,\mathrm{d}\nu' = \sum_{n=-\infty}^{+\infty} \int_{(2n-1)\nu_\mathrm{N}}^{(2n+1)\nu_\mathrm{N}} F(\nu')W(\nu-\nu')\,\mathrm{d}\nu' = \\ &= \int_{-\nu_\mathrm{N}}^{+\nu_\mathrm{N}} \sum_{n=-\infty}^{+\infty} F(n\Delta t^{-1}+\nu')W(\nu-\nu')\,\mathrm{d}\nu' \equiv \\ &\equiv \int_{-\nu_\mathrm{N}}^{+\nu_\mathrm{N}} F^A(\nu')W(\nu-\nu')\,\mathrm{d}\nu', \end{aligned} \tag{52}$$

where ν_N is equal to $1/(2\Delta t)$ and is usually known as the *Nyquist frequency*. Thus the full $F(\nu)$ is equivalent to an *aliased Fourier transform*

$$F^A(\nu) = \sum_{n=-\infty}^{+\infty} F(n\Delta t^{-1}+\nu) \tag{53}$$

defined only over $(-\nu_\mathrm{N}, +\nu_\mathrm{N})$. Similarly, we may define an *aliased power spectrum* for a stochastic process

$$P^A(\nu) = \sum_{n=-\infty}^{+\infty} P(n\Delta t^{-1}+\nu). \tag{54}$$

Obviously, many different forms of $F(\nu)$ could give rise to the same $F^A(\nu)$, and likewise many different forms of $P(\nu)$ could give rise to the same $P^A(\nu)$. This is the sense in which the frequencies $[n(\Delta t)^{-1}\pm\nu]$ are aliases of each other. The behavior of the spectral windows for the unequally spaced data case cannot be described so neatly in terms of complete aliasing.

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The heart of the matter in any practical application is clearly the particular data spacing involved, and the particular spectral window it gives rise to. Also, the fact that the spectral window depends only on the choice of times, t_k, allows one, at least in principle, the possibility of designing a spectral window by the choice of these times. In particular, it may be possible, with the same total number of data points, to reduce the effect of aliasing by carefully choosing non-commensurate intervals between successive t_k. It appears to be nearly impossible to handle this analytically, so we have performed some computer experiments with various unequal data spacings to see what spectral windows are produced.

We have distributed 25 points over the same time interval with spacings ranging from equally spaced, to one with two points at each end and the rest at the center. The intermediate spacings represent varying degrees of concentration toward the center. The spacing Δt_k between t_k and t_{k+1} is proportional to

$$k^{-1/\alpha}, \qquad (k = 1 \ldots 12)$$
$$(25 - k)^{-1/\alpha}, \qquad (k = 13 \ldots 24),$$

where α is a constant. Figure 1 shows the spectral windows obtained for various values of α. The frequency unit is the Nyquist frequency for equal spaced data, i.e. $12.5/T$, where T is the total data length, and we have plotted the data out to five times the Nyquist frequency to show the aliasing peaks for equal spacing, and the way in which these are reduced when the data spacing is changed. In terms of lack of side lobes on the central peak, and the reduction of aliasing peaks, the case $\alpha = 2$ seems to be about the best. We have searched at even higher frequencies in this case and find no peak as high as 30% of the central peak out to 18.86 times the 'Nyquist frequency', none as high as 50% out to 39.33 times the 'Nyquist frequency' and none as high as 60% to the limit of our search, 92 times the 'Nyquist frequency'. When possible, this seems to be an effective way of reducing aliasing.

Unfortunately, one cannot often preselect the spacing of one's data. This is more often determined by outside influences of a non-scientific nature. To give an idea of what may be obtained in practice, Figure 2 shows the power spectral window for data obtained by Kinman (1973) in a long series of photometric observations of the quasi-stellar object 3C345. Because the data covers a long period of time, the central peak is very sharp and well defined. In addition, there are three small secondary peaks, with the rest of the spectral window being very close to zero. These three small peaks occur at periods of precisely one year, one lunar synodic month and one calendar month. It is satisfying that these are just the periods at which one might expect something unusual to happen, since it is well known that any astronomical object has a seasonal appearance in the night sky, so that a one year periodicity is bound to show up in any long series of astronomical observations. In addition, photometric observations can be made with precision only at times of the dark of the Moon, hence the appearance of the synodic month. The appearance of the calendar month is a little more surprising, but it must be remembered that observatories do run their affairs according

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to the civil calendar and that scheduling of observing time is done in units of the ordinary week, even though account is taken of the periods of dark Moon time. The important thing to notice is that the power spectral window is very well behaved at frequencies other than those at which we expect some trouble.

4. Physical Normalization

It was pointed out in the previous section that one usually wishes to normalize the result of a mathematical operation such as the Fourier transform so that one obtains a result which is directly relevant to a physical property of the system one is investigating. We have, so far, shown that

$$\begin{aligned} &\text{(a)}\ F_T(\nu) = TF(\nu)^*\gamma_T(\nu), \\ &\text{(b)}\ F_N(\nu) = NF(\nu)^*\gamma_N(\nu), \\ &\text{(c)}\ \langle F_T F_T^*(\nu)\rangle = \operatorname{var}(f)T^2P(\nu)^*\gamma_T(\nu)^2, \\ &\text{(d)}\ \langle F_N F_N^*(\nu)\rangle = \operatorname{var}(f)N^2P(\nu)^*\gamma_N(\nu)^2. \end{aligned} \tag{55}$$

We may now consider what the effect will be of applying the finite and discrete transforms to various function types. We shall restrict our attention here to the three function types: 'good' functions, in the sense used by Lighthill (*op. cit.*), periodic functions, and stochastic functions.

'Good' functions (or non-periodic determinate functions) are those for which

$$\int_{-\infty}^{+\infty} |f(t)|\ \mathrm{d}t \tag{56}$$

exists, and thus the Fourier transform exists, without the need for introducing generalized functions. An example would be the profile of a spectral line or a group of spectral lines. Periodic functions include multiply periodic functions. An example would be the light curve of a pulsating variable star. Stochastic functions, of course, represent noise of various origins, including the nearly white noise associated with observational errors.

To obtain an meaningful physical normalization, we shall require that a finite, non-zero, physically meaningful result be obtained in the limit as T (or N) tends to infinity. Other approaches are possible, but this one is convenient and has the incidental advantage of showing us the effect of changing the data length on the Fourier transform for different function types.

We first note that if $U_T(\nu)$ is any function with an integral of unity, and with the property that as $T \to \infty$, it becomes narrower and narrower, tending in the limit to a delta function, then

$$\lim_{T\to\infty} Q(\nu)^* U_T(\nu) = Q(\nu). \tag{57}$$

Since $\gamma_T(\nu)$ has an integral of $1/T$, it follows from (55a) and (57) that the finite

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transform of a good function tends to the value of $F(\nu)$ as $T\to\infty$. In this sense, the good function is well-behaved, satisfying the requirement that a meaningful result be obtained as $T\to\infty$ so that no additional physical normalization is required.

If $f(t)$ is a periodic function

$$f(t) = ae^{i2\pi\nu_0 t} + be^{-i2\pi\nu_0 t}, \tag{58}$$

then

$$F_T(\nu) = a\delta_T(\nu + \nu_0) + b\delta_T(\nu - \nu_0). \tag{59}$$

In the limit as $T\to\infty$, this becomes a delta function at $-\nu_0$ and another at $+\nu_0$. If one wishes to have the *area* under the peaks at $\nu=\pm\nu_0$ give the corresponding amplitude, then, since $\delta_T(\nu)$ has a unit integral, no further normalization is required. On the other hand, if one wants the *size of the peak* to give the corresponding amplitude, then it will be necessary to divide by T as a normalizing factor [since $\delta_T(0)=T$] to obtain

$$\hat{a} = \lim_{T\to\infty}\frac{1}{T}F_T(-\nu_0), \qquad \hat{b} = \lim_{T\to\infty}\frac{1}{T}F_T(\nu_0). \tag{60}$$

In other words, the amplitude of the peak increases directly as T, the area under the peak stays constant with T.

Finally, from (55c), and (57), it is clear that $\langle F_T F_T^*\rangle$ increases directly as T, so that the estimate

$$\widehat{P(\nu)} = \frac{1}{\text{var}\,(f)}\frac{1}{T}F_T(\nu)F_T^*(\nu) \tag{61}$$

will tend to the true value of $P(\nu)$ as $T\to\infty$. Alternatively, we may say that the rms value of $F_T(\nu)$ goes as $T^{1/2}$ for a stochastic function.

Imagine, then, an $f(t)$ composed of a *combination* of good functions, periodic functions and stochastic functions. It is clear from the respective behavior of their $F_T(\nu)$ that the relative contribution of each to the finite Fourier transforms will vary with the data length T. This is a complicated way of saying, for example, that it is possible to detect a periodic signal in the presence of noise if observations are continued long enough. It also shows the importance of removing periodic and other trends from stochastic processes before analysing them.

The situation with the discrete transform is a good deal more difficult because of the phenomenon of aliasing, and because the spectral windows $\delta_N(\nu)$ and $|\delta_N(\nu)|^2$ cannot be normalized to an integral of unity. For equally spaced data with complete aliasing, then, if we agree to accept the aliased spectrum $F^A(\nu)$ or $P^A(\nu)$ as a physically meaningful result, the same discussion we gave above for the continuous case also applies to this case, i.e. $F_N(\nu)$ for a good function tends to a constant limit as $N\to\infty$, that for a periodic function goes as N, and that for a stochastic functions goes as $N^{1/2}$. Therefore,

$$\hat{a} = \frac{1}{N}F_N(-\nu_0), \qquad \hat{b} = \frac{1}{N}F_N(\nu_0) \tag{62}$$

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and

$$\widehat{P^A(\nu)} = \frac{1}{\operatorname{var}(f)}\frac{1}{N}F_N(\nu)F_N^*(\nu). \tag{63}$$

For the case of unequal data spacing, where complete aliasing is not present to simplify the situation, it is not possible to make a priori judgements as to the normalization properties of the three types of function, except that since $\delta_N(0)=N$ for all data spacing, one expects Equation (62) also to be valid here, provided the data spacing is not extremely pathological.

Because of these problems of physical normalization, the non-integrability of $\delta_N(\nu)$ and the different treatment required for different types of function, it has proved most convenient in practice to program the calculation of $N^{-1}F_N(\nu)$ rather than $F_N(\nu)$. Then the resulting spectral window $\gamma_N(\nu)$ is always normalized to $\gamma_N(0)=1$. The interpretation of the amplitudes in the transform or power spectrum obtained must then be made in terms of the particular model one has for $f(t)$ as discussed in the previous section. In practice, this rarely is a problem, for Fourier analysis or power spectrum analysis is often used primarily as an exploratory tool to discover the significant periodicities or characteristic time scales present. More detailed analysis is then carried out on the basis of a specific physical model suggested by the exploratory analysis.

5. Conclusions

We conclude that it is possible to use the transform $F_N(\nu)$ for arbitrary data spacings in Fourier and power spectrum analysis with results that are comparable to analysis with equal data spacing. The main difference is that whereas aliasing can be predicted in advance for the case of equal data spacing, it must be analysed after the fact, in terms of the actual t_k used and the resultant spectral window, for the case of unequal data spacing.

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Appendix: Calculation of $F_N(\nu)$

This appendix contains explicit instructions for calculating the Fourier transform $F_N(\nu)$ using FORTRAN. It is assumed that the data are stored so that F(I) contains the datum and T(I) the time of the I'th observation. There are N observations. It is necessary to choose (i) a *frequency interval* DF, and (ii) high and low *frequency indices*, KH and KL so that the transform will be calculated at all frequencies from $\nu=$ KL*DF to $\nu=$ KH*DF. KL and KH are, of course, integers. In practice, since one must know

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the spectral window, one needs information in the vicinity of $\nu=0$, so KL is almost always 0. Also, it is useful to cover the spectral window fairly densely, so DF should not be too large. The real part of $F_N(\nu)$ is stored as FR(K), where $\nu=K*DF$, the imaginary part is FI(K), and the amplitude squared is FF(K) (although we normally divide this by N^2, so as to normalize the spectral window to unit amplitude at zero frequency). The real part of the spectral window is D(K), the imaginary part G(K) and the (normalized) amplitude squared $[|\gamma_N(\nu)|^2]$ is GG(K). PI has the value π. The basic program is then

```
     DO1K = KL, KH
     FR(K) = 0.
     FI(K) = 0.
     D(K) = 0.
     G(K) = 0.
     A = 2.*PI*K*DF
     DO2I = 1, N
     A = A*T(I)
     C = COS(A)
     S = SIN(A)
     FR(K) = FR(K) + F(I)*C
     FI(K) = FI(K) + F(I)*S
     D(K) = D(K) + C
     G(K) = G(K) + S
   2 CONTINUE
     FF(K) = (FR(K)*FR(K) + FI(K)*FI(K))/N*N
     GG(K) = (D(K)*D(K) + G(K)*G(K))/N*N
   1 CONTINUE
```

Note that FF(K) and GG(K) have been normalized by dividing by N^2. What is printed or punched as output depends on specific needs. As an initial investigation, we normally use a plotter to give FF(K) and GG(K) graphically, and also output the numerical values of all calculated quantities. Finally, note that it is usual in most investigations to remove the mean and the first moment (trend) of a set of data before subjecting it to a Fourier analysis. This avoids bias from non-physical zeros in the data, and from long-term secular changes which are perhaps not of interest. It should be borne in mind, however, that this process does distort to some extent the low frequency end of the derived spectrum, since, among other things, it forces the power at zero frequency to be zero.

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